Hadron interactions from lattice QCD

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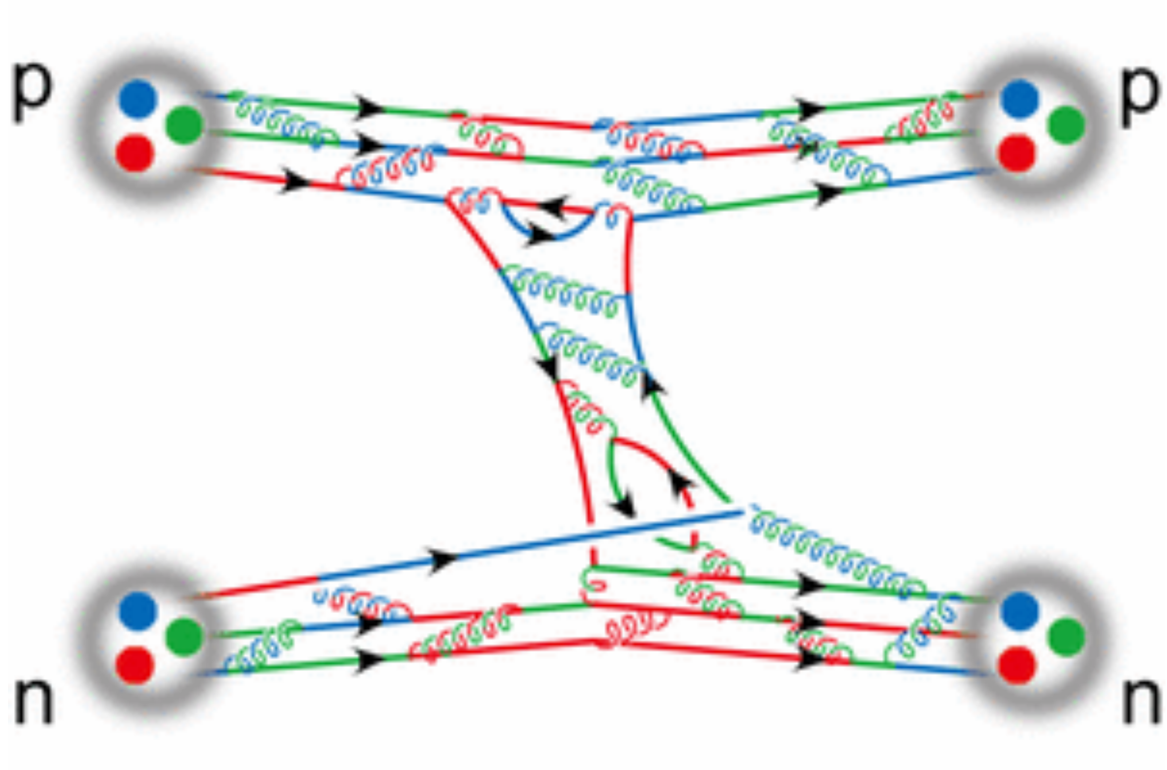
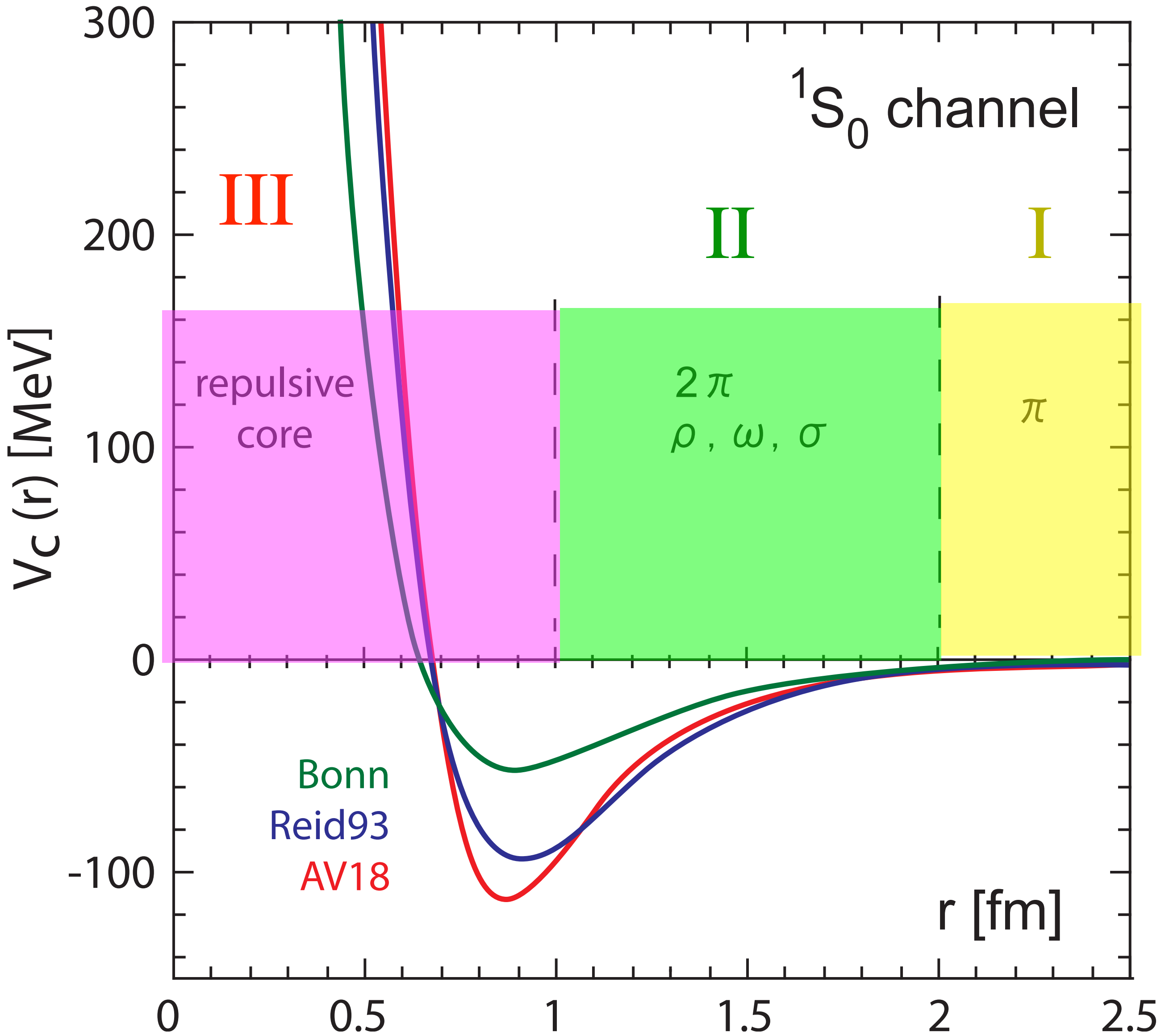


GGI Workshop “New Frontiers in Lattice Gauge Theory”
GGI, Firenze, Italy, September 12, 2012

1. Introduction

How can we extract hadronic interaction from lattice QCD ?

Ex. **Phenomenological NN potential**
(~40 parameters to fit 5000 phase shift data)



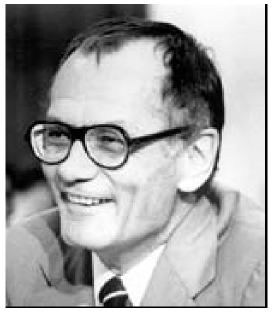
I One-pion exchange

Yiukawa(1935)

II Multi-pions

Taketani et al.(1951)

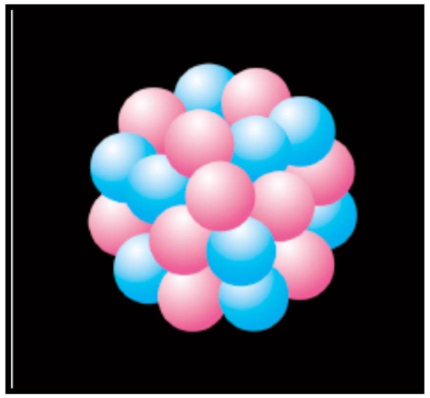
III Repulsive core

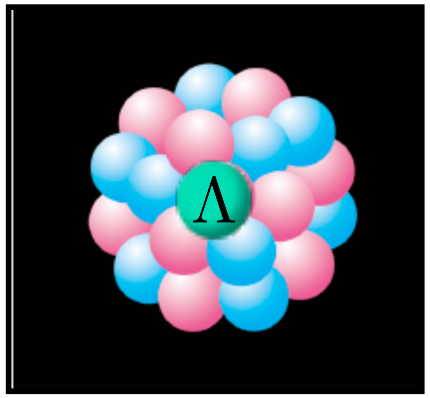


Jastrow(1951)

Nuclear force is a basis for understanding ...

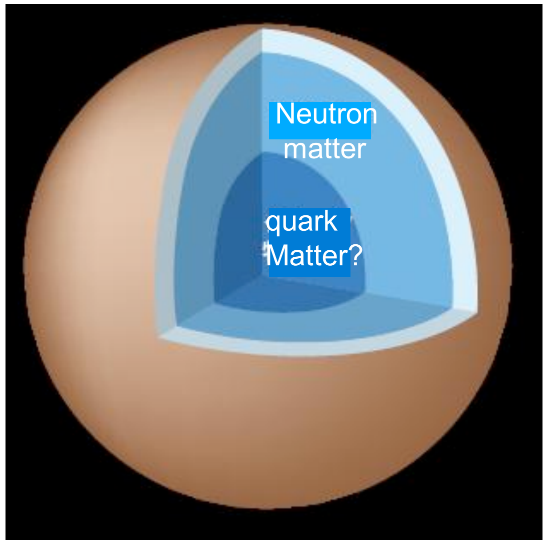
- Structure of ordinary and hyper nuclei



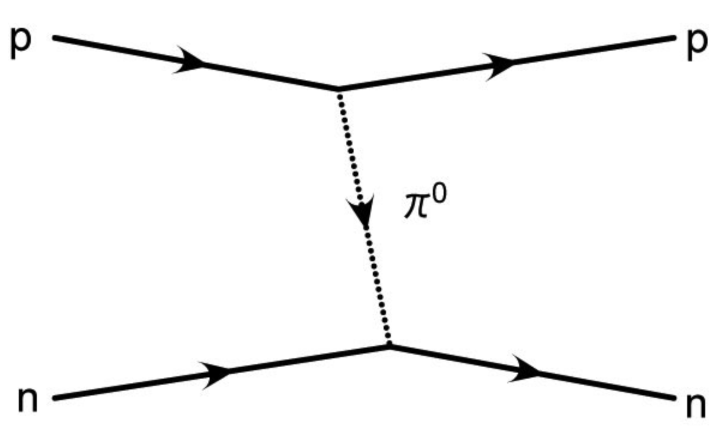


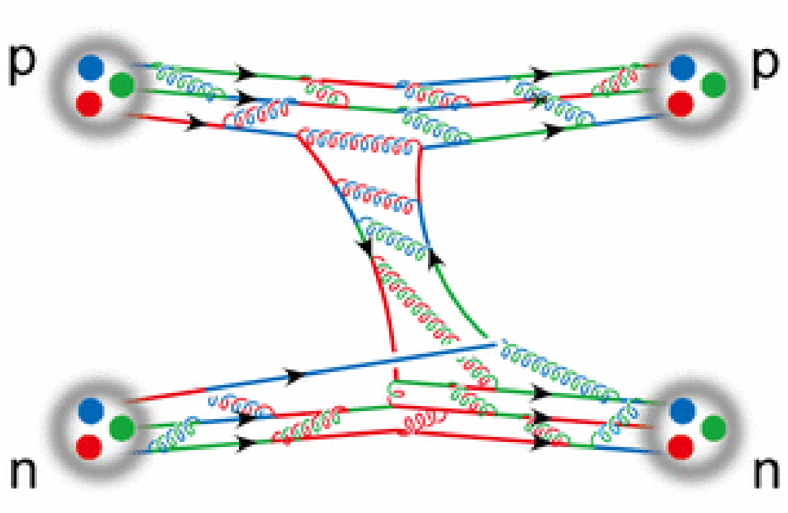
- Ignition of Type II SuperNova

- Structure of neutron star



Can we extract a nuclear force in (lattice) QCD ?





