Progress on B-physics lattice calculations by ETMC

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on behalf of ETM Collaboration

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Outline

- ETMC computation
  - Method based on **Ratios** of heavy-light \((h, \ell/s)\) observables using relativistic quarks and exact knowledge of static limit for the appropriate ratios
  - Interpolation of \((h, \ell/s)\) observables to the b-region from the charm region and the static limit

- Application of **Ratio method** to \(b\)-quark mass, decay constants and Bag parameters.

- Summary

**ETMC**, JHEP **1201** (2012) 046; JHEP **1004** (2010) 049; N. Carrasco and A. Shindler @ LAT2012

ETMC – $N_f = 2$ twisted-mass formulation

- Mtm lattice regularization of $N_f = 2$ QCD action is

[Frezzotti, Grassi, Sint, Weisz, JHEP 2001; Frezzotti, Rossi, JHEP 2004]

$$S_{N_f=2}^{\text{ph}} = S_L^{\text{YM}} + a^4 \sum_x \bar{\psi}(x) \left[ \gamma \cdot \tilde{\nabla} - i \gamma_5 \tau^3 \left( - \frac{a}{2} r \nabla^* \nabla + M_{\text{cr}}(r) \right) + \mu_q \right] \psi(x)$$

- $\psi$ is a flavour doublet, $M_{\text{cr}}(r)$ is the critical mass and $\tau^3$ acts on flavour indices

- From the “physical” basis (where the quark mass is real), the non-anomalous

$$\psi = \exp(i \pi \gamma_5 \tau^3 / 4) \chi, \quad \bar{\psi} = \bar{\chi} \exp(i \pi \gamma_5 \tau^3 / 4)$$

transformation brings the lattice action in the so-called “twisted” basis

$$S_{N_f=2}^{\text{tw}} = S_L^{\text{YM}} + a^4 \sum_x \bar{\chi}(x) \left[ \gamma \cdot \tilde{\nabla} - \frac{a}{2} r \nabla^* \nabla + M_{\text{cr}}(r) + i \mu_q \gamma_5 \tau^3 \right] \chi(x)$$

- Unlike the standard Wilson regularization, in Mtm Wilson case the subtracted Wilson operator $- \frac{a}{2} r \nabla^* \nabla + M_{\text{cr}}(r)$ is “chirally rotated” w.r.t. the quark mass offers important advantages...
ETMC – $N_f = 2$ twisted-mass formulation

- Automatic $O(a)$ improvement for the physical quantities
- Dirac-Wilson matrix determinant is positive
  and (lowest eigenvalue)$^2$ bounded from below by $\mu q^2$
- Simplified (operator) renormalization ...
  - Multiplicative quark mass renormalization
  - No RC for pseudoscalar decay constant (PCAC)
- $O(a^2)$ breaking of parity and isospin

Frezzotti, Rossi, JHEP 2004;
ETMC $N_f = 2$ simulations

- $a = \{0.054, 0.067, 0.085, 0.098\}$ fm
- $m_{ps} \in \{270, 600\}$ MeV
- $L \in \{1.7, 2.8\}$ fm, $m_{ps}L \geq 3.5$
ETMC $N_f = 2$ simulations

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a\mu_\ell$</th>
<th>$a\mu_s$ (valence)</th>
<th>$a\mu_h$ (valence)</th>
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<tbody>
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<td>3.80</td>
<td>0.0080, 0.0110</td>
<td>0.0175, 0.0194, 0.0213</td>
<td>0.1982, $\ldots$, 0.8536</td>
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<td>3.90</td>
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<td>0.1828, $\ldots$, 0.7873</td>
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<td>4.05</td>
<td>0.0030, 0.0060, 0.0080</td>
<td>0.0139, 0.0154, 0.0169</td>
<td>0.1572, $\ldots$, 0.6771</td>
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<td>4.20</td>
<td>0.0020, 0.0065</td>
<td>0.0116, 0.0129, 0.0142</td>
<td>0.13315, $\ldots$, 0.4876</td>
</tr>
</tbody>
</table>

- $\mu_\ell \in [\sim m_s/6, \sim m_s/2]$
- $\mu_h \in [\sim m_c, \sim 3m_c]$
Smearing techniques improve signal; reduce the coupling between the ground and excited states; safe good plateaux at earlier times; absolutely necessary for obtaining safe plateau in the calculation of 3-point correlation functions (when large heavy quark mass (> 1 GeV) are employed).

