On the Phase Diagram of $\mathcal{N} = 4$ Lattice Gauge Theory

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Overview

- Supersymmetry on the lattice: The problem
- Topological twisting of supersymmetric gauge theories
- The most supersymmetric gauge theory in (3+1)-dimensions
- Conformal symmetry of the continuum theory
- Is the lattice theory conformal at large distances?
- Numerical results

Supersymmetry

Forget the lattice for a moment

Supersymmetry is invariance under transformations that take bosons ϕ to fermions ψ and vice versa

Simple example: supersymmetric quantum mechanics

$$S = \int dt \left[\frac{1}{2}(\partial_t \phi)^2 + \bar{\psi}\partial_t \psi + \frac{1}{2}W'(\phi)^2 + \bar{\psi}W''(\phi)\psi \right]$$

Invariant under

$$\begin{aligned} \delta\phi &= \bar{\psi}\epsilon \\ \delta\psi &= (\partial_t \phi - W'(\phi))\epsilon \end{aligned}$$

"Topological twisting"

Supersymmetric invariance of the action can be made trivial by a Q with $Q^2 = 0$:

$$\begin{array}{rcl} Q\phi & = & \psi \\ Q\psi & = & 0 \\ Q\bar{\psi} & = & B \\ QB & = & 0 \end{array}$$

and

$$S = Q\left\{\int dt\bar{\psi}\left[-(\partial_t\phi + W'(\phi)) - \frac{1}{2}B\right]\right\}$$

Supersymmetry on a space-time lattice

Two supersymmetry transformations generate a time translation: $\delta_1\delta_2 \sim \partial_t$
In quantum field theory this becomes a *space-time translation*: $\delta_1\delta_2 \sim \hat{P}$

Here is the basic problem: A space-time lattice is only invariant under discrete translations – *will obviously break supersymmetry!*

Can we cure the problem by defining a kind of "discrete supersymmetry"?

There is not even discrete supersymmetry

There is no unique definition of a derivative on a lattice

Examples:

Backward derivative: $a\Delta^- f(x) \equiv f(x) - f(x-a)$

Forward derivative: $a\Delta^+ f(x) \equiv f(x+a) - f(x)$

Perhaps a translation should be either a forward or backward **lattice** translation?

But lattice derivatives do not not satisfy the *Leibniz rule*, a crucial ingredient in proving supersymmetry invariance

Topological field theory

Is there a way out?

Supersymmetric quantum mechanics holds the clue

Recall:

$$S = \int dt \left[\frac{1}{2}(\partial_t \phi)^2 + \bar{\psi}\partial_t \psi + \frac{1}{2}W'(\phi)^2 + \bar{\psi}W''(\phi)\psi \right]$$

can be written

$$S = Q\left\{ \int dt \bar{\psi} \left[-(\partial_t \phi + W'(\phi)) - \frac{1}{2}B \right] \right\}$$

with $Q^2 = 0$.

Let us replace (for example) $\partial_t \rightarrow \Delta^-$:

$$S = Q \sum_t \bar{\psi} \left[-(\Delta^- \phi + W'(\phi)) - \frac{1}{2} B \right]$$

This is clearly supersymmetric: $QS = Q^2[\ldots] = 0$

Lattice Supersymmetry

Remarkably, the same trick works for a very special supersymmetric gauge theory in (3+1)-dimensions: the $\mathcal{N} = 4$ theory

Now it gets a bit technical

We work on a four-dimensional Euclidean lattice with Lorentz symmetry $SO(4)$

We combine ("twist") the fermions according to a combination of Lorentz symmetry and an $SO(4)$ subgroup of a global rotation – **R symmetry**.

This mixes up Lorentz indices with fermion indices.

There are four four-spinors (Majorana). Think of them as a 4×4 matrix Ψ and then expand i Dirac's γ-matrices:

$$\Psi = \eta I + \psi_\mu \gamma_\mu + \chi_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\psi}_\mu \gamma_5 \gamma_\mu + \overline{\eta} \gamma_5$$

Four of the scalars get twisted too: they combine into a 4-vector B_μ

Convenient combination is the *complex* $\mathcal{A}_\mu = A_\mu + iB_\mu$

The lattice no-go Theorem

One massless fermion on a four-dimensional lattice becomes *sixteen* fermions

For supersymmetry we need a perfect match (bosons) $\leftrightarrow$ (fermions)

This is yet one more obstacle against lattice supersymmetry

Here the theory cures itself: we precisely *need* sixteen massless fermions!

