Radial Quantization for Conformal Field Theories on the Lattice

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New Frontiers in Lattice Gauge Theory
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Motivation

- (near) Conformal Field Theories are important
  - BSM walking technicolor
  - AdS/CFT weak-strong duality
  - Model building

- Lattice difficulty: scales are (nearly) exponential.

- Hypercubic vs Radial Lattice

$$a < \Delta r < L \quad \text{vs} \quad a < \Delta log(r) < L$$

Early History

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. This leads to an operator formulation for Euclidean quantum field theory. A covariant thermodynamics is developed, with which the Hagedorn spectrum can be obtained, given further hypotheses. The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For d > 2 these correspond however, to curved spaces. The result is verified for the spherical model

Outline

- Conformal Field Theory
- Lattice Radial Quantization
- 3-D Ising model at Tc
- Conclusion & Future Directions

Conformal Field Theories

O(d+1,1) adds Dilations and Inversion to Poincare transformations

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

Algebra:

$$K_\mu : (inv \to trans \to inv)$$

$$[K_\mu, \mathcal{O}(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

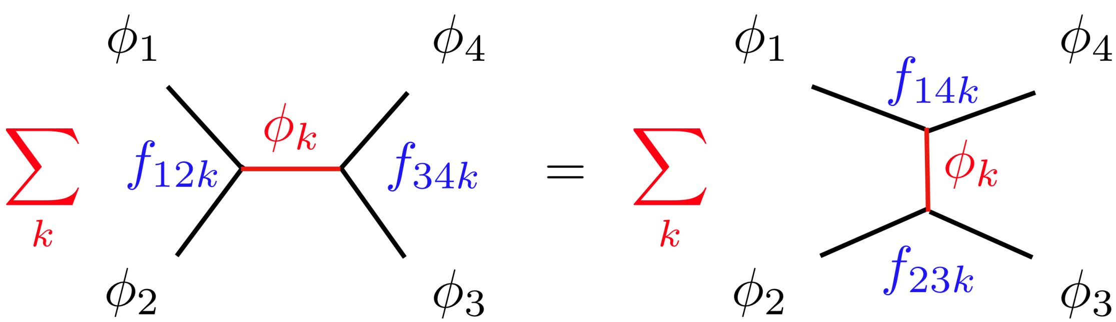
$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

CFT are highly constrained

1. More than hyper scaling (scale invariance).
2. 2 and 3 point correlators are determined.
3. OPE & factorization may fixed the theory completely*?