Plan of my talk

1. Introduction
2. Our strategy
3. Example: Nuclear potential
4. Inelastic Scattering (work in progress)
5. Demonstration (as a conclusion)

2. Our Strategy

Our strategy in lattice QCD

Full details: Aoki, Hatsuda & Ishii, PTP123(2010)89.

Step 1 define (Equal-time) Nambu-Bethe-Salpeter (NBS) Wave function

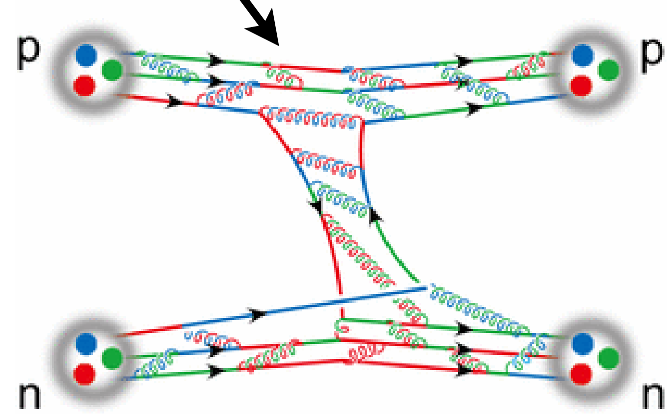
Spin model: Balog et al., 1999/2001

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0 | N(\mathbf{x}+\mathbf{r},0) N(\mathbf{x},0) | NN, W_k \rangle$$

$W_k = 2\sqrt{\mathbf{k}^2 + m_N^2}$

energy

$N(x) = \varepsilon_{abc} q^a(x) q^b(x) q^c(x)$: local operator



Important property

partial wave $\varphi_{\mathbf{k}}^l \to A_l \dfrac{\sin(kr - l\pi/2 + \delta_l(k))}{kr}$ $r = |\mathbf{r}| \to \infty$

Lin et al., 2001; CP-PACS, 2004/2005

$\delta_l(k)$ scattering phase shift (phase of the S-matrix) in QCD !

How can we extract it ?

cf. Maiani-Testa theorem

cf. Luescher's finite volume method

Step 2

define non-local but energy-independent “potential” as

$$\left[\epsilon_k - H_0\right] \varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, U(\mathbf{x},\mathbf{y}) \varphi_{\mathbf{k}}(\mathbf{y})$$

$\mu = m_N/2$

reduced mass

$$\epsilon_k = \frac{\mathbf{k}^2}{2\mu} \qquad H_0 = \frac{-\nabla^2}{2\mu}$$

non-local potential

Properties & Remarks

1. Potential itself is NOT an observable. Using this freedom, we can construct a non-local but energy-independent potential as

inner product

$$U(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'}^{W_k, W_{k'} \leq W_{\mathrm{th}}} \left[\epsilon_k - H_0\right] \varphi_{\mathbf{k}}(\mathbf{x}) \eta^{-1}_{\mathbf{k},\mathbf{k}'} \varphi^{\dagger}_{\mathbf{k}'}(\mathbf{y})$$

$\eta^{-1}_{\mathbf{k},\mathbf{k}'}$: inverse of $\eta_{\mathbf{k},\mathbf{k}'} = (\varphi_{\mathbf{k}}, \varphi_{\mathbf{k}'})$

$\varphi_{\mathbf{k}}$ is linearly independent.

For $^{\forall} W_{\mathbf{p}} < W_{\mathrm{th}} = 2m_N + m_\pi$ (threshold energy)

$$\int d^3y\, U(\mathbf{x},\mathbf{y}) \phi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'} \left[\epsilon_k - H_0\right] \varphi_{\mathbf{k}}(x) \eta^{-1}_{\mathbf{k},\mathbf{k}'} \eta_{\mathbf{k}',\mathbf{p}} = \left[\epsilon_p - H_0\right] \varphi_{\mathbf{p}}(x)$$

Proof of existence (cf. Density Functional Theory)

Of course, potential satisfying this is not unique. (Scheme dependence. cf. running coupling)

2. Non-relativistic approximation is NOT used. We just take the specific (equal-time) flame.

Step 3 expand the non-local potential in terms of derivative as

$$U(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\nabla)\delta^3(\mathbf{x}-\mathbf{y})$$

$$V(\mathbf{x},\nabla) = V_0(r) + V_\sigma(r)(\sigma_\mathbf{1}\cdot\sigma_\mathbf{2}) + V_T(r)S_{12} + V_{\rm LS}(r)\mathbf{L}\cdot\mathbf{S} + O(\nabla^2)$$

LO LO LO NLO NNLO

spins

tensor operator $S_{12} = \frac{3}{r^2}(\sigma_\mathbf{1}\cdot\mathbf{x})(\sigma_\mathbf{2}\cdot\mathbf{x}) - (\sigma_\mathbf{1}\cdot\sigma_\mathbf{2})$

$V_A(\mathbf{x})$ local and energy independent coefficient function
(cf. Low Energy Constants(LOC) in Chiral Perturbation Theory)

Step 4 extract the local potential at LO as

$$V_{\rm LO}(\mathbf{x}) = \frac{[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x})}{\varphi_{\mathbf{k}}(\mathbf{x})}$$

Step 5 solve the Schroedinger Eq. in the infinite volume with this potential.

→ phase shifts and biding energy below inelastic threshold

(We can calculate the phase shift at all angular momentum.)

$\delta_L(k)$ exact by construction

$\delta_L(p \neq k)$ approximated one by the derivative expansion

expansion parameter

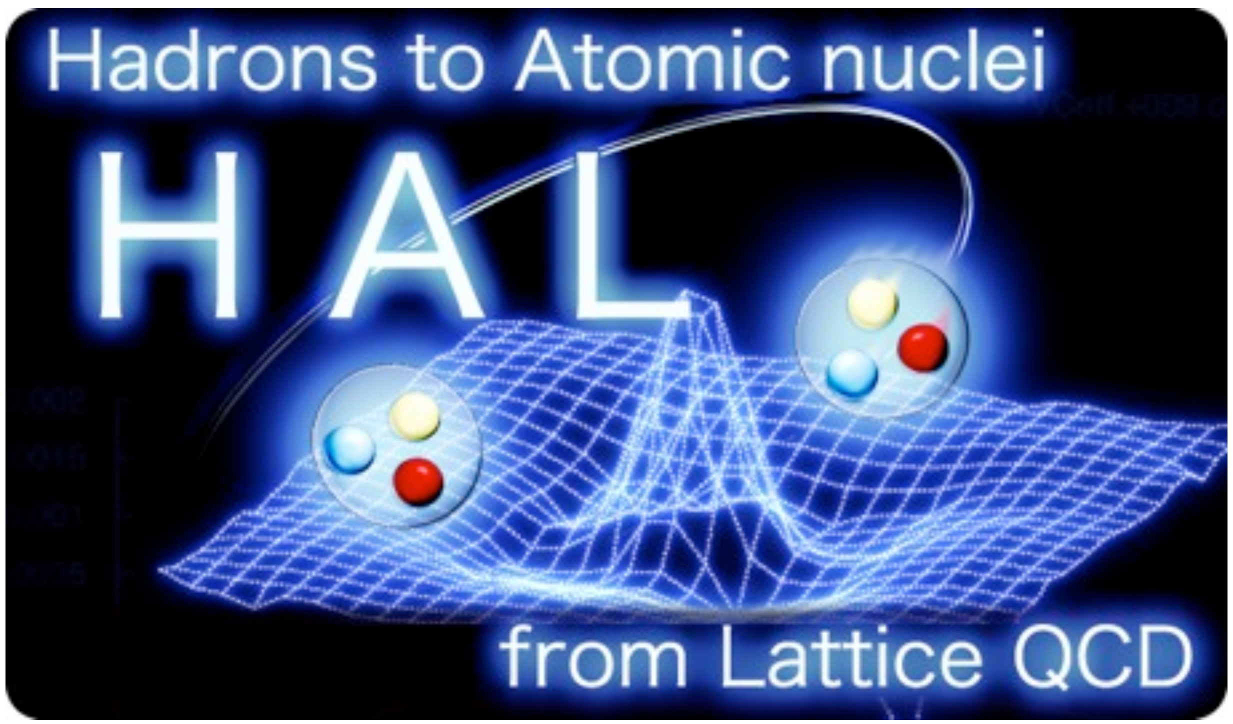
$$\frac{W_p - W_k}{W_{\rm th} - 2m_N} \simeq \frac{\Delta E_p}{m_\pi}$$

We can check a size of errors at LO of the expansion. (See later).
We can improve results by extracting higher order terms in the expansion.

This procedure gives a new method to extract phase shift from QCD.
(by-pass Maiani-Testa theorem, using space correlation)

HAL QCD method

HAL QCD Collaboration



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Our strategy

Potentials from lattice QCD → Nuclear Physics with these potentials → Neutron stars Supernova explosion

3. Example:Nuclear potential

Extraction of NBS wave function

NBS wave function

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0|N(\mathbf{x}+\mathbf{r},0)N(\mathbf{x},0)|NN, W_k\rangle$$

Potential

$$[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, U(\mathbf{x},\mathbf{y})\varphi_{\mathbf{k}}(\mathbf{y})$$

4-pt Correlation function

source for NN

$$F(\mathbf{r}, t-t_0) = \langle 0|T\{N(\mathbf{x}+\mathbf{r},t)N(\mathbf{x},t)\}\underline{\overline{\mathcal{J}}(t_0)}|0\rangle$$

complete set for NN

$$\begin{aligned} F(\mathbf{r}, t-t_0) &= \langle 0|T\{N(\mathbf{x}+\mathbf{r},t)N(\mathbf{x},t)\}\sum_{n,s_1,s_2} \underline{|2N, W_n, s_1, s_2\rangle\langle 2N, W_n, s_1, s_2|}\overline{\mathcal{J}}(t_0)|0\rangle + \cdots \\ &= \sum_{n,s_1,s_2} A_{n,s_1,s_2}\varphi^{W_n}(\mathbf{r})e^{-W_n(t-t_0)}, \quad A_{n,s_1,s_2} = \langle 2N, W_n, s_1, s_2|\overline{\mathcal{J}}(0)|0\rangle. \end{aligned}$$

ground state saturation at large t

$$\lim_{(t-t_0)\to\infty} F(\mathbf{r}, t-t_0) = A_0\underline{\varphi^{W_0}(\mathbf{r})}e^{-W_0(t-t_0)} + O(e^{-W_{n\neq 0}(t-t_0)})$$

NBS wave function

This is a standard method in lattice QCD and was employed for our first calculation.