Employ "optimal" source: $\Phi_{source}^W(w) = w\Phi_S + (1 - w)\Phi_L$; Check vs. GEVP – when two states matter – seems OK (in progress).
**Ratio method**

- we use correlators with relativistic quarks
- $c$-mass region computations are reliable (‘small’ discr. errors)
- construct HQET-inspired ratios of the observable of interest at successive (nearby) values of the heavy quark mass ($\mu_h^{(n)} = \lambda \mu_h^{(n-1)}$)
- ratios show smooth chiral and continuum limit behaviour
- ratios at the $\infty$-mass (static) point are exactly known ($= 1$)
- physical values of the observable at the $b$-mass point is related to its $c$-like value by a *chain* of the ratios ending up at the static point: use HQET-inspired interpolation
b-quark mass computation - 1

- observing that \( \lim_{\mu_p \to \infty} \left( \frac{M_{h\ell}}{\mu_p} \right) = \text{constant} \) (HQET)

- construct (taking \( \frac{\bar{\mu}_h^{(n)}}{\bar{\mu}_h^{(n-1)}} = \lambda \)):

\[
y(\bar{\mu}_h^{(n)}, \lambda; \bar{\mu}_\ell, a) \equiv \frac{M_{h\ell}(\bar{\mu}_h^{(n)}; \bar{\mu}_\ell, a)}{M_{h\ell}(\bar{\mu}_h^{(n-1)}; \bar{\mu}_\ell, a)} \cdot \frac{\bar{\mu}_h^{(n-1)}}{\bar{\mu}_h^{(n)}} \cdot \frac{\rho(\bar{\mu}_h^{(n-1)}, \mu^*)}{\rho(\bar{\mu}_h^{(n)}, \mu^*)} = \\
= \lambda^{-1} \frac{M_{h\ell}(\bar{\mu}_h^{(n)}; \bar{\mu}_\ell, a)}{M_{h\ell}(\bar{\mu}_h^{(n)}; \bar{\mu}_\ell, a)} \cdot \frac{\rho(\bar{\mu}_h^{(n)}/\lambda, \mu^*)}{\rho(\bar{\mu}_h^{(n)}, \mu^*)}, \quad n = 2, \ldots, N
\]

\( \mu_p = \rho(\bar{\mu}_h, \mu^*) \bar{\mu}_h(\mu^*) \) (with \( \bar{\mu}_h \leftarrow \overline{\text{MS}} \) scheme)

\( \rho(\bar{\mu}_h, \mu^*) \) known up to \( N^3\text{LO} \) – relevant only for the ’1/\( \bar{\mu}_h \)’ interpolation

→ In the static limit (and in CL) obviously:

\[
\lim_{\bar{\mu}_h \to \infty} y(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, a = 0) = 1
\]
**b-quark mass computation - 2**

- *Triggering* input mass: $M_{h\ell}(\mu_h^{(1)})$ PS meson mass (at $\mu_h^{(1)} \sim m_c$) affected by (tolerably) small cutoff effects.

- Ratios $y(\mu_h^{(n)}; \lambda; \bar{\mu}_\ell, a)$ have small discretisation errors.
fit ansatz $y(\bar{\mu}_h) = 1 + \frac{\eta_1}{\bar{\mu}_h} + \frac{\eta_2}{\bar{\mu}_h^2}$ (inspired by HQET)

Determine $K$ (integer) such that $M_{hu/d}(\bar{\mu}_h^{(K+1)}) \equiv M_B^{\text{expt}}$:

$$y(\bar{\mu}_h^{(2)}) y(\bar{\mu}_h^{(3)}) \ldots y(\bar{\mu}_h^{(K+1)}) = \lambda^{-K} \frac{M_{hu/d}(\bar{\mu}_h^{(K+1)})}{M_{hu/d}(\bar{\mu}_h^{(1)})} \cdot \left[ \frac{\rho(\bar{\mu}_h^{(1)}; \mu^*)}{\rho(\bar{\mu}_h^{(K+1)}; \mu^*)} \right]$$

(strong cancellations of perturbative factors in the ratios)

One adjusts $(\lambda, \bar{\mu}_h^{(1)})$ such that $K$ integer

our calculation: $\lambda = 1.1784$ and $\bar{\mu}_h^{(1)} = 1.14$ GeV (in $\overline{\text{MS}}, 2$ GeV)

$\bar{\mu}_b = \lambda^K \bar{\mu}_h^{(1)}$ ($K = 9$)

$y$ deviates from its static value $\sim 1\%$ for $\bar{\mu}_h \leq m_b$.