A beautiful formalism

Beautiful for topological field theory experts, but I will skip most details here

Just this: the natural language is in terms of "five dimensions" where all bosons are lumped into complex gauge variables $\mathcal{A}_a$ and the fermions combine into $(\eta, \psi_a, \chi_{ab})$

Don't worry about this 'five-dimensional' language: *we're in four dimensions*

The five complex gauge fields are easy to understand from dimensional reduction of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory

Let us first look at the continuum $\mathcal{N} = 4$ theory

One can now do as for supersymmetric quantum mechanics: find a $\mathcal{Q}$ with $\mathcal{Q}^2 = 0$ so that the action is essentially

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\,\left(\chi_{ab}\mathcal{F}_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_b] - \frac{1}{2}\eta d\right)$$

where $\mathcal{F}_{ab}$ is the complexified field strength:

$$\mathcal{F}_{ab} = [\mathcal{D}_a, \mathcal{D}_b], \quad \overline{\mathcal{F}}_{ab} = [\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b]$$

and the complex covariant derivatives are

$$\mathcal{D}_a = \partial_a + \mathcal{A}_a, \quad \overline{\mathcal{D}}_a = \partial_a + \overline{\mathcal{A}}_a$$

The supersymmetry transformation is

$$
\begin{aligned}
\mathcal{Q}\,\mathcal{A}_a &= \psi_a \\
\mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}
\tag{1}
$$

The action is identical to the continuum $\mathcal{N} = 4$ theory

Invariance under the remaining 15 supercharges is not manifest but it is there

I am not being totally honest

Even in the continuum we need one tiny extra piece added to the action:

$$S_{\text{closed}} = -\frac{1}{8}\int \text{Tr}\ \epsilon_{mnpqr}\chi_{qr}\overline{\mathcal{D}}_p\chi_{mn}$$

This is not $\mathcal{Q}$-exact, but it is $\mathcal{Q}$-invariant on account of the Bianchi identity

$$\epsilon_{mnpqr}\overline{\mathcal{D}}_p\overline{\mathcal{F}}_{qr} = 0$$

The extra term is still topological

Can we transcribe this to the lattice?

It works out precisely as one could have dreamt it would

We introduce complexified Wilson links that live in the *algebra* of the gauge group:

$$\mathcal{A}_a(x) \to \mathcal{U}_a(\mathbf{n}) = \sum_{C=1}^{N^2} T^C \mathcal{U}_a^C(\mathbf{n})$$

We need five links in four dimensions. Introduce an additional body diagonal

$$\begin{aligned}
\widehat{\boldsymbol{\mu}}_1 &= (1,0,0,0) \\
\widehat{\boldsymbol{\mu}}_2 &= (0,1,0,0) \\
\widehat{\boldsymbol{\mu}}_3 &= (0,0,1,0) \\
\widehat{\boldsymbol{\mu}}_4 &= (0,0,0,1) \\
\widehat{\boldsymbol{\mu}}_5 &= (-1,-1,-1,-1)
\end{aligned}$$

The fields all transform under ordinary non-complexified $U(N)$ transformations

$$\mathcal{U}_a(\mathbf{n}) \to G(\mathbf{n})\mathcal{U}_a(\mathbf{n})G^\dagger(\mathbf{n}+\widehat{\boldsymbol{\mu}}_a)$$

Supersymmetry then dictates that $\psi_a(\mathbf{n})$ lives on a link and transforms accordingly

The local fermion $\eta(\mathbf{n})$ lives on a site and transforms as such

The fermions χ_{ab} live on corner variables (or new links leading from the origin out to $\widehat{\boldsymbol{\mu}}_a+\widehat{\boldsymbol{\mu}}_b$) and transform as

$$\chi_{ab}(\mathbf{n}) \to G(\mathbf{n}+\widehat{\boldsymbol{\mu}}_a+\widehat{\boldsymbol{\mu}}_b)\chi_{ab}(\mathbf{n})G^\dagger(\mathbf{n})$$

Covariant derivatives are

$$\mathcal{D}_a^{(+)} f_b(\mathbf{n}) \quad = \quad \mathcal{U}_a(\mathbf{n}) f_b(\mathbf{n}+\widehat{\boldsymbol{\mu}}_a) - f_b(\mathbf{n})\mathcal{U}_a(\mathbf{n}+\widehat{\boldsymbol{\mu}}_b)\ ,$$

$$\overline{\mathcal{D}}^{(-)}_a f_a(\mathbf{n}) \quad = \quad f_a(\mathbf{n})\overline{\mathcal{U}}_a(\mathbf{n}) - \overline{\mathcal{U}}_a(\mathbf{n}-\widehat{\boldsymbol{\mu}}_a) f_a(\mathbf{n}-\widehat{\boldsymbol{\mu}}_a) \tag{2}$$

Now we can define the invariant action!