Improved method

Ishii *et al.* (HALQCD), PLB712(2012) 437

normalized 4-pt Correlation function

$$R(\mathbf{r},t) \equiv F(\mathbf{r},t)/(e^{-m_N t})^2 = \sum_n A_n \varphi^{W_n}(\mathbf{r}) e^{-\Delta W_n t}$$

$$\Delta W_n = W_n - 2m_N = \frac{\mathbf{k}_n^2}{m_N} - \frac{(\Delta W_n)^2}{4m_N}$$

$$-\frac{\partial}{\partial t} R(\mathbf{r},t) = \left\{ H_0 + U - \frac{1}{4m_N}\frac{\partial^2}{\partial t^2} \right\} R(\mathbf{r},t)$$

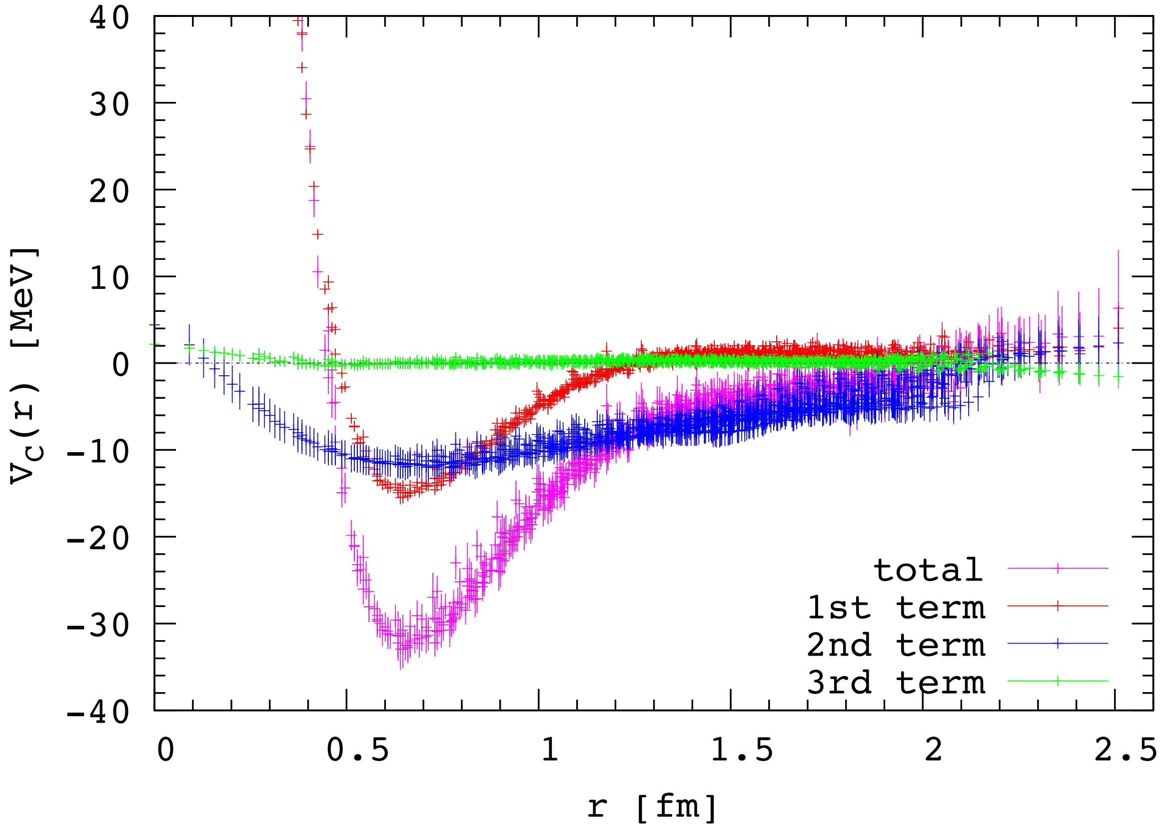
potential

Leading Order

$$\left\{ -H_0 - \frac{\partial}{\partial t} + \frac{1}{4m_N}\frac{\partial^2}{\partial t^2} \right\} R(\mathbf{r},t) = \int d^3r' \, U(\mathbf{r},\mathbf{r}') R(\mathbf{r}',t) = V_C(\mathbf{r}) R(\mathbf{r},t) + \cdots$$

1st **2nd** **3rd** **total**

3rd term(relativistic correction) is negligible.

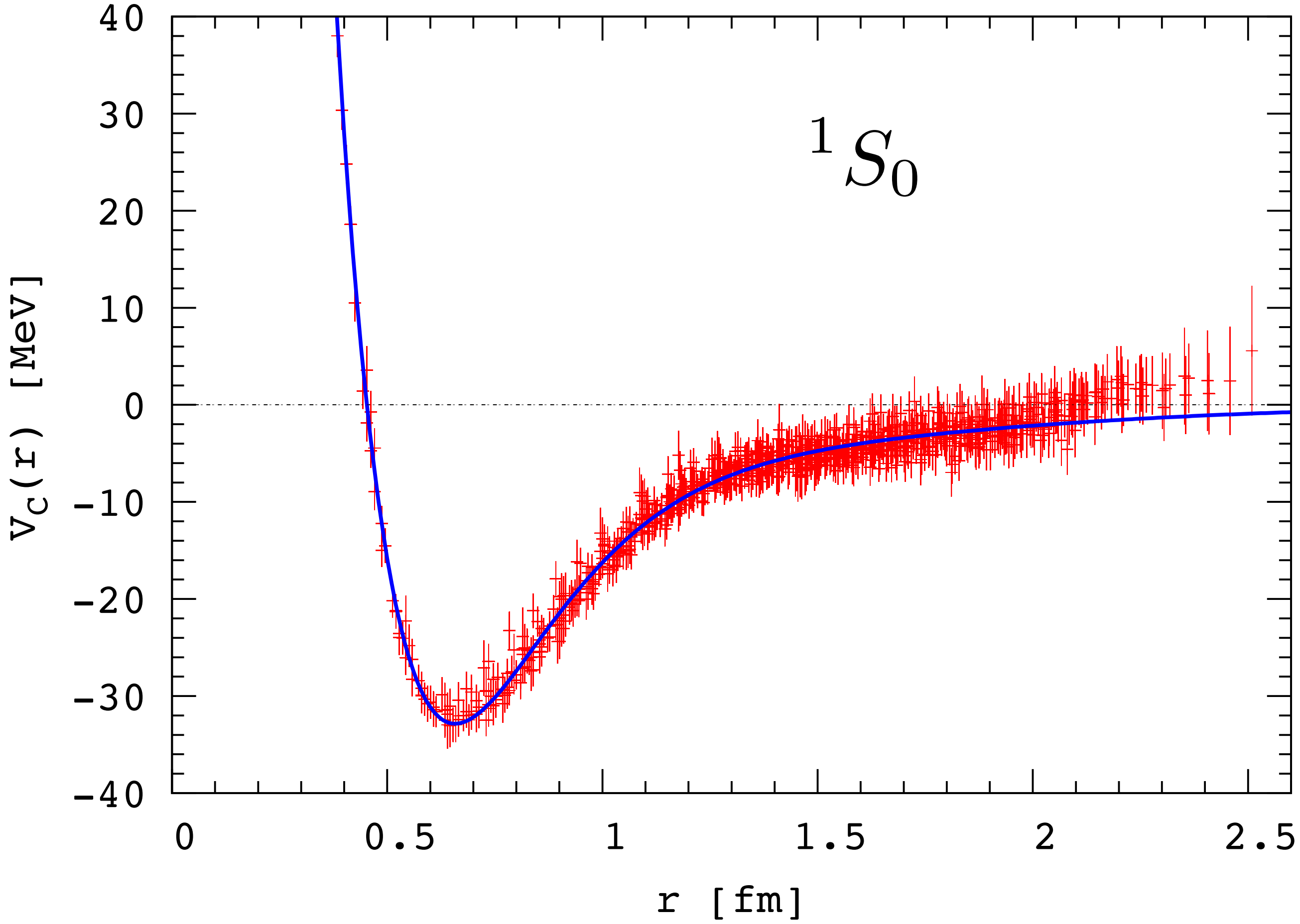


Ground state saturation is no more required ! (advantage over finite volume method.)

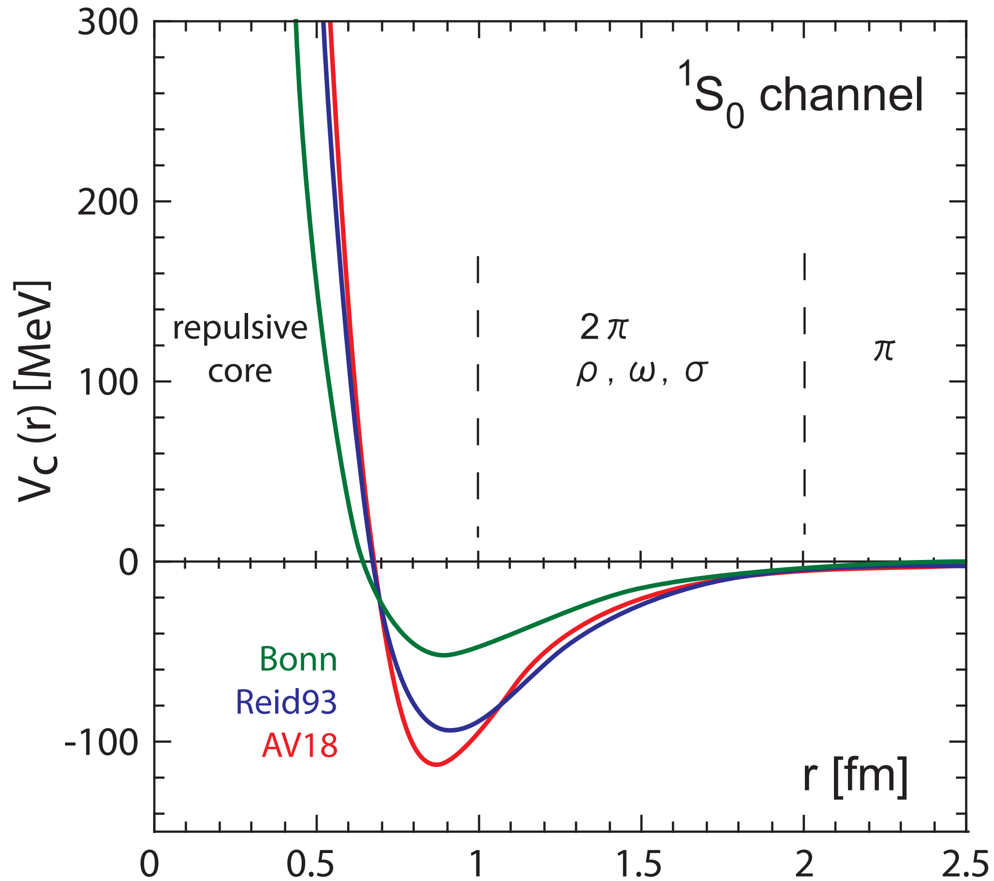
NN potential

2+1 flavor QCD, spin-singlet potential (in preparation)