curvature denotes a large $1/\bar{\mu}_h^2$ contribution to ratios $y$. 
\[ y(\bar{\mu}_h) = 1 + \frac{\eta_1}{\bar{\mu}_h} + \frac{\eta_2}{\bar{\mu}_h^2} \]

- Determine \( K \) (integer) such that \( M_{hu/d}(\bar{\mu}^{(K+1)}_h) \equiv M_B^{\text{expt}} \):
  \[ y(\bar{\mu}^{(2)}_h) y(\bar{\mu}^{(3)}_h) \ldots y(\bar{\mu}^{(K+1)}_h) = \lambda^{-K} \frac{M_{hu/d}(\bar{\mu}^{(K+1)}_h)}{M_{hu/d}(\bar{\mu}^{(1)}_h)} \cdot \left[ \frac{\rho(\bar{\mu}^{(1)}_h, \mu^*)}{\rho(\bar{\mu}^{(K+1)}_h, \mu^*)} \right] \]
  (strong cancellations of perturbative factors in the ratios)

- One adjusts \((\lambda, \bar{\mu}^{(1)}_h)\) such that \( K \) integer

- our calculation: \( \lambda = 1.1784 \) and \( \bar{\mu}^{(1)}_h = 1.14 \) GeV (in \( \overline{\text{MS}}, \) 2 GeV)

  \[ \bar{\mu}_b = \lambda^K \bar{\mu}^{(1)}_h \ (K = 9) \]

- \( y \) deviates from its static value \( \sim 1\% \) for \( \bar{\mu}_h \leq m_b \).

- curvature denotes a large \( 1/\bar{\mu}_h^2 \) contribution to ratios \( y \).
b-quark mass computation - 5

\[ y(\bar{\mu}_h) = 1 + \frac{\eta_1}{\bar{\mu}_h} + \frac{\eta_2}{\bar{\mu}_h^2} \]  
(Comparison with less precise data from ETMC-2011 paper)

- Determine \( K \) (integer) such that \( M_{hu/d}(\bar{\mu}_h^{(K+1)}) \equiv M_B^{\text{expt}} \):
  \[ y(\bar{\mu}_h^{(2)}) y(\bar{\mu}_h^{(3)}) \ldots y(\bar{\mu}_h^{(K+1)}) = \lambda^{-K} \frac{M_{hu/d}(\bar{\mu}_h^{(K+1)})}{M_{hu/d}(\bar{\mu}_h^{(1)})} \cdot \left[ \frac{\rho(\bar{\mu}_h^{(1)}, \mu^*)}{\rho(\bar{\mu}_h^{(K+1)}, \mu^*)} \right] \]
  (strong cancellations of perturbative factors in the ratios)

- One adjusts \((\lambda, \bar{\mu}_h^{(1)})\) such that \( K \) integer

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- \( y \) deviates from its static value \( \sim 1\% \) for \( \bar{\mu}_h \leq m_b \).

- curvature denotes a large \( 1/\bar{\mu}_h^2 \) contribution to ratios \( y \).
b-quark mass - results

- \( m_b(m_b, \overline{\text{MS}})_{N_f=2} = 4.35(12) \text{ GeV} \) (PRELIMINARY!)

- Compatible result for \( m_b \) when \((hs)\)-data and \( M_{B_s}^{\text{expt}} \) as input are used

- Main source of uncertainty of the ETMC result is due to quark mass RC and scale setting uncertainties; stat & fit errors very small.
$f_B$ and $f_{Bs}$

- HQET behaviour: $\lim_{\mu_h \to \infty} f_{h\ell}(s)\sqrt{\mu_h^{pole}} = \text{constant}$

- Construct

$$z_\ell(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, a) \equiv \lambda^{1/2} \frac{f_{h\ell}(\bar{\mu}_h, \bar{\mu}_\ell, a)}{f_{h\ell}(\bar{\mu}_h/\lambda, \bar{\mu}_\ell, a)} \cdot \frac{C_A^{stat}(\mu^*, \bar{\mu}_h/\lambda)}{C_A^{stat}(\mu^*, \bar{\mu}_h)} \cdot \frac{[\rho(\bar{\mu}_h, \mu^*)]^{1/2}}{[\rho(\bar{\mu}_h/\lambda, \mu^*)]^{1/2}}$$

$$z_s(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, \bar{\mu}_s, a) \equiv \lambda^{1/2} \frac{f_{hs}(\bar{\mu}_h, \bar{\mu}_\ell, \bar{\mu}_s, a)}{f_{hs}(\bar{\mu}_h/\lambda, \bar{\mu}_\ell, \bar{\mu}_s, a)} \cdot \frac{C_A^{stat}(\mu_b^*, \bar{\mu}_h/\lambda)}{C_A^{stat}(\mu_b^*, \bar{\mu}_h)} \cdot \frac{[\rho(\bar{\mu}_h, \mu^*)]^{1/2}}{[\rho(\bar{\mu}_h/\lambda, \mu^*)]^{1/2}}$$