We copy the procedure from the continuum:

$$S = \sum_{\mathbf{n}} \mathrm{Tr}\; \mathcal{Q}\Big(\chi_{ab}(\mathbf{n})\mathcal{D}^{(+)}_a\mathcal{U}_b(\mathbf{n}) + \eta(\mathbf{n})\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(\mathbf{n}) - \frac{1}{2}\eta(\mathbf{n})d(\mathbf{n})\Big)\ .$$

This is manifestly $\mathcal{Q}$-symmetric

I'm cheating slightly again

Just like in the continuum theory, there is an extra term

$$S_{closed} = \frac{1}{2}\epsilon_{abcde}\chi_{de}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_a+\widehat{\boldsymbol{\mu}}_b+\widehat{\boldsymbol{\mu}}_c)\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_c)$$

An *exact lattice analog of the Bianchi identity*

$$\epsilon_{abcde}\overline{\mathcal{D}}_c^{(-)}\overline{\mathcal{F}}_{ab}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_c) = 0$$

makes this additional term $\mathcal{Q}$-invariant on the lattice

The exactly preserved lattice supersymmetry

$$
\begin{aligned}
\mathcal{Q}\,\mathcal{U}_a &= \psi_a \\
\mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\overline{\mathcal{U}}_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}
$$

This is just like in the continuum theory

However, on the lattice this is the *only* exactly preserved supersymmetry

Why is this theory so interesting?

Probably the only four-dimensional gauge theory that is ultraviolet *finite*

Its β-function vanishes to all orders in perturbation theory: $\beta(g) = 0$

The theory is related to deep structures in mathematics ('The Geometric Langlands Program')

The theory is conformal even at the quantum level

The AdS/CFT correspondence gives exact predictions at strong coupling

Plus much more....!

To simulate we must (slightly) break supersymmetry

In ordinary lattice gauge theory the continuum limit is reached through the Renormalization Group

The lattice spacing (ultraviolet cut-off) a is replaced by a physical scale

Here the continuum limit is reached in a much more peculiar way

We add to the action a term

$$S_M = \mu_L^2 \sum_{\mathbf{n}} \left(\frac{1}{N} \mathrm{Tr}(\mathcal{U}_a^\dagger(\mathbf{n}) \mathcal{U}_a(\mathbf{n})) - 1 \right)^2$$

This breaks supersymmetry, but we can live with it (all counterterms under control)

Eventually we will take $\mu_L \to 0$; this restores supersymmetry

By keeping μ_L finite we decouple the trace mode of the scalars while the traceless scalar modes feel only a quartic potential

In this way we retain the *flat directions* of the scalars – an important aspect of the supersymmetric gauge theory

The additional term has another remarkable effect: it defines for us a continuum limit

How continuum gauge fields arise

To get the correct naive continuum limit we must expand

$$\mathcal{U}_a(\mathbf{n}) \;=\; \mathbf{1} + a\mathcal{A}_a(\mathbf{n})$$

This restores continum Yang-Mills theory and continuum Yang-Mills gauge invariance

The additional mass term precisely achieves this!

It can happen because we have complexified gauge links $\mathcal{U}_a = e^{A_a + iB_a}$

This is the *naive continuum limit* – what about the quantum theory?