a=0.09fm, L=2.9fm $m_\pi \simeq 700$ MeV



phenomenological potential



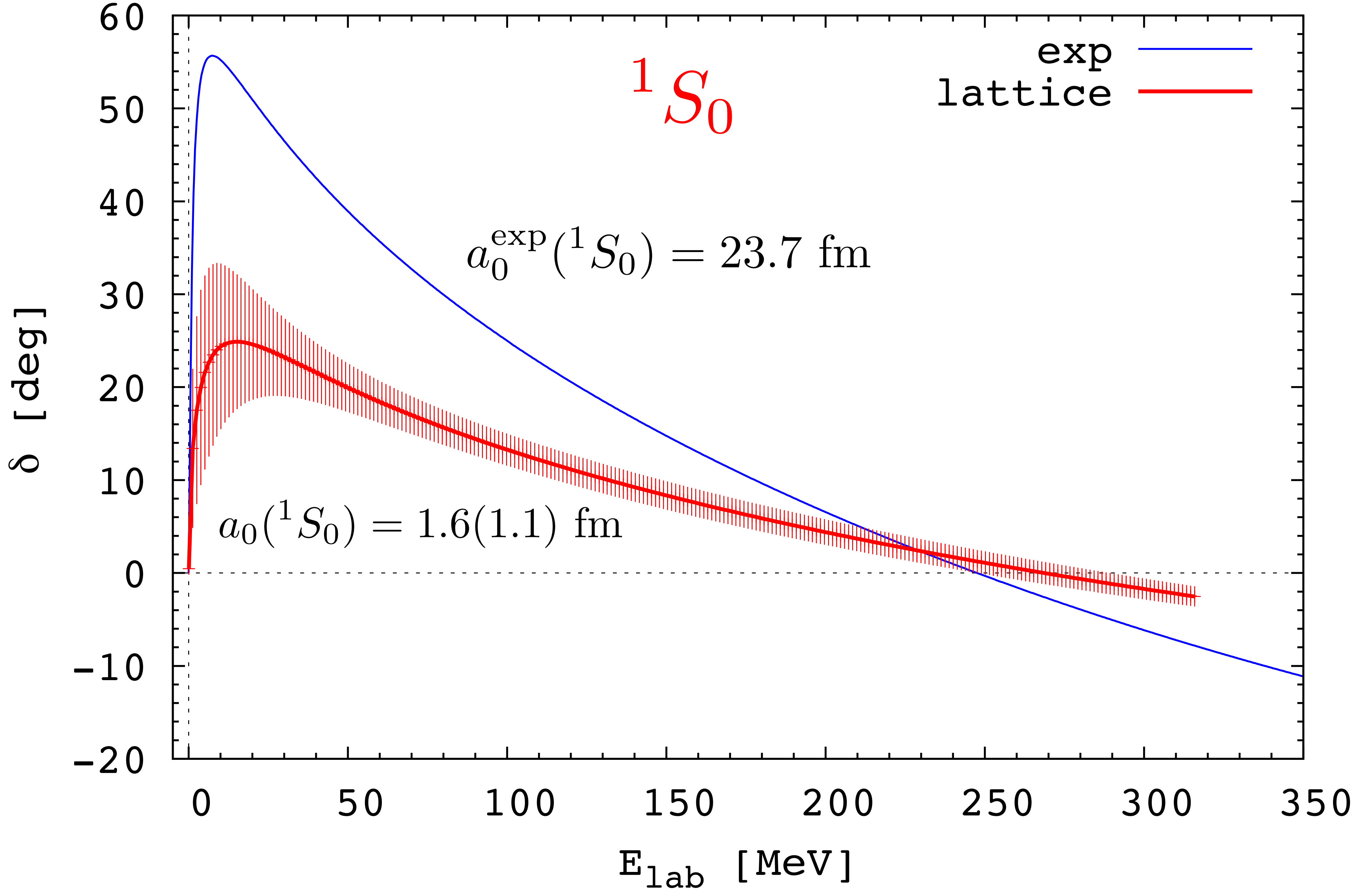
Qualitative features of NN potential are reproduced !

(1)attractions at medium and long distances
(2)repulsion at short distance(repulsive core)

1st paper(quenched QCD): Ishii-Aoki-Hatsuda, PRL90(2007)0022001

This paper has been selected as one of 21 papers in Nature Research Highlights 2007.

NN potential → phase shift



It has a reasonable shape. The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass.

Convergence of velocity expansion

If the higher order terms are large, LO potentials determined from NBS wave functions at different energy become different.(cf. LOC of ChPT).

Numerical check in quenched QCD

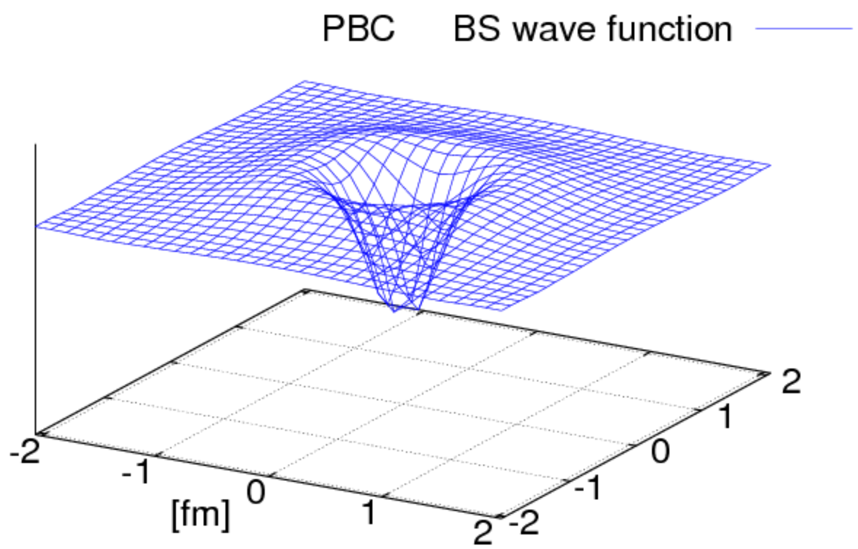
$m_\pi \simeq 0.53$ GeV

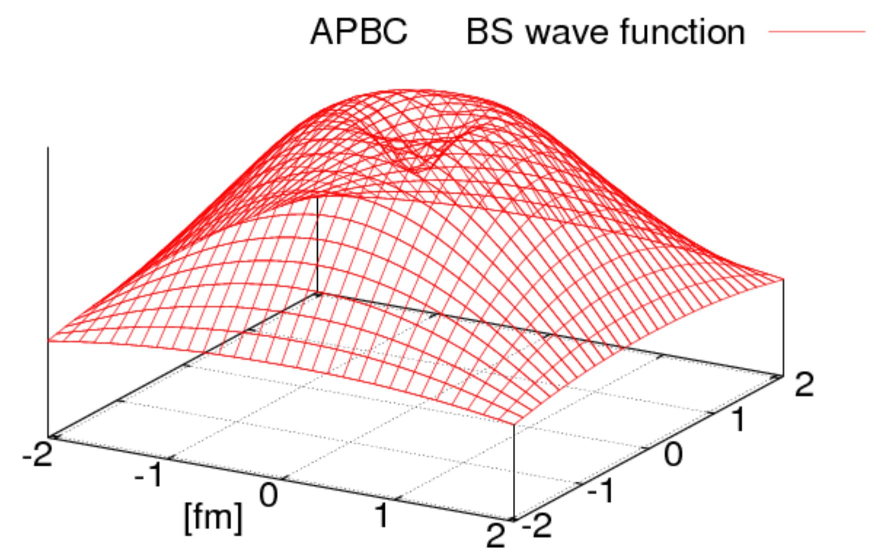
a=0.137fm

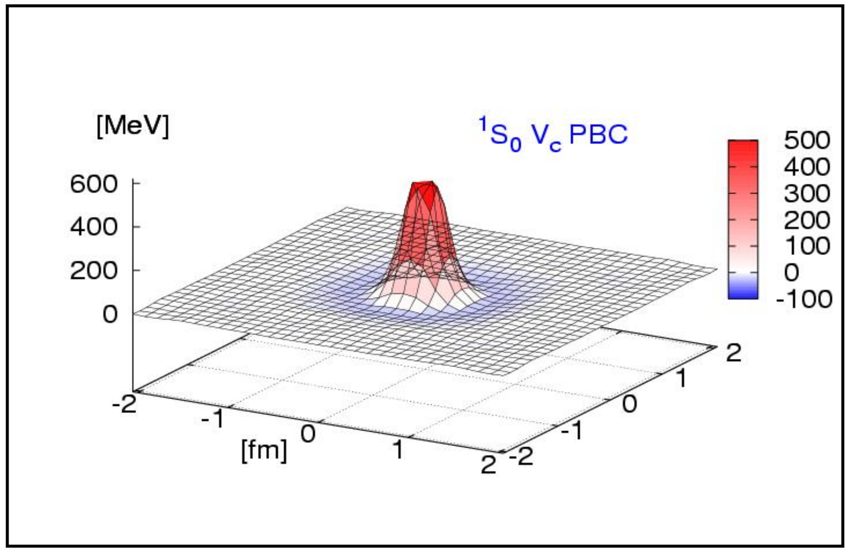
K. Murano, N. Ishii, S. Aoki, T. Hatsuda

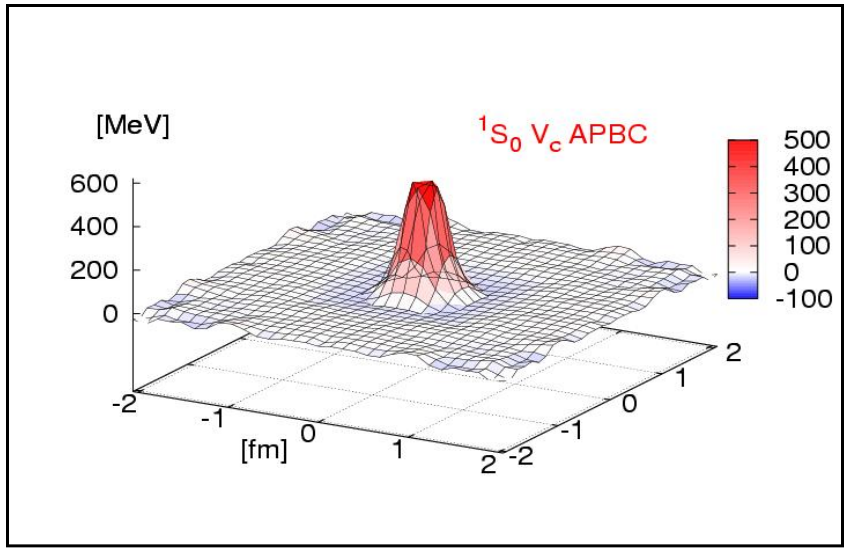
PTP 125 (2011)1225.