$C_A^{stat}(\mu^*, \bar{\mu}_h)$ known up to $N^2$LO – relevant for the interpolation in '1/\bar{\mu}_h' fit

$\rightarrow$ In the static limit (and in CL) obviously:

$$\lim_{\bar{\mu}_h \to \infty} z_{\ell/s}(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, a = 0) = 1$$

$$\lim_{\bar{\mu}_h \to \infty} \frac{z_s(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, a = 0)}{z_\ell(\bar{\mu}_h, \lambda; \bar{\mu}_\ell, a = 0)} = 1$$
- Ratios $z_s(\bar{\mu}_h^{(n)}, \lambda; \bar{\mu}_\ell, a)$ have small discretisation effects (from $\sim 1\%$ for the smallest to $\sim 3\%$ for the largest heavy quark masses).

- At Triggering point $f_{hs}(\bar{\mu}_h^{(1)})$ pseudoscalar decay constant (with $\bar{\mu}_h^{(1)} \sim m_c$) is affected by (tolerably) small cutoff effects.
• fit ansatz $z_s(\bar{\mu}_h) = 1 + \frac{\zeta_1}{\bar{\mu}_h} + \frac{\zeta_2}{\bar{\mu}_h^2}$

• use $z_s(\bar{\mu}_h^{(2)}) z_s(\bar{\mu}_h^{(3)}) \ldots z_s(\bar{\mu}_h^{(K+1)}) = \chi^{K/2} \frac{f_{hs}(\bar{\mu}_h^{(K+1)})}{f_{hs}(\bar{\mu}_h^{(1)})} \left[ \frac{C_{A}^{\text{stat}}(\bar{\mu}_h^{(1)}, \mu^*)}{C_{A}^{\text{stat}}(\bar{\mu}_h^{(K+1)}, \mu^*)} \right] \left[ \frac{\rho(\bar{\mu}_h^{(K+1)}, \mu^*)}{\rho(\bar{\mu}_h^{(1)}, \mu^*)} \right]^{1/2}$

• for $\bar{\mu}_h^{(K+1)} = \chi^K \bar{\mu}_h^{(1)} = m_b$, identify $f_{hs}(\bar{\mu}_h^{(K+1)})$ with $f_{Bs}$

$\rightarrow f_{Bs} = 234(6) \text{ MeV (PRELIMINARY)}$

(principal uncertainty from scale setting; other errors (stat+fit) < 1 %)
- fit ansatz \( z_s(\bar{\mu}_h) = 1 + \frac{\zeta_1}{\bar{\mu}_h} + \frac{\zeta_2}{\bar{\mu}_h^2} \)
- use \( z_s(\bar{\mu}_h^{(2)}) z_s(\bar{\mu}_h^{(3)}) \ldots z_s(\bar{\mu}_h^{(K+1)}) = \lambda^{K/2} \frac{f_{h_s}(\bar{\mu}_h^{(K+1)})}{f_{h_s}(\bar{\mu}_h^{(1)})} \cdot \left[ \frac{C_{A}^{\text{stat}}(\bar{\mu}_h^{(1)}, \mu^*)}{C_{A}^{\text{stat}}(\bar{\mu}_h^{(K+1)}, \mu^*)} \right] \left[ \frac{\rho(\bar{\mu}_h^{(K+1)}, \mu^*)}{\rho(\bar{\mu}_h^{(1)}, \mu^*)} \right]^{1/2} \\
- for \( \bar{\mu}_h^{(K+1)} = \lambda^K \bar{\mu}_h^{(1)} = m_b \), identify \( f_{h_s}(\bar{\mu}_h^{(K+1)}) \) with \( f_{Bs} \)

\[ f_{Bs} = 234(6) \text{ MeV (PRELIMINARY)} \]

(principal uncertainty from scale setting; other errors (stat+fit) < 1 % )
$f_{B_s}/f_B$ @ triggering point