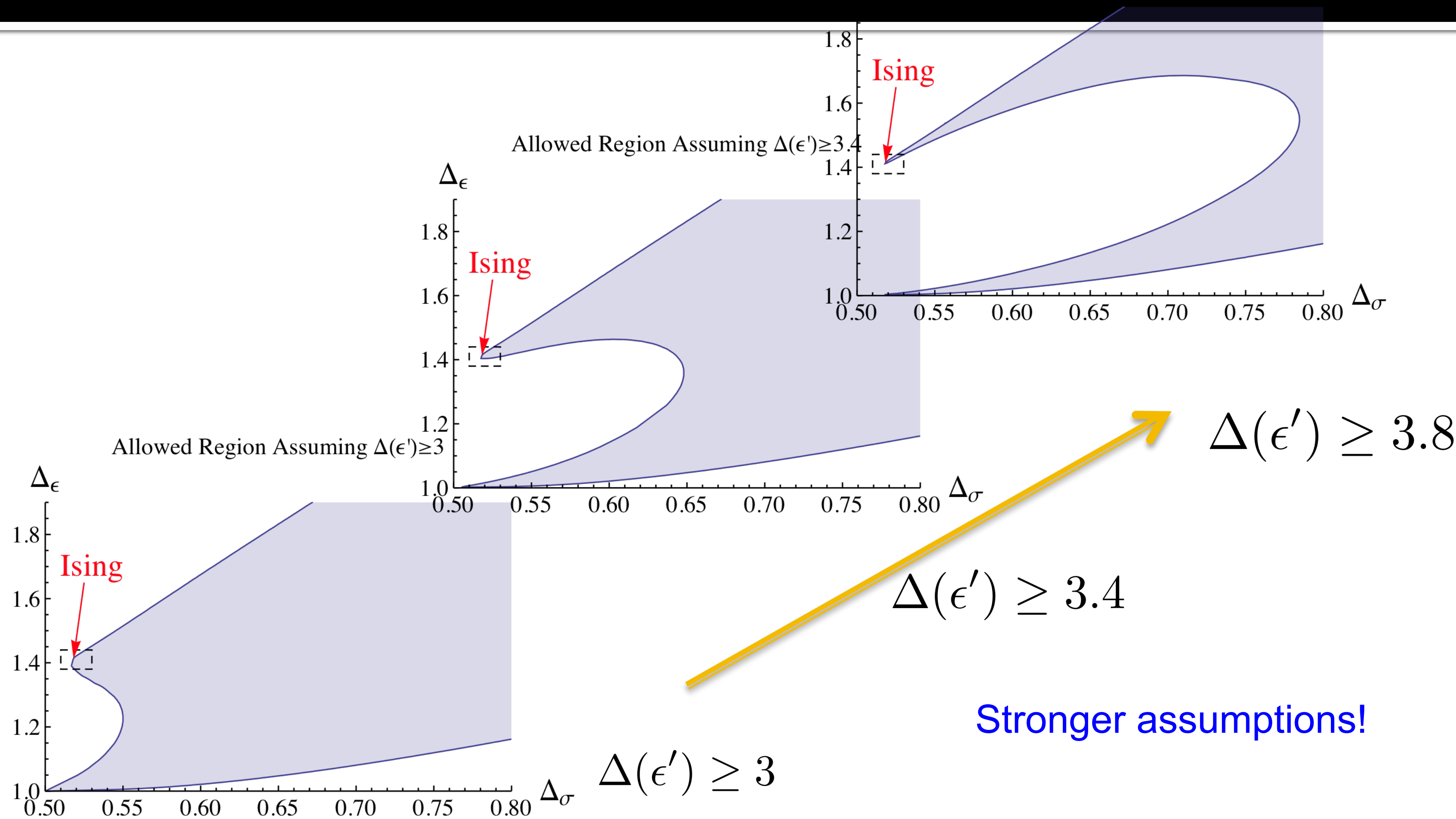
$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$



* "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Inequalities from Bootstrap*



•"Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Radial Quantization

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"}time\text{"} \quad \tau = log(r)\ , \quad \text{"}mass\text{"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

$$
\begin{aligned}
r_1^{\Delta} r_2^{\Delta}\langle \phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^{\Delta}} \\
&\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\
&= Ce^{-\tau\Delta}
\end{aligned}
$$

With $|x_1 - x_2|^2 = r_1 r_2 [r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^{\Delta}$

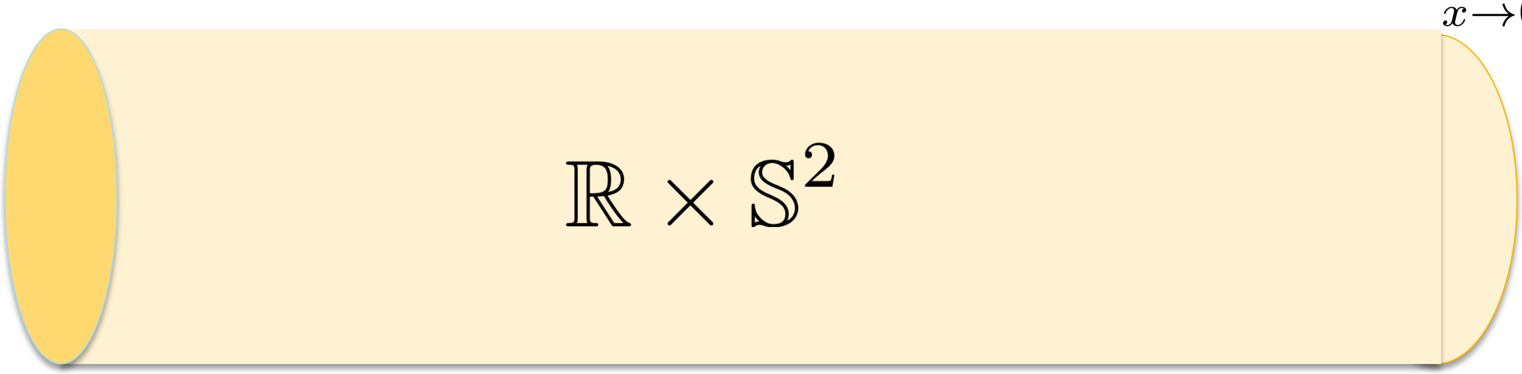
as $\tau = \log(r_2) - \log(r_1) \to \infty$