Only numerical simulations can tell

My next slides will show some of our results so far

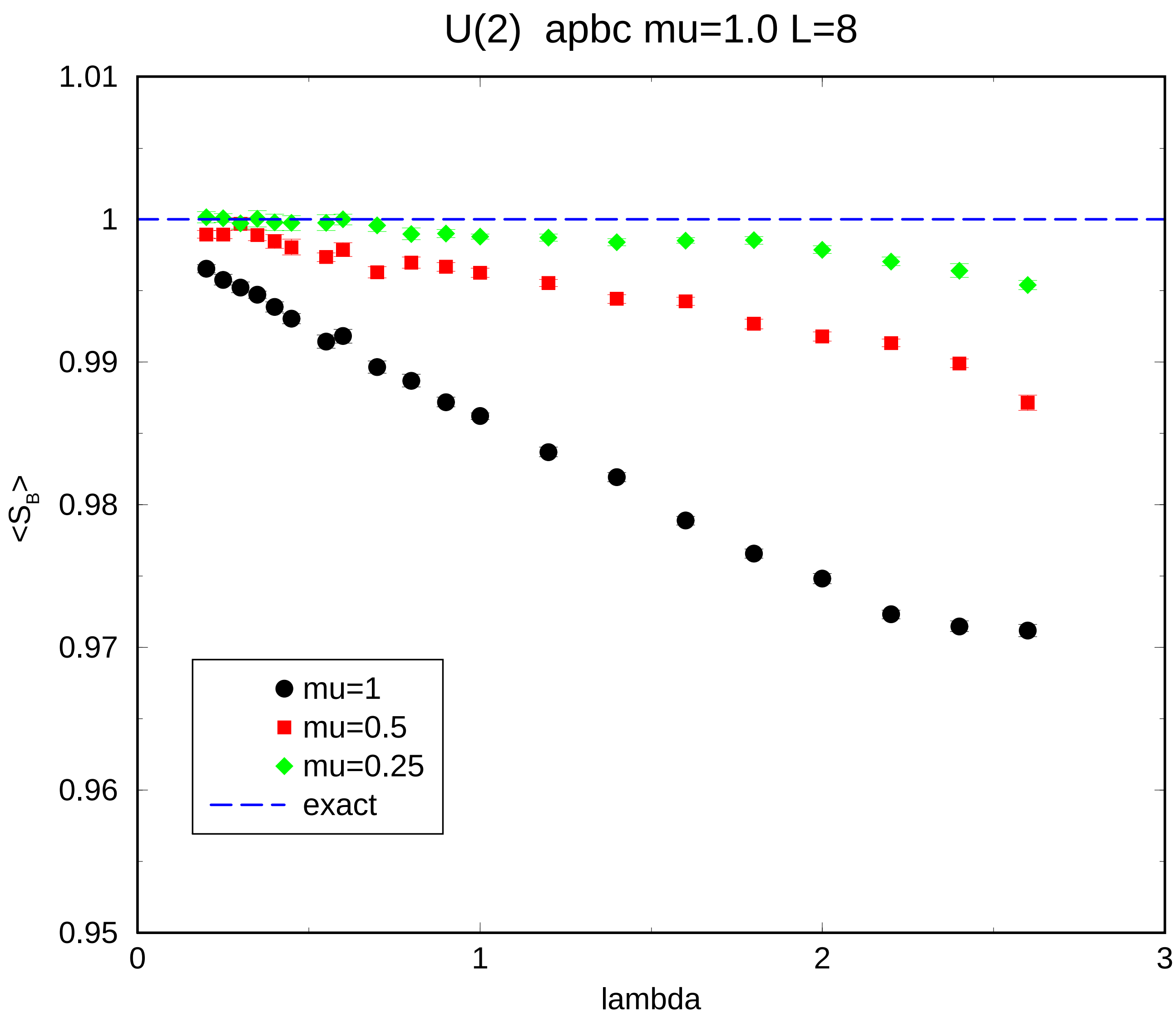
Supersymmetry is apparently well preserved

This theory has a potential *sign problem*: When we integrate out the fermionic fields we get Pfaffian which is not real