$\bar{f}_{\ell} (\text{GeV})$

$[f_{hs}/f_{h\ell}]_{\text{trig}}$

- $f_{hs}/f_{h\ell}$ @ triggering point ($\sim m_c$)

Linear fit, Linear + Log and HMChPT vs. light quark mass

$\rightarrow$ increase systematic uncertainty: $f_{hs}/f_{hu/d} = 1.18(1)_{\text{stat}}(4)_{\text{syst}}$
\( f_{Bs}/f_B \) @ triggering point

- \( f_{hs}/f_{h\ell} \) @ triggering point (\( \sim m_c \))

Linear fit, Linear + Log and HMChPT vs. light quark mass \( \rightarrow \) increase systematic uncertainty: \( f_{hs}/f_{hu/d} = 1.18(1)_{\text{stat}}(4)_{\text{syst}} \)

- Try to smooth out the light quark dependence in the \( f_{hs}/f_{h\ell} \) ratio:

  fit \( (f_{hs}/f_{h\ell})^{\text{trig}} \times (f_{s\ell}/f_{K}^{\text{exp}}) \) vs. \( \bar{\mu}_\ell \) using linear fit ansatz (consistent with SU(2) ChPT; A. Roessl NPB 1999.)

  \( \rightarrow \) datapoints with larger error due to the scale uncertainty necessary for converting \( f_{s\ell} \) in phys. units: \( f_{hs}/f_{hu/d} = 1.18(3) \)
\[ f_{B_s}/f_B \]

- double ratio \( \zeta = z_s/z_\ell \) shows no significant \( \mu_\ell \) dependence at successive values of the heavy quark mass up to 3 GeV and small discr. effects.

- double ratio \( \zeta = z_s/z_\ell \) vs. \((1/\bar{\mu}_h)\): very weak dependence

- apply similar method for \( z_s/z_\ell \) to reach the b-quark mass point ...

\[ f_{B_s}/f_B = 1.19(05) \text{ (PRELIMINARY!)} \]

(principal source of uncertainty due to systematics in the fit ansatz @ triggering mass point)
Results & Comparisons - I

- $f_{Bs}(ETMC - 2012) = 234(06) \text{ MeV (PRELIMINARY!)}$
  - vertical lines show average over $N_f = 2 + 1$ results ...
  - no significant dependence on dynamical strange degree of freedom (within the present precision)

- $f_{Bs}/f_B(ETMC - 2012) = 1.19(05) \text{ MeV (PRELIMINARY!)}$
Results & Comparisons - II

- \( f_B(\text{ETMC - 2012}) = 197(10) \text{ MeV} \) (PRELIMINARY!)
  
  \( f_B = f_{Bs} / (f_{Bs}/f_B) \)
Ratio method for the $\Delta B = 2$ operators

- **QCD**
  \[
  O_1 = [\bar{b}^\alpha \gamma_\mu (1 - \gamma_5) q^\alpha][\bar{b}^\beta \gamma_\mu (1 - \gamma_5) q^\beta] \\
  O_2 = [\bar{b}^\alpha (1 - \gamma_5) q^\alpha][\bar{b}^\beta (1 - \gamma_5) q^\beta] \\
  O_3 = [\bar{b}^\alpha (1 - \gamma_5) q^\beta][\bar{b}^\beta (1 - \gamma_5) q^\alpha] \\
  O_4 = [\bar{b}^\alpha (1 - \gamma_5) q^\alpha][\bar{b}^\beta (1 + \gamma_5) q^\beta] \\
  O_5 = [\bar{b}^\alpha (1 - \gamma_5) q^\beta][\bar{b}^\beta (1 + \gamma_5) q^\alpha]
  \]

- **HQET**
  \[
  \tilde{O}_1 = [\bar{h}^\alpha \gamma_\mu (1 - \gamma_5) q^\alpha][\bar{h}^\beta \gamma_\mu (1 - \gamma_5) q^\beta] \\
  \tilde{O}_2 = [\bar{h}^\alpha (1 - \gamma_5) q^\alpha][\bar{h}^\beta (1 - \gamma_5) q^\beta] \\
  \tilde{O}_3 = -\tilde{O}_2 - (1/2) \tilde{O}_1 \\
  \tilde{O}_4 = [\bar{h}^\alpha (1 - \gamma_5) q^\alpha][\bar{h}^\beta (1 + \gamma_5) q^\beta] \\
  \tilde{O}_5 = [\bar{h}^\alpha (1 - \gamma_5) q^\beta][\bar{h}^\beta (1 + \gamma_5) q^\alpha]
  \]