3-d Ising

$$Z_{Ising} = \sum_{\{s_{\tau,i}=\pm 1\}} e^{\beta \sum_{\tau,i} s_{\tau+1,i} s_{\tau,i} + \beta \sum_{\tau <i,j>} s_{\tau,i} s_{\tau,j}}$$

$$\lim_{x\to 0} \mathcal{O}(x)|0\rangle$$

$$= |\mathcal{O}\rangle$$



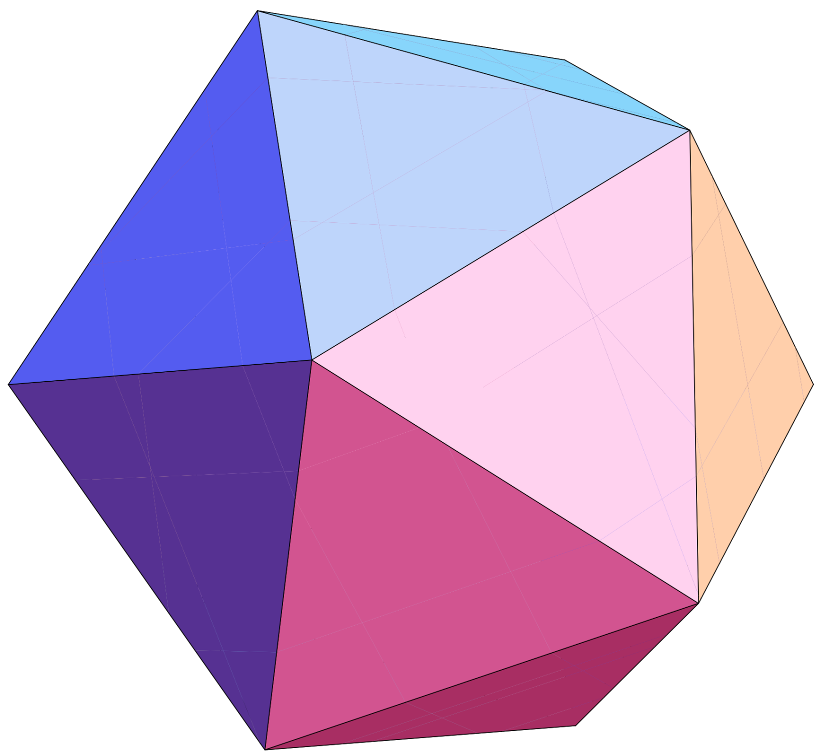
state-op corr

$(r = 0)$

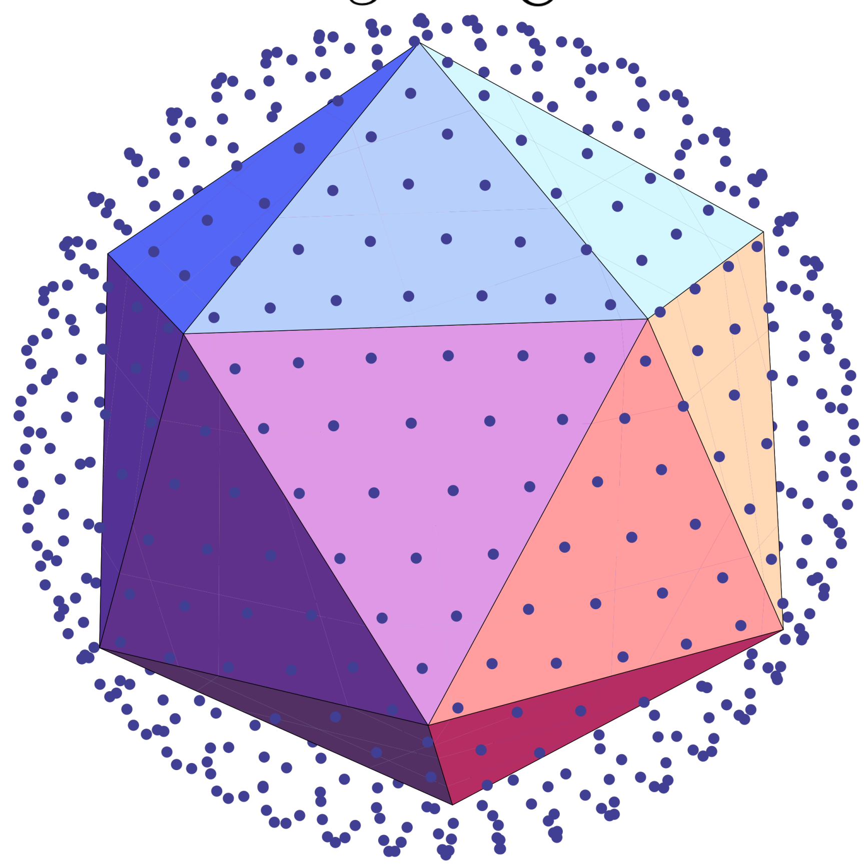
$+\infty$ $\quad \tau = \log r \quad$ $-\infty$