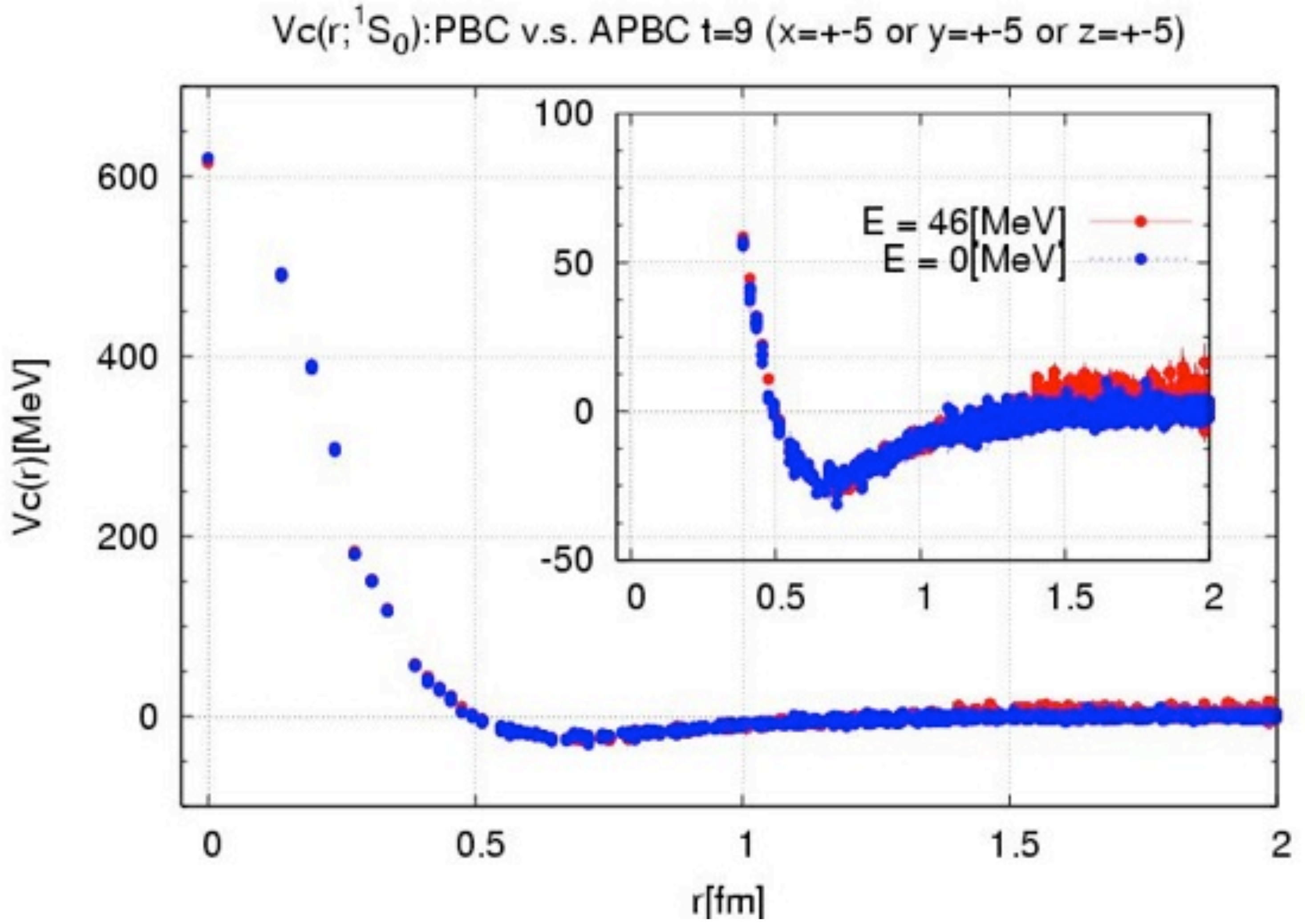
- PBC (E∼0 MeV)
- APBC (E∼46 MeV)

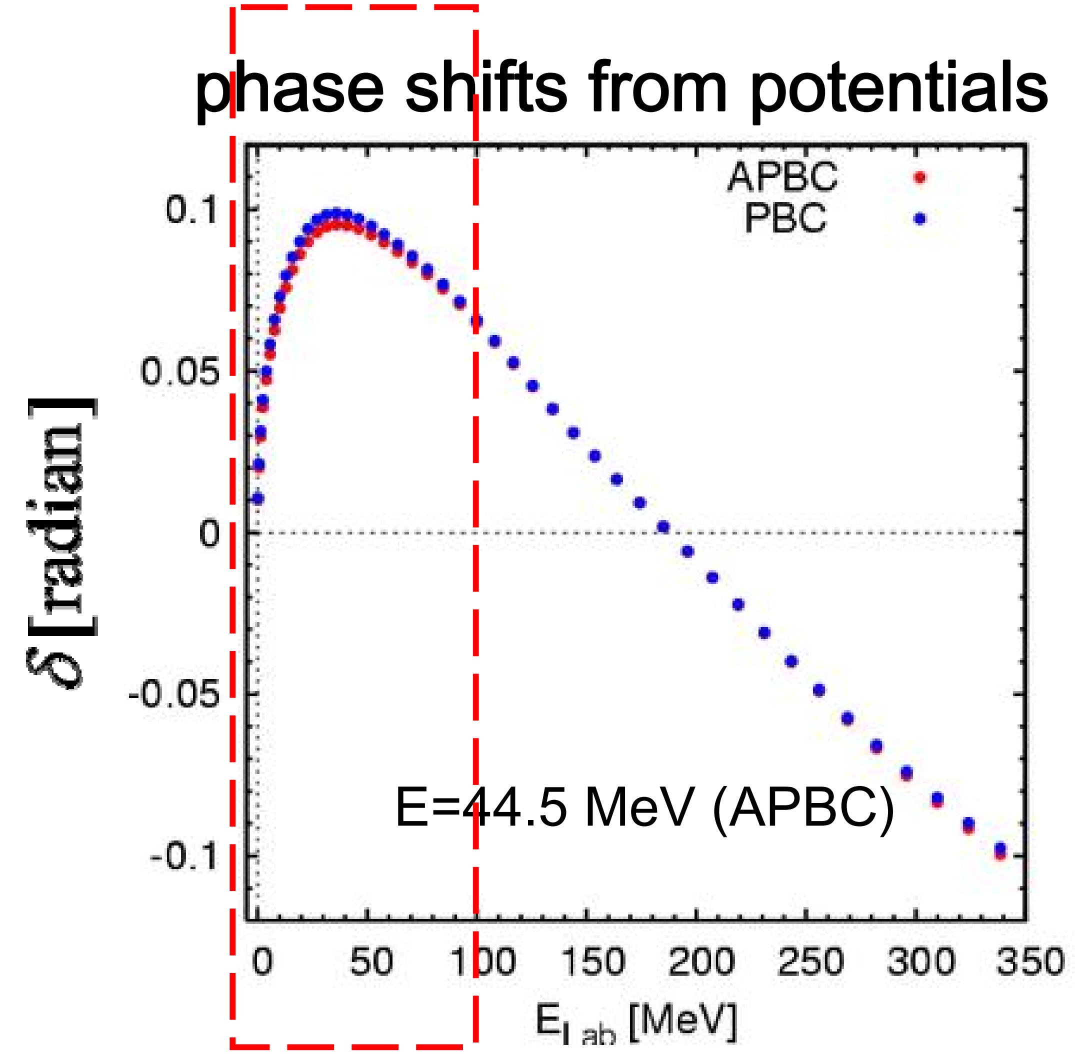












Higher order terms turn out to be very small at low energy in HAL scheme.

Need to be checked at lighter pion mass in 2+1 flavor QCD.

Note: convergence of the velocity expansion can be checked within this method.

(cf. convergence of ChPT, convergence of perturbative QCD)

4. Inelastic scattering (work in progress)

Inelastic scattering

1. Particle production

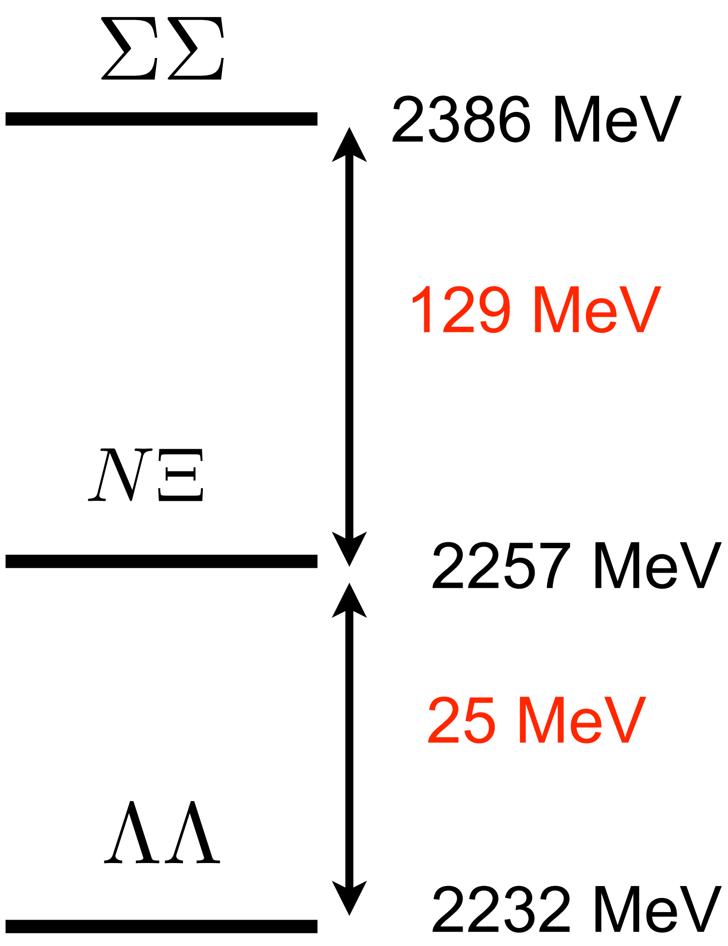
Ex.

$$NN \rightarrow NN, NN + \pi, NN + 2\pi, \cdots, NN + K\bar{K}, \cdots, NN + N\bar{N}, \cdots$$

2. Particle exchanges

Ex.