- **Matching between QCD and HQET operators:**
  \[
  [W_{QCD}^T(\mu_h, \mu)]^{-1}\langle \tilde{O}(\mu) \rangle_{\mu_h} = C(\mu_h) [W_{HQET}^T(\mu_h, \tilde{\mu})]^{-1}\langle \tilde{O}(\tilde{\mu}) \rangle + O(1/\mu_h) + \ldots
  \]
  \[
  [W_{\ldots}^T(\mu_1, \mu_2)]^{-1}: \text{evolution 5x5 matrices} \\
  C(\mu_h): \text{matching matrix}
  \]

(e.g. D.Becirevic, V.Gimenez, G.Martinelli, M.Papinutto, J.Reyes, JHEP 2002)
Calculation of $B_{Bd/s}$

- $B$-bag parameters encode the non-perturbative QCD contribution to the $B_d/s - \bar{B}_d/s$ mixing amplitude: $\Delta M_q \propto |V_{tq}^* V_{tb}|^2 M_{Bq} f_{Bq}^2 \hat{B}_{Bq}$

- ETMC Calculation: [Frezzotti and Rossi, JHEP 2004] use mixed action; Osterwalder-Seiler valence quarks; suitable combinations of maximally twisted valence quarks ensure both
  - continuum-like renormalisation pattern for the 4-fermion operators
  - automatic $O(a)$-improvement.

(application to the $K$-sector: ETMC, Phys.Rev.D 2011; ETMC, 1207.1287)

(see talk by R. Frezzotti)

- $R_{BB} = \frac{C_{PO1P}(x_0)}{8/3 C_{PA}(x_0) C_{AP}(x_0)} \rightarrow B_B$

- We present results at three values of lattice spacing $a \in [0.1, 0.067]$ fm; a finer lattice spacing $a = 0.054$ fm in progress.
Ratio method for the $\Delta B = 2$ operators

• set:
  \[ \tilde{\Theta}(\mu_h, \mu, \tilde{\mu}) \equiv (W^T_{QCD}(\mu_h, \mu)C(\mu_h)W^T_{\text{HQET}}(\mu_h, \tilde{\mu}))^{-1}\langle \tilde{O}(\mu) \rangle_{\mu_h} \equiv \]
  \[ [C_B(\mu_h, \mu, \tilde{\mu})]^{-1}\langle \tilde{O}(\mu) \rangle_{\mu_h} = \langle \tilde{O}(\tilde{\mu}) \rangle + \mathcal{O}(1/\mu_h) + \ldots \]

• $[W^T_{\ldots}(\mu_1, \mu_2)]^{-1}$ and $C(\mu_h)$ are $(3 \times 3 \oplus 2 \times 2)$ block-diagonal matrices

• For $B_{Bq}$ case, calculate ratios at successive values of $\mu_h^{(n)} = \lambda \mu_h^{(n-1)}$
  
  (need only $3 \times 3$ matrices):
  \[ w^{(n)}_\Theta = \frac{\Theta_j(\mu_h^{(n)}, \mu, \tilde{\mu})}{\Theta_j(\mu_h^{(n-1)}, \mu, \tilde{\mu})} \text{ for } j = 1, 2, 3 \]
  and construct the appropriate ratio chain.

• up to LL order $O_1$ and $\tilde{O}_1$ renormalise multiplicatively; need only $j = 1$
Ratio method for $B_{Bs}$, $B_{ Bd}$ and their ratio

- HQET predicts: $\lim_{\mu_{h}^{\text{pole}} \to \infty} B_{B_{d}/s} = \text{constant}$
- $\lim_{\tilde{\mu}_{h} \to \infty} w_{B_{d}/s}(\tilde{\mu}_{h}, \lambda; \tilde{\mu}_{\ell}, a = 0) = 1$
- form the chain for $B_{Bs}$ and $B_{Bd}$:
  \[
  w_{B_{d}/s}(\tilde{\mu}_{h}^{(2)}) w_{B_{d}/s}(\tilde{\mu}_{h}^{(3)}) \ldots w_{B_{d}/s}(\tilde{\mu}_{h}^{(K+1)}) = \frac{B_{B_{d}/s}(\tilde{\mu}_{h}^{(K+1)}, \mu, \tilde{\mu})}{B_{B_{d}/s}(\tilde{\mu}_{h}^{(1)}, \mu, \tilde{\mu})} \cdot \left[ \frac{C_{B}(\tilde{\mu}_{h}^{(1)}, \mu, \tilde{\mu})}{C_{B}(\tilde{\mu}_{h}^{(K+1)}, \mu, \tilde{\mu})} \right]
  \]
- work in a similar way; form the double ratios for $B_{Bs}/B_{Bd}$:
  \[
  \zeta_{w}(\tilde{\mu}_{h}^{(2)}) \zeta_{w}(\tilde{\mu}_{h}^{(3)}) \ldots \zeta_{w}(\tilde{\mu}_{h}^{(K+1)}) = \frac{w_{B_{s}}(\tilde{\mu}_{h}^{(K+1)}) w_{B_{d}}^{-1}(\tilde{\mu}_{h}^{(K+1)})}{w_{B_{s}}(\tilde{\mu}_{h}^{(1)}) w_{B_{d}}^{-1}(\tilde{\mu}_{h}^{(1)})}
  \]
- for $\tilde{\mu}_{h}^{(K+1)} = \lambda^{K} \tilde{\mu}_{h}^{(1)} = m_{b}$ get the physical values for the Bag parameters and of their ratio
compute the double ratio