Impossible to simulate by Monte Carlo methods except by reweighting

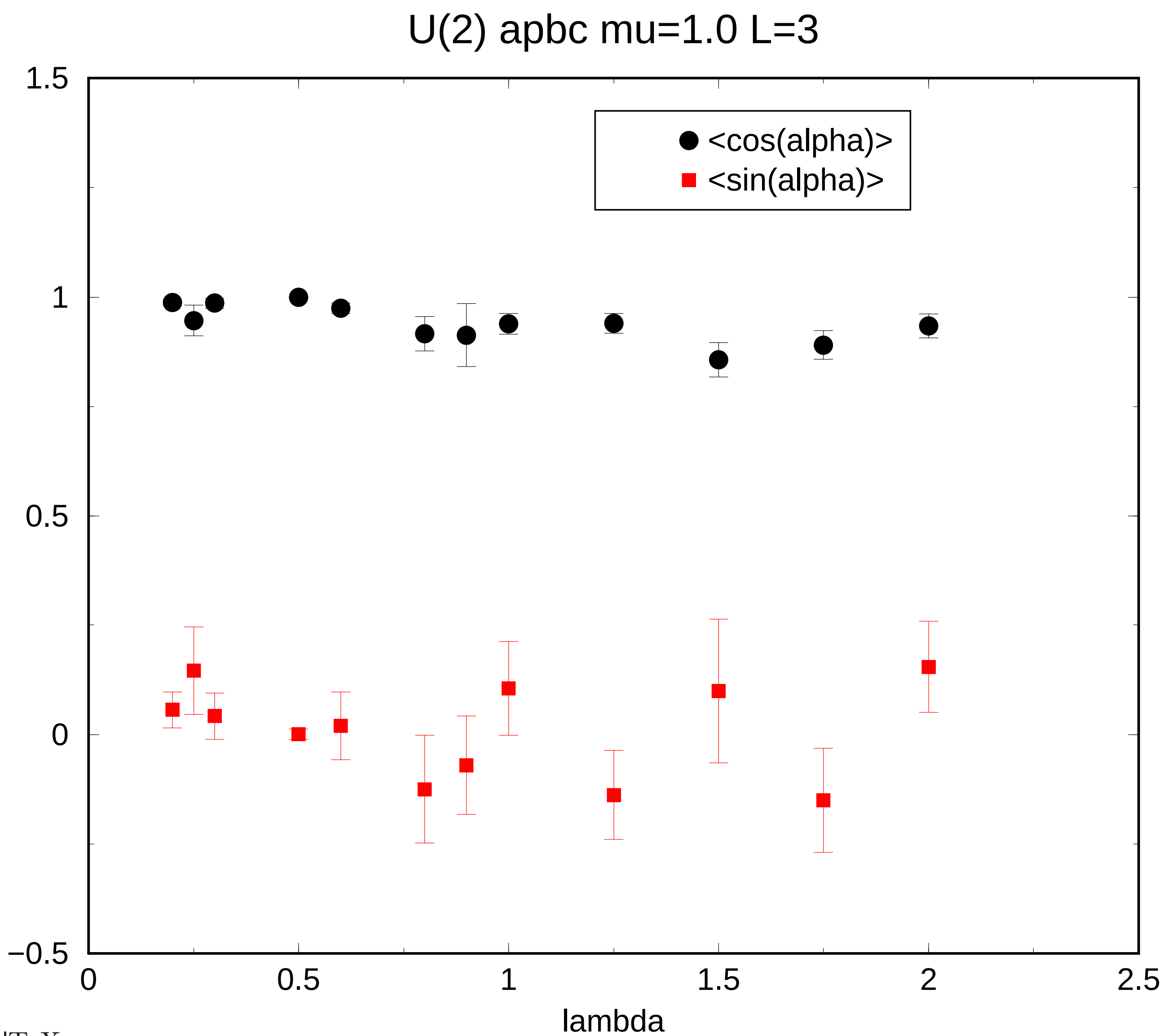
We first check what happens if we *phase quench*: simply ignore the phase

This breaks supersymmetry, but by how much? We compute the bosonic part of the action and compare with the exact answer $S_B/V = 9N^2/2 = 18$ for gauge group U(2)


U(2) apbc mu=1.0 L=8
1.01
1
0.99
0.98
0.97
0.96
0.95
<S_B>
0
1
2
3
lambda
mu=1
mu=0.5
mu=0.25
exact

Let's measure the phase of the Pfaffian

On very small lattices we can compute the phase we're discarding!


U(2) apbc mu=1.0 L=3
<cos(alpha)>
<sin(alpha)>
1.5
1
0.5
0
−0.5
0
0.5
1
1.5
2
2.5
lambda

Is there a string tension?

Essentially *all* other strongly coupled lattice gauge theories have a region with *confinement* where the static potential grows linearly at large distances:

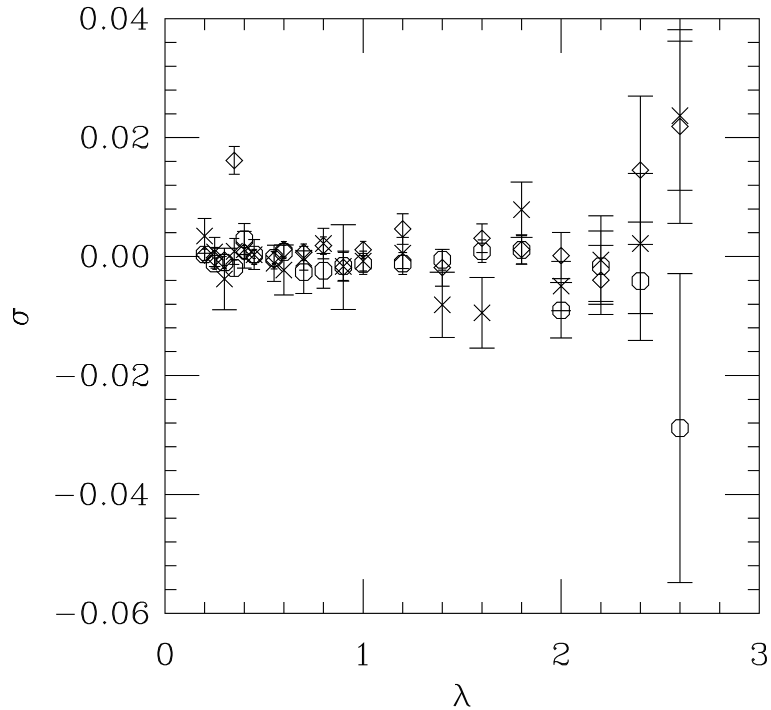
$$V(r) = -\frac{C}{r} + A + \sigma r$$

In a conformal theory there is no scale and hence the potential is trivial at all couplings:

$$V(r) = -\frac{C}{r}$$

We measure the static potential from the supersymmetric Wilson loop: the gauge-invariant product of links around a rectangular loop of spatial distance R (and large time T)

Using the ansatz on the top we extract the coefficient σ – the string tension



We find no string tension at all – consistent with conformal symmetry

The Polyakov line also indicates deconfinement

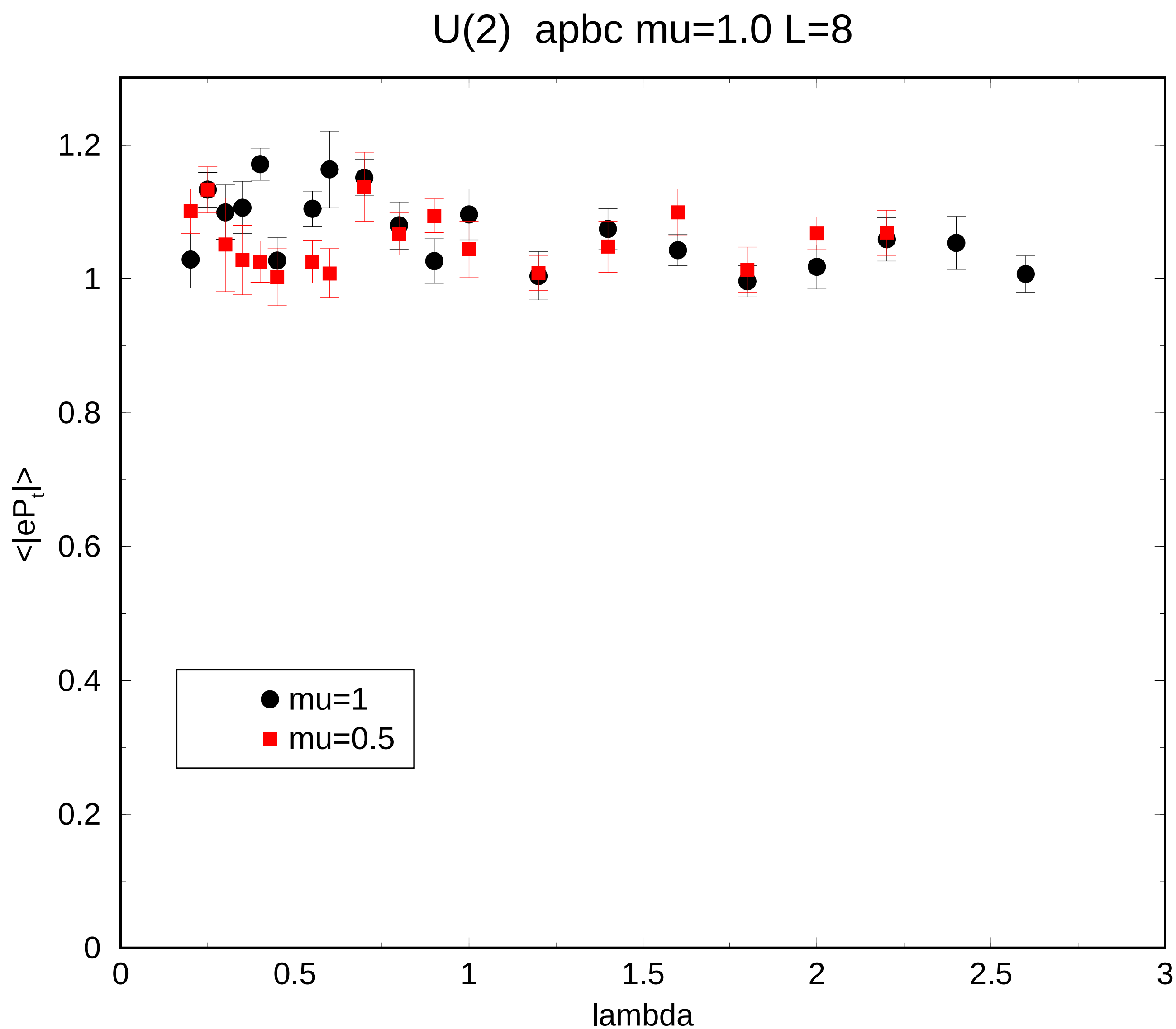
The product of links along the time-axis