$$\Lambda\Lambda \rightarrow \Lambda\Lambda, N\Xi, \Sigma\Sigma$$



NBS wave function : multi-channel

Aoki *et al.* (HALQCD), Proc. Jpn. Acad. Ser. B, Vol. 87(2011) 509

$$AB \to AB, CD$$

$$\psi_{AB}(\boldsymbol{r}, \boldsymbol{k}) = \lim_{\delta\to 0^+} \langle 0 | T\{\varphi_A(\boldsymbol{x}+\boldsymbol{r}, \delta)\varphi_B(\boldsymbol{x}, 0)\}|W\rangle, \qquad |W\rangle = c_{AB}|AB, W\rangle + c_{CD}|CD, W\rangle$$

$$\psi_{CD}(\boldsymbol{r}, \boldsymbol{q}) = \lim_{\delta\to 0^+} \langle 0 | T\{\varphi_C(\boldsymbol{x}+\boldsymbol{r}, \delta)\varphi_D(\boldsymbol{x}, 0)\}|W\rangle, \qquad W = E_k^A + E_k^B = E_q^C + E_q^D$$

$$|\mathbf{r}| \to \infty$$

$$\begin{pmatrix} \hat\psi^l_{AB}(r,k) \\ \hat\psi^l_{CD}(r,q) \end{pmatrix} \simeq \begin{pmatrix} j_l(kr) & 0 \\ 0 & j_l(qr) \end{pmatrix} \begin{pmatrix} c_{AB} \\ c_{CD} \end{pmatrix} + \begin{pmatrix} n_l(kr) + i j_l(kr) & 0 \\ 0 & n_l(qr) + i j_l(qr) \end{pmatrix}$$

$$\times\, O(W) \begin{pmatrix} e^{i\delta_l^1(W)} \sin\delta_l^1(W) & 0 \\ 0 & e^{i\delta_l^2(W)} \sin\delta_l^2(W) \end{pmatrix} O^{-1}(W) \begin{pmatrix} c_{AB} \\ c_{CD} \end{pmatrix}$$

$\delta_l^1(W)$, $\delta_l^2(W)$: phase shifts for anglura momentum l

$$O(W) = \begin{pmatrix} \cos\theta(W) & -\sin\theta(W) \\ \sin\theta(W) & \cos\theta(W) \end{pmatrix}$$

$\theta(W)$: mixing angle

NBS wave function : multi-particles

Work in progress

scalar fields $\varphi + \varphi + \varphi \longrightarrow \varphi + \varphi + \varphi$

NBS wave function $\Psi^3_{W,c}(\{\mathbf{x}\}) = \langle 0 | \underline{\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\varphi(\mathbf{x}_3)} | W, c\rangle_{\rm in}$ c:quantum numbers

$$\varphi^3(\{\mathbf{x}\})$$

Jacobi coordinate $\mathbf{r} = 2(\mathbf{x}_1 - \mathbf{x}_2),\ \mathbf{s} = (2\mathbf{x}_3 - (\mathbf{x}_1 + \mathbf{x}_2))/\sqrt{3}$

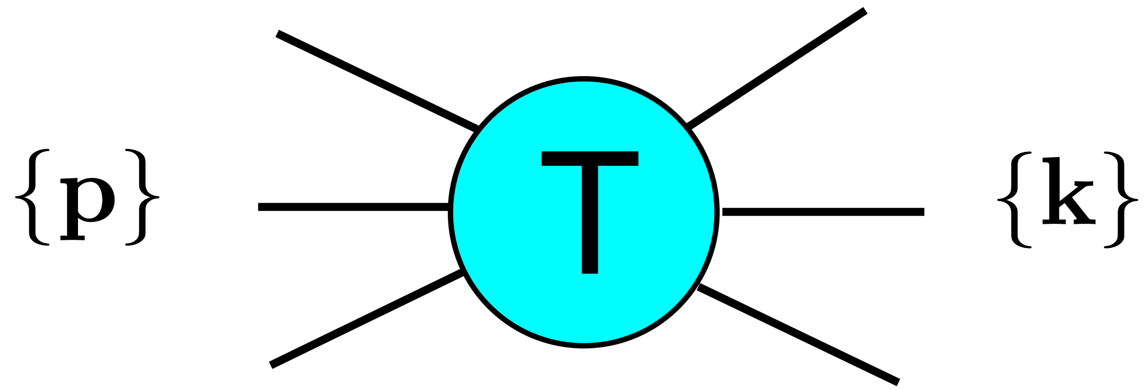
$|\mathbf{r}|, |\mathbf{s}| \to \infty$

$$\Psi^3_{W,c}(\{\mathbf{x}\}) \simeq \frac{{}_0\langle 0|\varphi^3(\{\mathbf{x}\})|W,c\rangle_0}{Z_3(W)} + \sum_{\sigma\in\text{perm.}} \sum_{l_1 m_1 l_2 m_2} i^{l_1+l_2} I_{l_1 l_2}(r,s,k_r,k_s)$$

$$\times Y_{l_1 m_1}(\Omega_{\mathbf{p}_r})\overline{Y_{l_1 m_1}(\Omega_{\mathbf{k}_r})}Y_{l_2 m_2}(\Omega_{\mathbf{p}_s})\overline{Y_{l_2 m_2}(\Omega_{\mathbf{k}_s})}$$

$$I_{l_1 l_2}(r,s,k_r,k_s) \propto [n_{l_1}(p_r r) + i j_{l_1}(p_r r)][n_{l_2}(p_s s) + i j_{l_2}(p_s s)]\underline{T^{3\leftarrow 3}_{l_1 l_2}(p_r,p_s,k_r,k_s)}$$

on-shell T-matrix



Construction of energy-independent potential for inelastic scattering

$$NN \rightarrow NN + n\pi$$

I_n

$$W_{\rm th}^n = 2m_N + nm_\pi$$

$$E_W^n = \frac{\mathbf{p}_1^2}{2m_N} + \frac{\mathbf{p}_2^2}{2m_N} + \sum_{i=1}^{n} \frac{\mathbf{k}_i^2}{2m_\pi}$$

$$W = \sqrt{m_N^2 + \mathbf{p}_1^2} + \sqrt{m_N^2 + \mathbf{p}_2^2} + \sum_{i}^{n} \sqrt{m_\pi^2 + \mathbf{k}_i^2}$$

I_2

$$W_{\rm th}^2 = 2m_N + 2m_\pi$$

$$\mathbf{p}_1 + \mathbf{p}_2 + \sum_{i=1}^{n} \mathbf{k}_i = 0$$

momentum conservation

I_1

$$W_{\rm th}^1 = 2m_N + m_\pi$$

$$W \simeq W_{\rm th}^n + E_W^n$$

non-relativistic approximation

I_0

$$W_{\rm th}^0 = 2m_N$$

NBS wave function $W \geq W_{\rm th}^n$

$$\varphi_{W,c_l}^{kl}([\mathbf{x}]_k) = \langle 0|N(\mathbf{x},0)N(\mathbf{x}+\mathbf{x}_0,0)\prod_{i=1}^{k}\pi(\mathbf{x}+\mathbf{x}_i,0)|NN+l\pi,W,c_l\rangle_{\rm in},\ \underline{k,l \leq n},$$
$$= 0, \qquad \text{otherwise},$$

OR

$$\varphi_{W,c_l}^{kl}([\mathbf{x}]_k) = \langle 0|N(\mathbf{x},0)N(\mathbf{x}+\mathbf{x}_0,0)\prod_{i=1}^{k}\pi(\mathbf{x}+\mathbf{x}_i,0)|NN+l\pi,W,c_l\rangle_{\rm in},\ \underline{l \leq n},$$
$$= 0, \qquad \text{otherwise},$$

For both choices $\varphi_{W,c_l}^{kl}([\mathbf{x}]_k) \simeq 0, \quad k > n$ $\quad |\mathbf{x}_i - \mathbf{x}_j| \to \infty$

$$[\mathbf{x}]_k = \{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k\}$$

vector of NBS wave functions

$$\varphi^{i}_{W,c_i} \equiv \left(\varphi^{0i}_{W,c_i}([\mathbf{x}]_0), \varphi^{1i}_{W,c_i}([\mathbf{x}]_1), \cdots, \varphi^{ni}_{W,c_i}([\mathbf{x}]_n), \cdots\right)$$

metric(inner product)

$$\eta^{ij}_{W_1W_2,c_id_j} = \left(\varphi^{i}_{W_1,c_i}, \varphi^{j}_{W_2,d_j}\right)$$

$$\equiv \sum_{k=0}^{\infty} \int \prod_{i=0}^{k} d^3x_i \, (\varphi^{ki}_{W_1,c_i})^{\dagger}([\mathbf{x}]_k)\varphi^{kj}_{W_2,d_j}([\mathbf{x}]_k).$$

linearly independent

inverse

$$\sum_{k,W,e_k} (\eta^{-1})^{ik}_{W_1W,c_ie_k} \cdot \eta^{kj}_{WW_2,e_kd_j} = \delta^{ij}\delta_{W_1W_2}\delta_{c_id_j}$$

brackets

$$\langle \varphi^{i}_{W,c_i} | [\mathbf{x}]_k \rangle = \sum_{m,W_1,d_m} (\eta^{-1})^{im}_{WW_1,c_id_m} (\varphi^{km}_{W_1,d_m})^{\dagger}([\mathbf{x}]_k)$$

$$\langle [\mathbf{x}]_k | \varphi^{i}_{W,c_i} \rangle = \varphi^{ki}_{W,c_i}([\mathbf{x}]_k),$$

which satisfy

$$\langle \varphi^{i}_{W_1,c_i} | \varphi^{j}_{W_2,d_j} \rangle \equiv \sum_{k=0}^{\infty} \int \prod_{l=0}^{k} d^3x_l \, \langle \varphi^{ki}_{W_1,c_i} | [\mathbf{x}]_k \rangle \langle [\mathbf{x}]_k | \varphi^{kj}_{W_2,d_j} \rangle$$

$$= \delta^{ij}\delta_{W_1W_2}\delta_{c_id_j}.$$

coupled channel equation

$$(E_W^k - H_0^k)\varphi_{W,c_i}^{ki}([\mathbf{x}]_k) = \sum_{l=0}^{\infty} \int \prod_{n=1}^{l} d^3y_n \, U^{kl}([\mathbf{x}]_k, [\mathbf{y}]_l)\varphi_{W,c_i}^{li}([\mathbf{y}]_l)$$

$$\langle [\mathbf{x}]_k | (E_W - H_0) | [\mathbf{y}]_l \rangle \equiv (E_W^k - H_0^k)\delta_{kl} \prod_{n=1}^{k} \delta^3(\mathbf{x}_n - \mathbf{y}_n)$$

$$\langle [\mathbf{x}]_k | U | [\mathbf{y}]_l \rangle \equiv U^{kl}([\mathbf{x}]_k, [\mathbf{y}]_l),$$

$$(E_W - H_0)|\varphi_{W,c_i}^i\rangle = U|\varphi_{W,c_i}^i\rangle.$$

construction of U

$$U = \sum_{i,W,c_i} (E_W - H_0)|\varphi_{W,c_i}^i\rangle\langle\varphi_{W,c_i}^i|,$$

$$U|\varphi_{W,c_i}^i\rangle = \sum_{j,W_1,d_j} (E_{W_1} - H_0)|\varphi_{W_1,d_j}^j\rangle\langle\varphi_{W_1,d_j}^j|\varphi_{W,c_i}^i\rangle = (E_W - H_0)|\varphi_{W,c_i}^i\rangle.$$

Hermiticity

$$U^{ij}_{W_1 W_2, c_i d_j} \equiv \langle \varphi^i_{W_1,c_i} | U | \varphi^j_{W_2,d_j} \rangle = \langle \varphi^i_{W_1,c_i} | (E_{W_2} - H_0) | \varphi^j_{W_2,d_j} \rangle,$$

⇕ **Hermite if** $W_1 = W_2$

$$(U^\dagger)^{ij}_{W_1 W_2, c_i d_j} = \overline{\langle \varphi^j_{W_2,c_j} | (E_{W_1} - H_0) | \varphi^i_{W_1,d_i} \rangle} = \langle \varphi^i_{W_1,c_i} | (E_{W_1} - H_0) | \varphi^j_{W_2,d_j} \rangle.$$

Extension to arbitrary channels is straightforward.

k: any operators, l: any states

Non-local potential U describes all QCD processes.