\[ \zeta_w = w_{B_s}/w_{B_d}(\bar{\mu}_h^{(n)}, \bar{\mu}_\ell) \]

for values of \( \bar{\mu}_\ell \) and extrapolate to CL @ \( u/d \) at each \( \bar{\mu}_h^{(n)} \)

- \( \zeta_w(\bar{\mu}_h^{(n)}, \mu_\ell) \) values show no significant \( \mu_\ell \) dependence and cutoff effects are small
- very weak dependence on heavy quark mass
- fit ansatze

\[ \zeta_w(\bar{\mu}_h) = 1 + c_1/\bar{\mu}_h ( + c_2/\bar{\mu}_h^2) \]

\[ B_{B_s}/B_{B_d} = 1.03(2) \quad (at \quad \bar{\mu}_h^{(K+1)} = \lambda^K \bar{\mu}_h^{(1)} = m_b) \quad (PRELIMINARY!) \]

- For Comparison

\[ B_{B_s}/B_{B_d} (FNAL/MILC – 2012) = 1.06(11); \]

\[ B_{B_s}/B_{B_d} (HPQCD – 2009) = 1.05(07) \]
\[ \frac{V_{td}}{V_{ts}} = \xi \left( \frac{\Delta M_d M_{Bs}}{\Delta M_s M_{Bd}} \right)^{1/2} \]

- \[ \xi = (f_{Bs} / f_{Bd}) \sqrt{B_{Bs} / B_{Bd}} \]

- Form the ratio \( \zeta_{\xi}(\bar{\mu}_h^{(n)}, \bar{\mu}_\ell) \) at successive values of \( \bar{\mu}_h^{(n)} = \lambda \bar{\mu}_h^{(n-1)} \);
  - Determine CL @ u/d at each \( \bar{\mu}_h^{(n)} \)

- \( \zeta_{\xi}(\bar{\mu}_h^{(n)}, \mu_\ell) \) show no significant \( \mu_\ell \) dependence and small cutoff effects

- \( \zeta_{\xi} \) vs. heavy quark mass \( (1/\bar{\mu}_h) \):
  - Very weak dependence

- Fit ansatze \( \zeta_{\xi}(\bar{\mu}_h) = 1 + c_1' / \bar{\mu}_h + c_2' / \bar{\mu}_h^2 \)  
  \[ \rightarrow \xi = 1.21(06) \text{ (at } \bar{\mu}_h^{(K+1)} = \lambda^K \bar{\mu}_h^{(1)} = m_b \text{)} \] (PRELIMINARY!)
  - (syst. error \( \sim 4\% \) due to fit ansatze @ triggering mass point)

- For comparison:
  - \( \xi(\text{FNAL/MILC - 2012}) = 1.268(63) \);  
  - \( \xi(\text{HPQCD - 2009}) = 1.258(33) \)
- smooth behaviour in the CL @ triggering point value ($\sim m_c$)

- ratios $w_{Bs}(\bar{\mu}_h^{(n)}, \bar{\mu}_\ell)$ show smooth behaviour with $\bar{\mu}_\ell$ and small discr. effects

- fit ansatz $w_{Bs}(\bar{\mu}_h) = 1 + w_1/\bar{\mu}_h + w_2/\bar{\mu}_h^2$

- *Similar analysis for $B_{B_d}$*
Results for B-Bag Parameters

• \( B_{B_s}/B_{B_d} = 1.03(2) \)
• \( \xi = 1.21(06) \)

• \( B_{B_d}(m_b, \overline{M_S}) = 0.87(05) \)
• \( B_{B_s}(m_b, \overline{M_S}) = 0.90(05) \)

(PRELIMINARY!)
Summary

• Ratio method uses relativistic quarks and an *obvious* value of the static limit. No static calculation needed.