Order s Refined Triangulated Icosahedron

$s = 1$

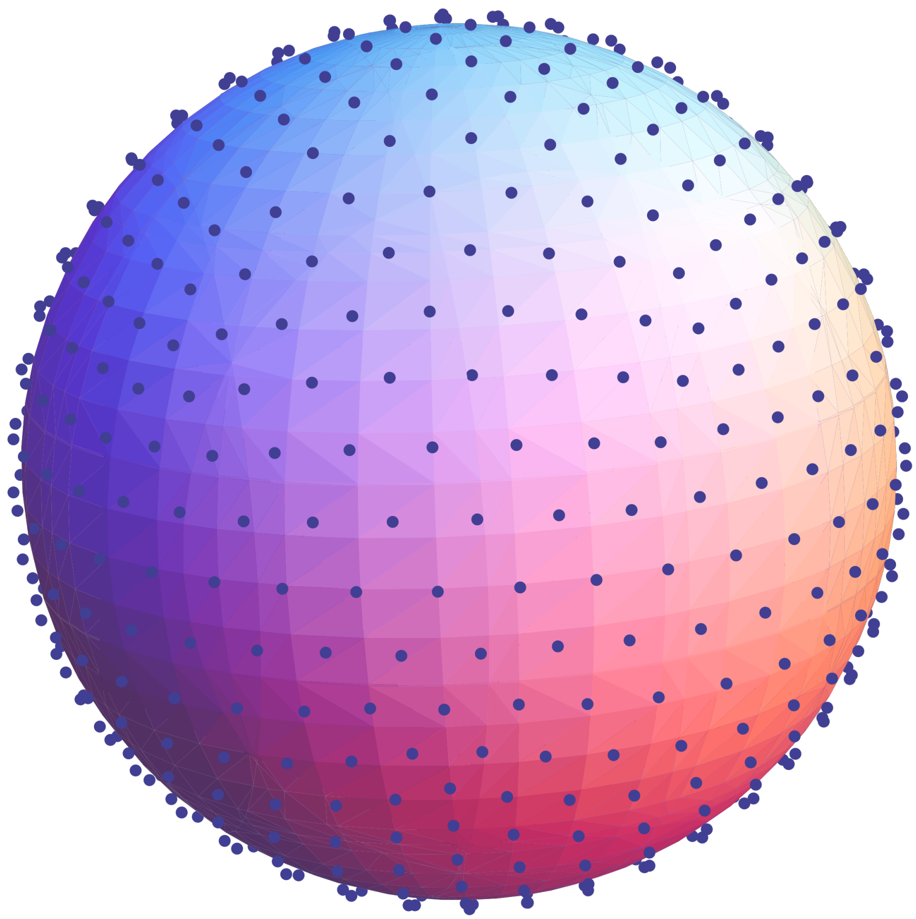


$s = 8$



Iscosahedral Symmetry Seduction is unique for l = 0, 1, 2

Fixed t lattice are s refined Icosahedrons



s = 8

vertices:
N = 10 + 2*s*s = 138

edges:
E = 3*N – 6

faces:
F = E – N + 2 = 2*N - 4

$$\text{Continuum limit is } s \to \infty \text{ at } \beta = \beta_{critical}$$

Primary operators 3-d Ising Model

Operator	Spin l	$\mathbb{Z}$	Δ	Exponent
s	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
s'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	$\Delta = 3$
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\mathrm{NR}}$

Low-lying primary operators of the 3D Ising model at criticality.