$$P_t[\mathcal{U}] \;=\; \prod[\mathcal{U}(\mathbf{1})\cdots\mathcal{U}(\mathbf{n}_{max})]$$

Note: this becomes the usual path-ordered exponential in the continuum limit

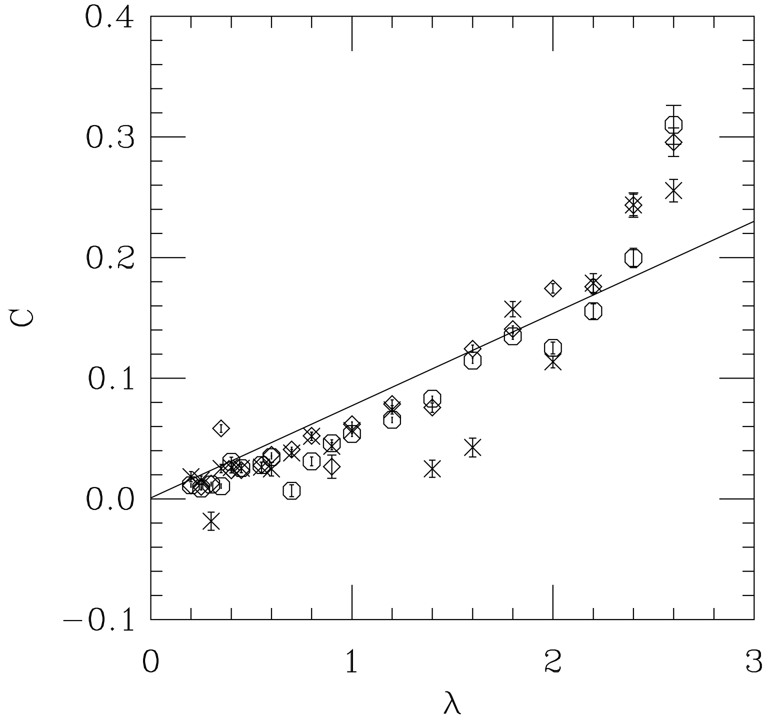
Since $\langle P_t[\mathcal{U}]\rangle\rangle = \exp[-F/T]$ a non-vanishing value indicates deconfined fundamental charges in this theory

Let us project onto the $SU(2)$ sector in order to disentangle the scalar trace mode


U(2) apbc mu=1.0 L=8
<|eP_t|>
lambda
0
0.2
0.4
0.6
0.8
1
1.2
0.5
1.5
2
2.5
3
mu=1
mu=0.5

A conformal theory has no scale

Recall: We found $V(r) = -\frac{C}{r}$ – what is C?