QFT(QCD) at given energy. ⇔ coupled channel quantum mechanics with energy-independent non-local potential U

5. Demonstration (as a conclusion)

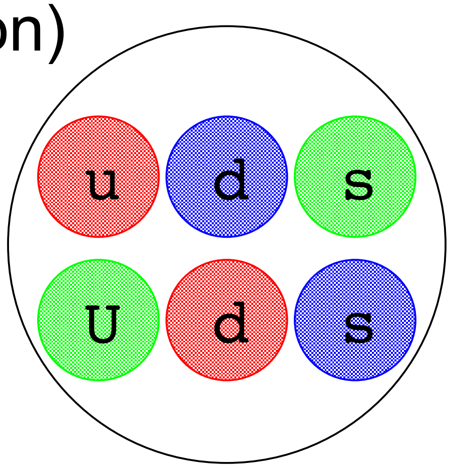
H-dibaryon in the flavor SU(3) symmetric limit

Inoue *et al.* (HAL QCD Coll.), PRL106(2011)162002

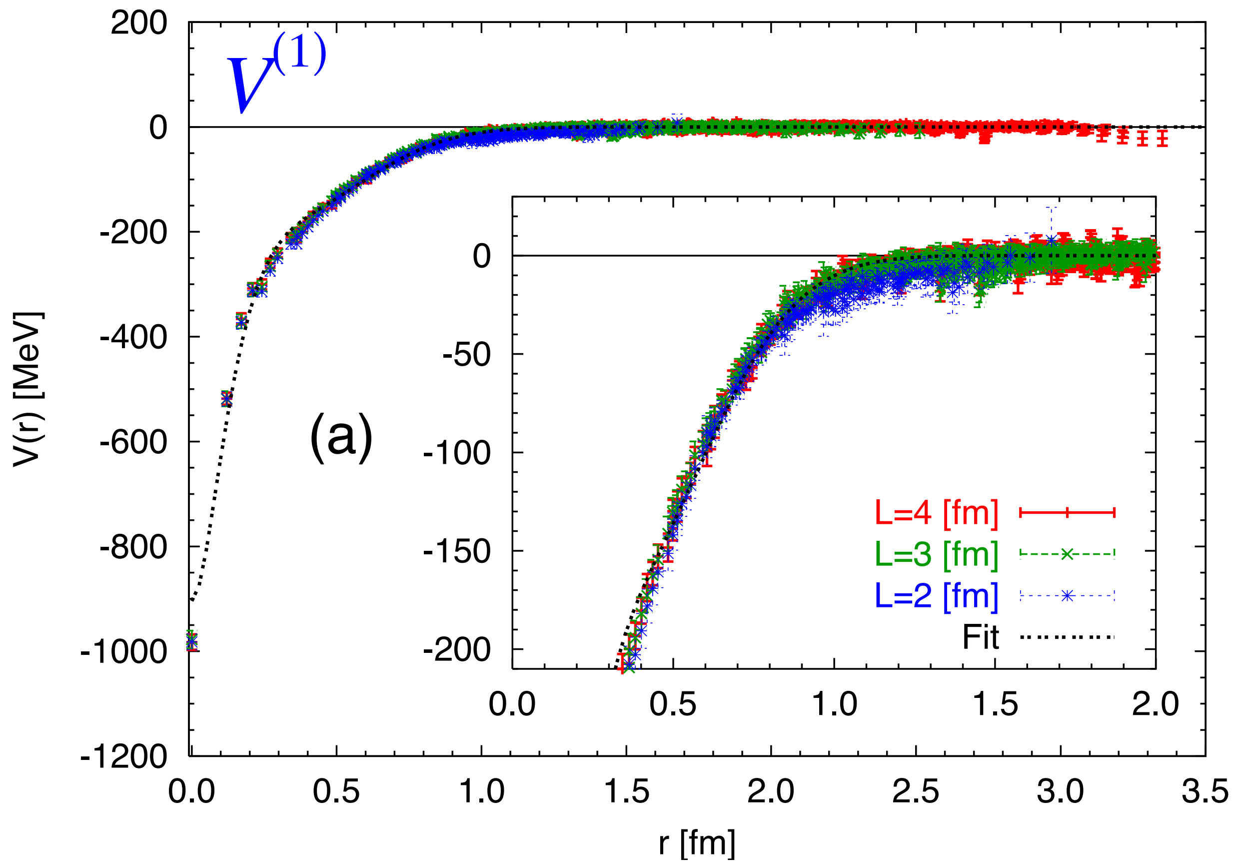
Attractive potential
in the flavor singlet channel

possibility of a bound state (H-dibaryon)

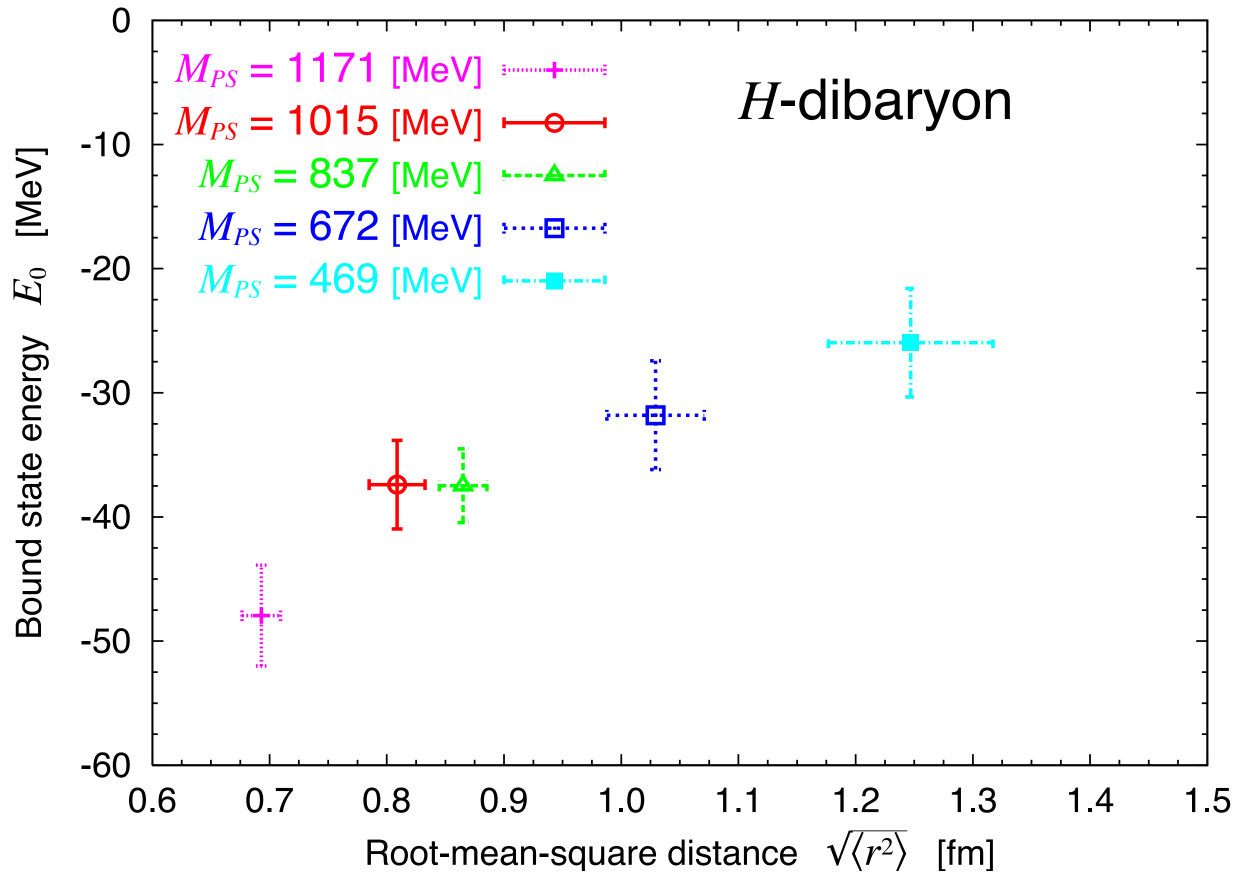
$$\Lambda\Lambda - N\Xi - \Sigma\Sigma$$