Primary l = 0

$$[K_\mu, \mathcal{O}(0)] = 0$$

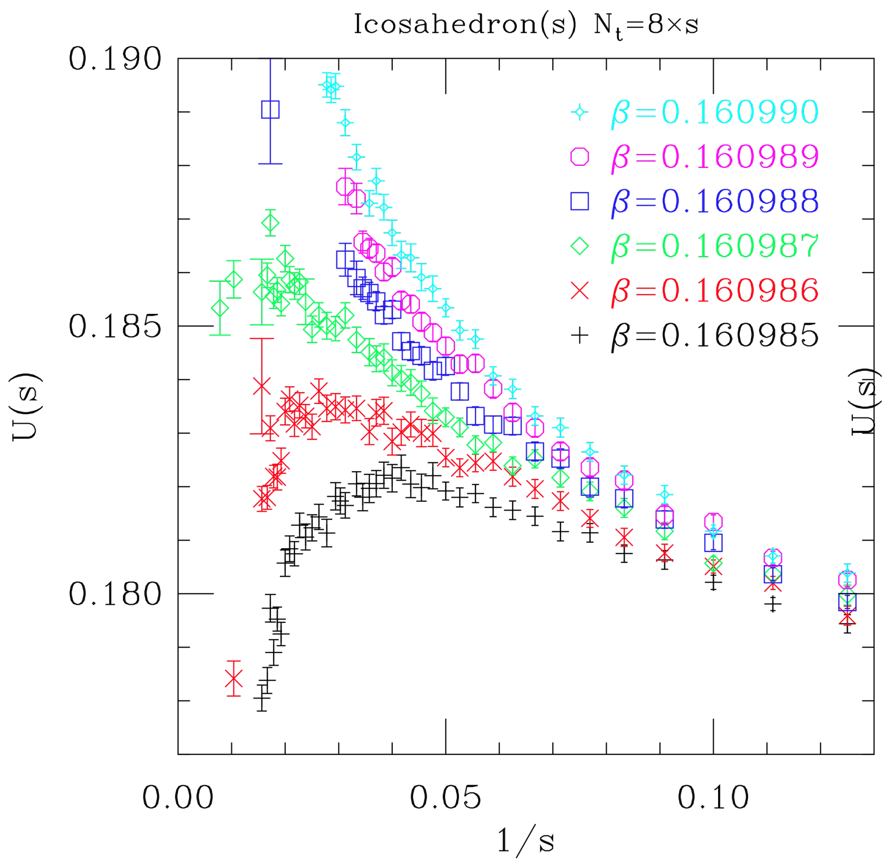
Descendants l > 0

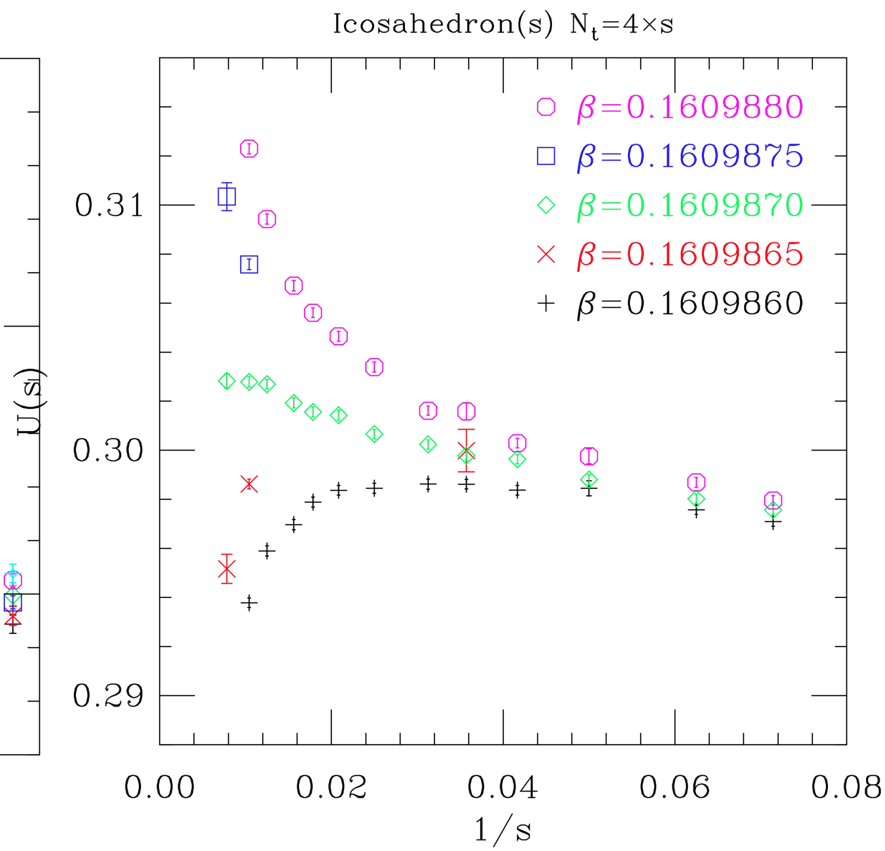
$$\mathcal{O}_{l+1}(x) = [P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}_l(x)$$