Fits well with $C = \lambda/(4\pi) = g^2 N/(4\pi)$ – as at weak coupling

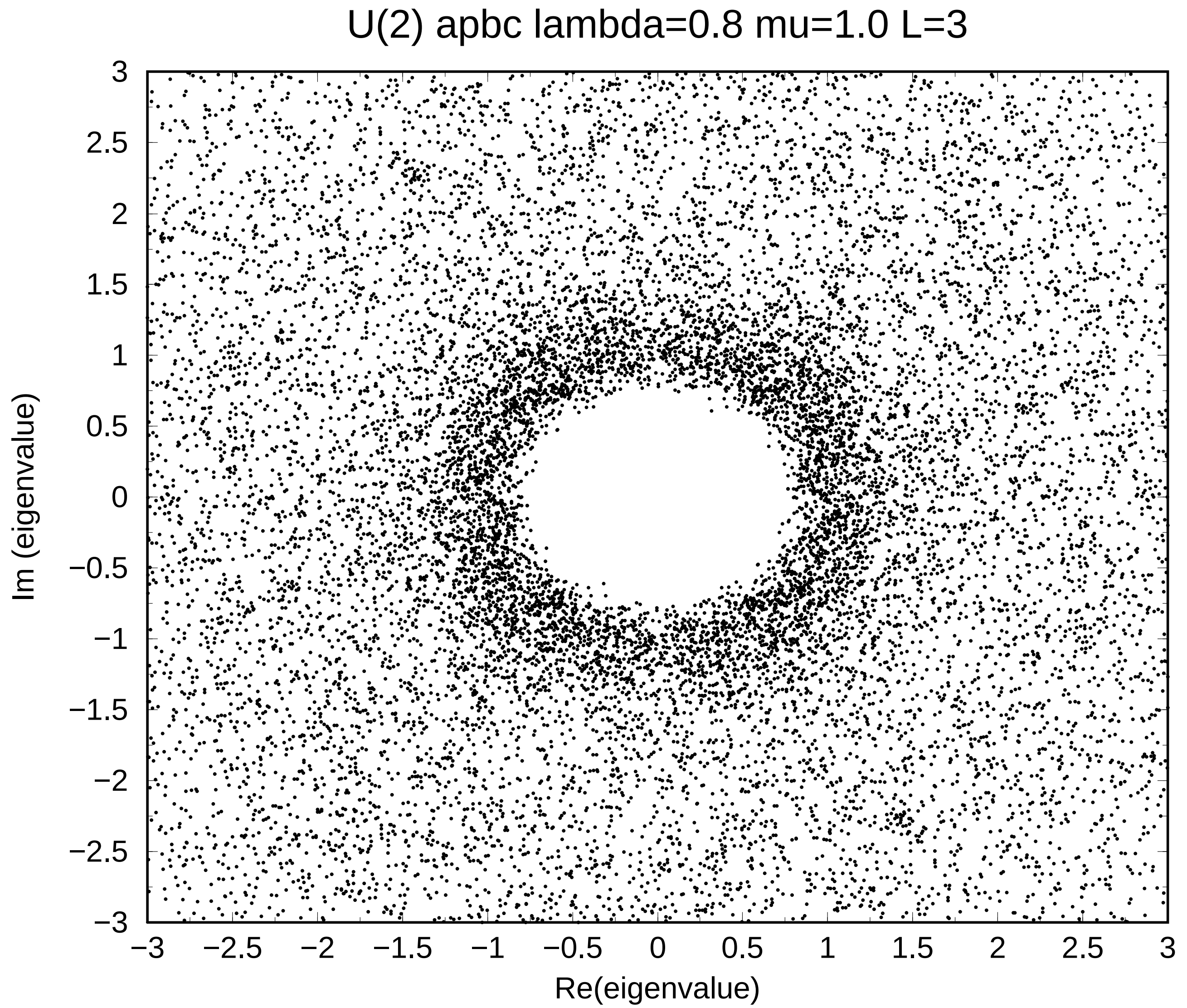
Chiral symmetry appears unbroken

Consider the eigenvalues of the Dirac operator

Attraction of eigenvalues towards the origin is required for a chiral condensate to form

We can measure the eigenvalues on small lattices just to get a feel for it

Naively we might expect close to free field behavior $\lambda \sim 1/L$ (also a *gap* cannot occur in a conformal theory at infinite volume)


U(2) apbc lambda=0.8 mu=1.0 L=3
Im (eigenvalue)
Re(eigenvalue)
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3 −2.5 −2 −1.5 −1 −0.5 0 0.5 1 1.5 2 2.5 3

The continuum limit in the quantum theory

We seem to be able to define a scale-free and conformal lattice gauge theory

The only scale in our simulations will then eventually be the lenght of the box L

At all couplings we have 'criticality': *like a line of critical points*

The continuum limit is then taken at any bare coupling by going to large distances

Conclusions

- We now know which supersymmetric gauge theories can be formulated on a space-time lattice
- A (super)conformal gauge theory in (3+1)-dimensions can be studied numerically
- Evidence that the theory is 'critical' on the whole coupling constant line
- First example of a strongly interacting lattice gauge theory that retains conformal symmetry