volume dependence

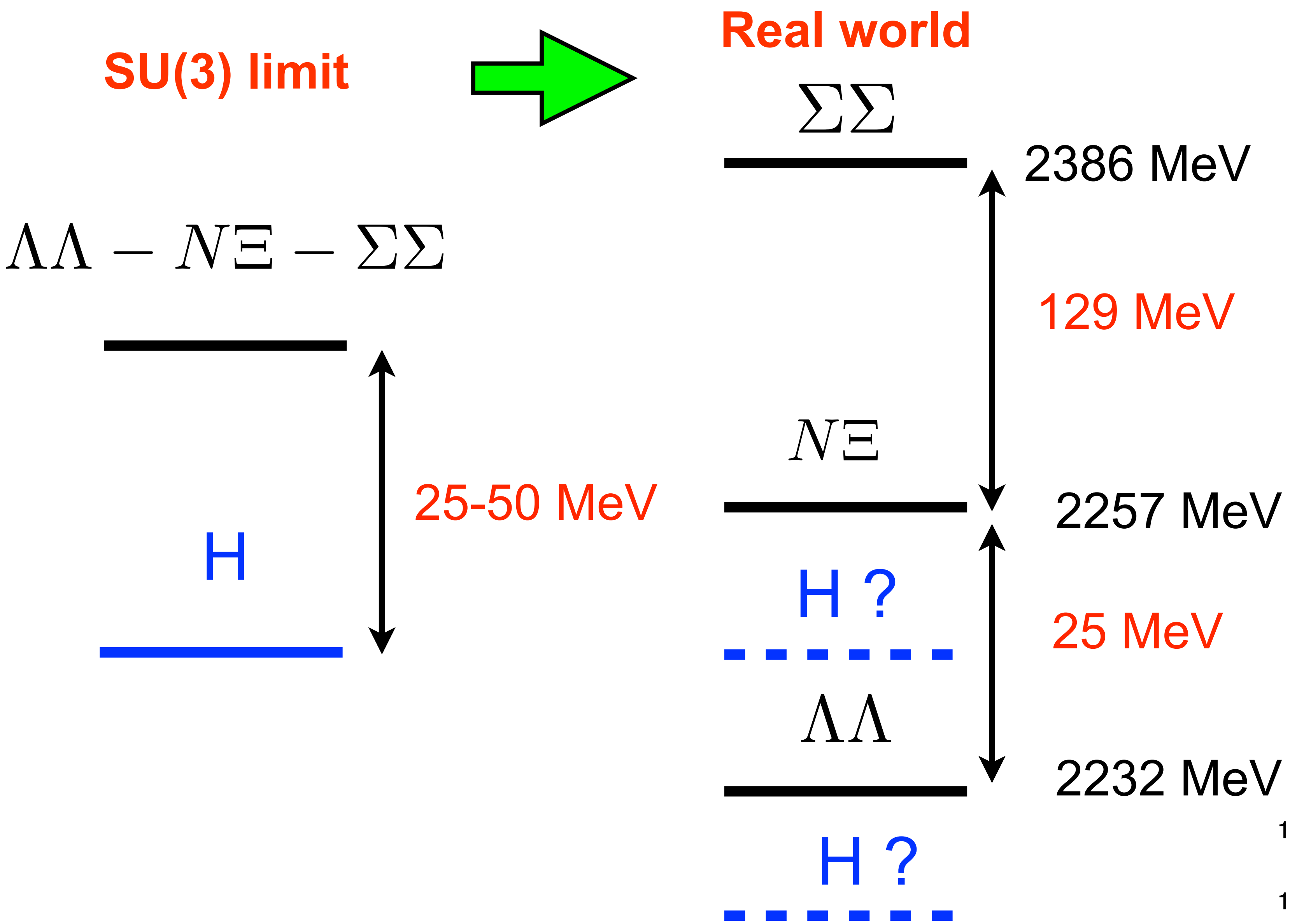


Solve Schroedinger equation
in the infinite volume

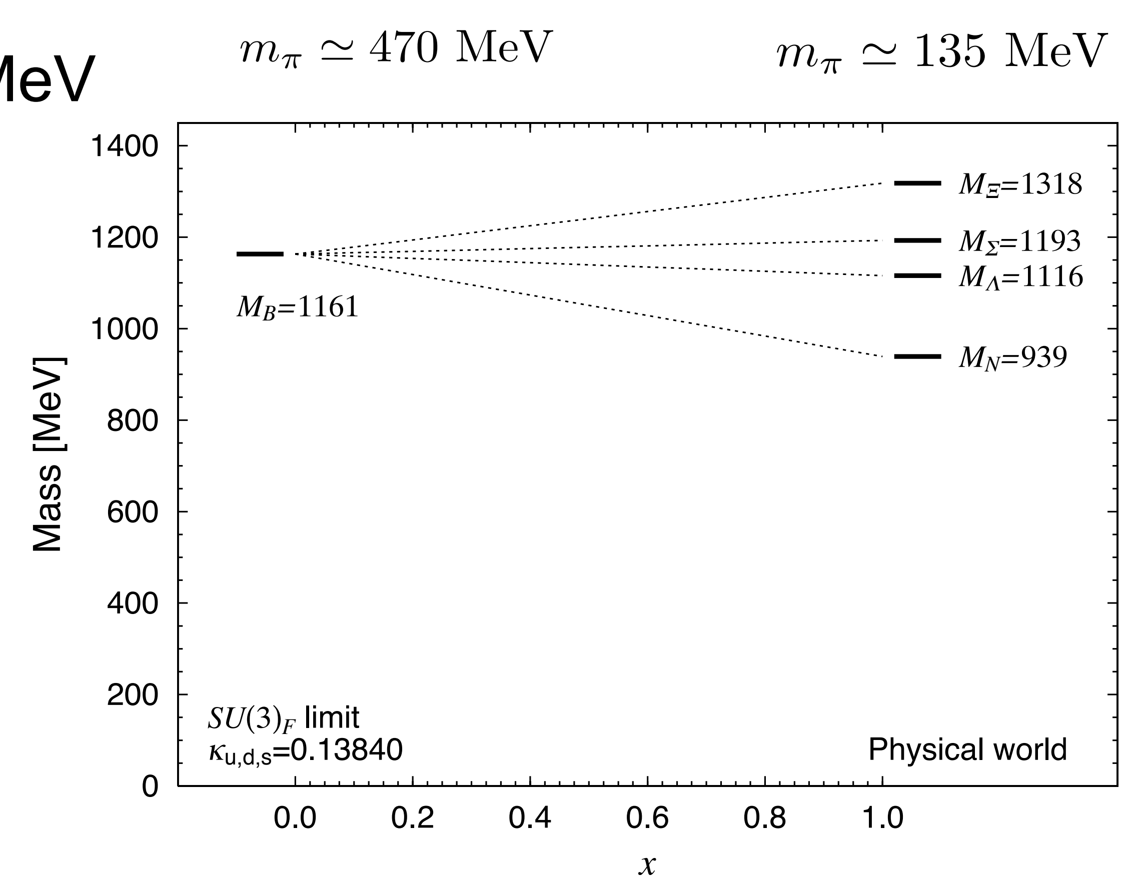


One bound state (H-dibaryon) exists.

H-dibaryon with the flavor SU(3) breaking



$$m_u = m_d \neq m_s$$



Our approximation for SU(3) breaking

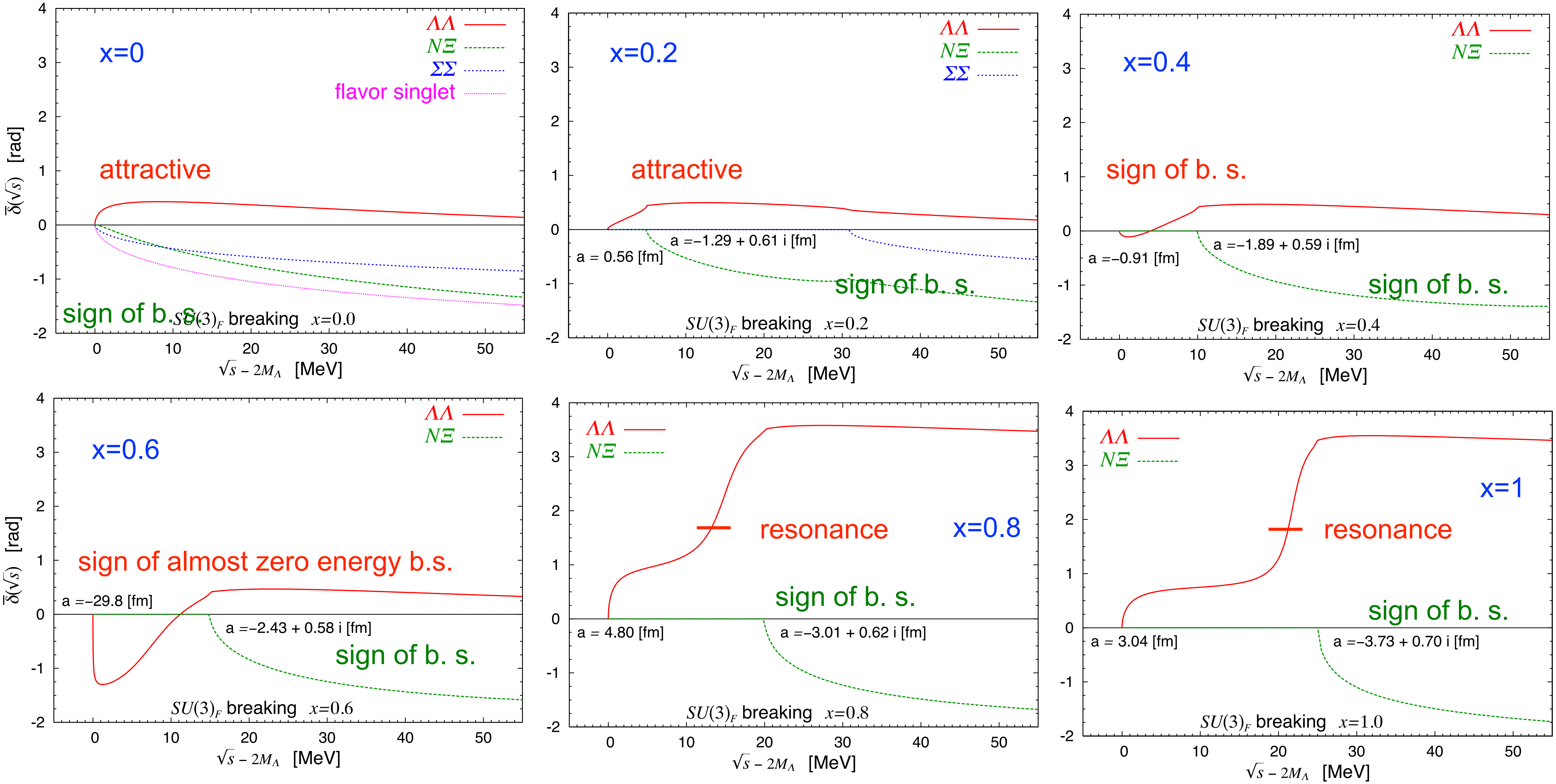
1. Linear interpolation of octet baryon masses

$$M_Y(x) = (1-x)M_Y^{\mathrm{SU(3)}} + xM_Y^{\mathrm{Phys}}$$

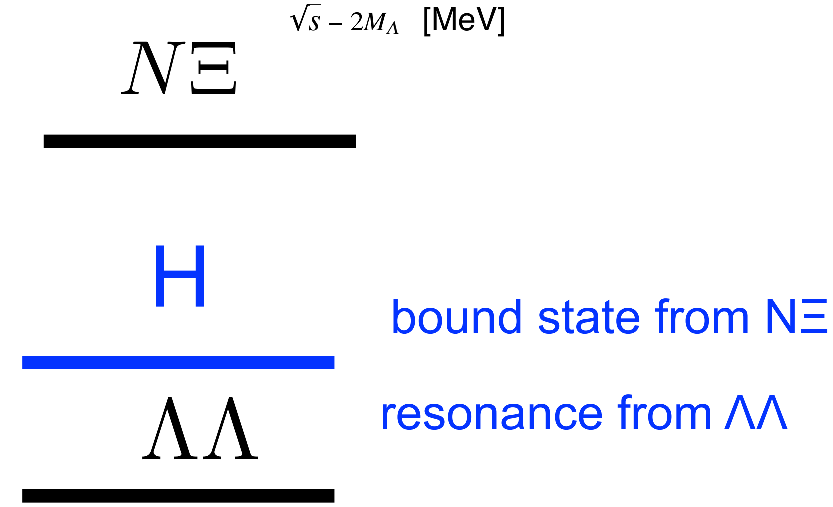
2. Potentials in particle basis in SU(3) limit

Phase shift

Inoue *et al.* (HAL QCD Coll.), arXiv:1112.5926[hep-lat]



H-dibaryon seems to become resonance at physical point.



H couples most strongly $N\Xi$.

$\Lambda\Lambda$ interaction is attractive.

H has a sizable coupling to $\Lambda\Lambda$ near and above the threshold.

Summary

- HAL QCD method is alternative to extract hadronic interactions in lattice QCD.
  - 2-particle elastic scattering(established).
  - asymptotic behavior of n-particle NBS wave function (in progress).
  - energy-independent non-local potential including inelastic scattering (in progress)
- Some Future directions
  - ex. rho resonance from pi-pi potential.
  - extension to weak interaction (work in progress).
    - Let us discuss this at GGI, if you are interested.