• Ratio method can be used for all observables whose static limit behaviour is known from HQET.

• ETMC \((N_f = 2)\) results for \(m_b, f_{Bs}, f_{B}, f_{Bs}/f_{B}, B_{Bs}, B_{Bd}, B_{Bs}/B_{Bd}\) and \(\xi\) are in the same ballpark of results from other collaborations.

• Repeat/extend the study to \(N_f = 2 + 1 + 1\) ensembles; work in progress; results available very soon using three lattice spacings \(a \in [0.09, 0.06] \text{ fm}\).
Thank you for your attention!
backup slides
b-quark mass ratios - phenom. indications

\[ y(\bar{\mu}_h) = 1 + \frac{\eta_1}{\mu_h} + \frac{\eta_2}{\mu_h^2} \]

- consider (HQET) \( M_{h\ell} = \mu_h^{\text{pole}} + \tilde{\Lambda} - \frac{(\lambda_1 + 3\lambda_2)}{2} \frac{1}{\mu_h^{\text{pole}}} + \mathcal{O}\left(\frac{1}{(\mu_h^{\text{pole}})^2}\right) \)

- and get \( y = 1 - \tilde{\Lambda} \frac{\lambda^{\text{pole}}}{\mu_h^{\text{pole}}} - 1 + \left(\frac{(\lambda_1 + 3\lambda_2)}{2}\right)(\lambda^{\text{pole}} + 1) + \tilde{\Lambda}^2 \lambda^{\text{pole}} \frac{\lambda^{\text{pole}} - 1}{(\mu_h^{\text{pole}})^2} \)

with \( \lambda^{\text{pole}} = \mu_h^{\text{pole}}(\bar{\mu}_h)/\mu_h^{\text{pole}}(\bar{\mu}_h/\lambda) = \lambda \rho(\bar{\mu}_h)/\rho(\bar{\mu}_h/\lambda) \)

- use phenomenological estimates for HQET parameters, as e.g.

\( \tilde{\Lambda} = 0.39(11) \text{ GeV} \),  \( \lambda_1 = -0.19(10) \text{ GeV}^2 \),  \( \lambda_2 = 0.12(2) \text{ GeV}^2 \)

Ratios for $f_{B_s}$ - Phenomenological Indications

$$z_s(\bar{\mu}_h) = 1 + \frac{\zeta_1}{\bar{\mu}_h} + \frac{\zeta_2}{\bar{\mu}_h^2}$$

(for $1/\bar{\mu}_h > 0.60$ estimated uncertainty on the black curve $\sim 0.03$)

- consider (HQET)
  $$\Phi_{hs}(\bar{\mu}_h, \mu_b^*) = \left( f_{hs} \sqrt{M_{hs}} \right)^{QCD} \frac{C_{stat}(\bar{\mu}_h, \mu_b^*)}{C^A_{stat}(\bar{\mu}_h, \mu_b^*)} = \Phi_0(\mu_b^*) \left( 1 + \frac{\Phi_1(\mu_b^*)}{\mu_h^{pole}} + \frac{\Phi_2(\mu_b^*)}{(\mu_h^{pole})^2} \right) + O\left( \frac{1}{(\mu_h^{pole})^3} \right)$$

- and get
  $$y_s^{1/2} z_s = \frac{\Phi_{hs}(\bar{\mu}_h)}{\Phi_{hs}(\bar{\mu}_h/\lambda)} = 1 - \Phi_1 \frac{\lambda^{pole} - 1}{\mu_h^{pole}} - \left( \Phi_2 (\lambda^{pole} + 1) - \Phi_1^2 \lambda^{pole} \right) \frac{\lambda^{pole} - 1}{(\mu_h^{pole})^2}$$

- use phenomenological values for HQET parameters
  $\tilde{\Lambda}_s = \tilde{\Lambda} + M_{B_s} - M_B$ ,  $\lambda_1 s = \lambda_1$ ,  $\lambda_2 s = \lambda_2$ ,  $\Phi_0 = 0.60$ GeV$^{3/2}$ and the estimates $\Phi_1 = -0.48$ GeV ,  $\Phi_2 = 0.08$ GeV$^2$ (→ values obtained from inputs at $B_s$ and $D_s$. )