Preliminary Numerical Work

- Swendsen Wang & Wolff cluster algorithm
- Binder $$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$
  - Fixes: $$\beta_{crit} = 0.16098684(7)$$
- Fix asymmetry of lattice by descendants.
  - $$\Delta_l = \Delta_0 + l \quad \text{for} \quad l = 0, 1, 2, \cdots$$
- Rough values of 3 primaries : η , ν , ω
- Much more is feasible with modest effort

Determining beta_critical





$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

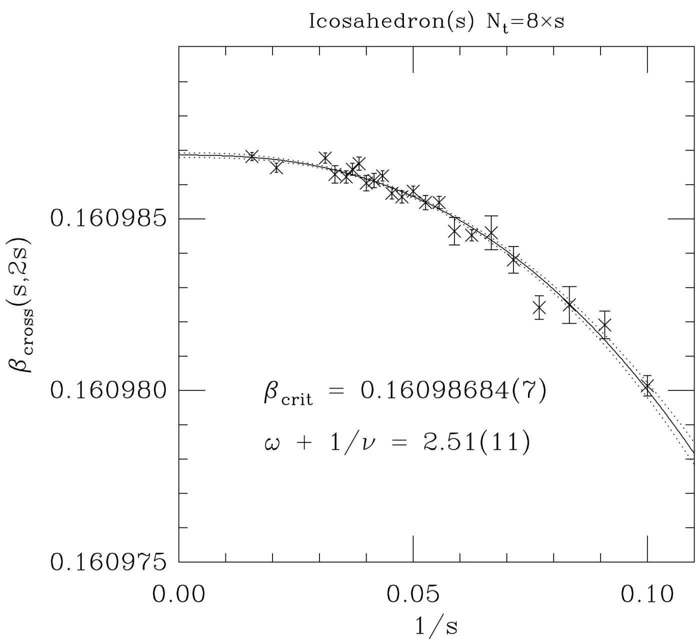
$$\beta_{crit} = 0.16098703(3)$$

Determining beta_critical

$$
\begin{aligned}
U_L(\beta) &= \widetilde{U}(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots \\
&= U^* + a_1(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots
\end{aligned}
$$

$$U(\beta, s=L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

$$\beta_{crit} = 0.16098703(3)$$



Observables

$$C_{lm}(t) = \sum_{t_0,x,y} Y^*_{lm}(\Omega_x)\langle s_{t+t_0,x}\ s_{t_0,y}\rangle Y_{lm}(\Omega_y)$$

cosh fit:

$$C_{lm}(t) = C[e^{-m_l t} + e^{-m_l(N_t-1-t)}]$$

for $\quad t = 0, \cdots, N_t - 1\ ,\ l = 0, 1, 2\ ,\ m = -l, \cdots, l$

$$m_l = \frac{c}{s}\Delta_l \qquad\qquad \Delta_l = \frac{1}{2} + \frac{\eta}{2} + l$$

After you adjust c = `speed of light` so $\Delta_{l+1} - \Delta_l = 1$

Improved cluster Estimator

Swendsen-Wang: Real space

$$g(x-y) = \langle s_x s_y \rangle \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \sum_{C_i} \Delta_{C_i}(x) \Delta_{C_i}(y)$$

$$\Delta_C(x) = \quad 1 \text{ if } x \in C \text{ else } 0$$

Wolff single cluster: momentum space

$$\widetilde{g}_{lm}(k) \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \frac{1}{|C|} \Big| \sum_{t,x \in C} e^{i2\pi kt/L_t} Y_{lm}(\Omega_x) \Big|^2$$

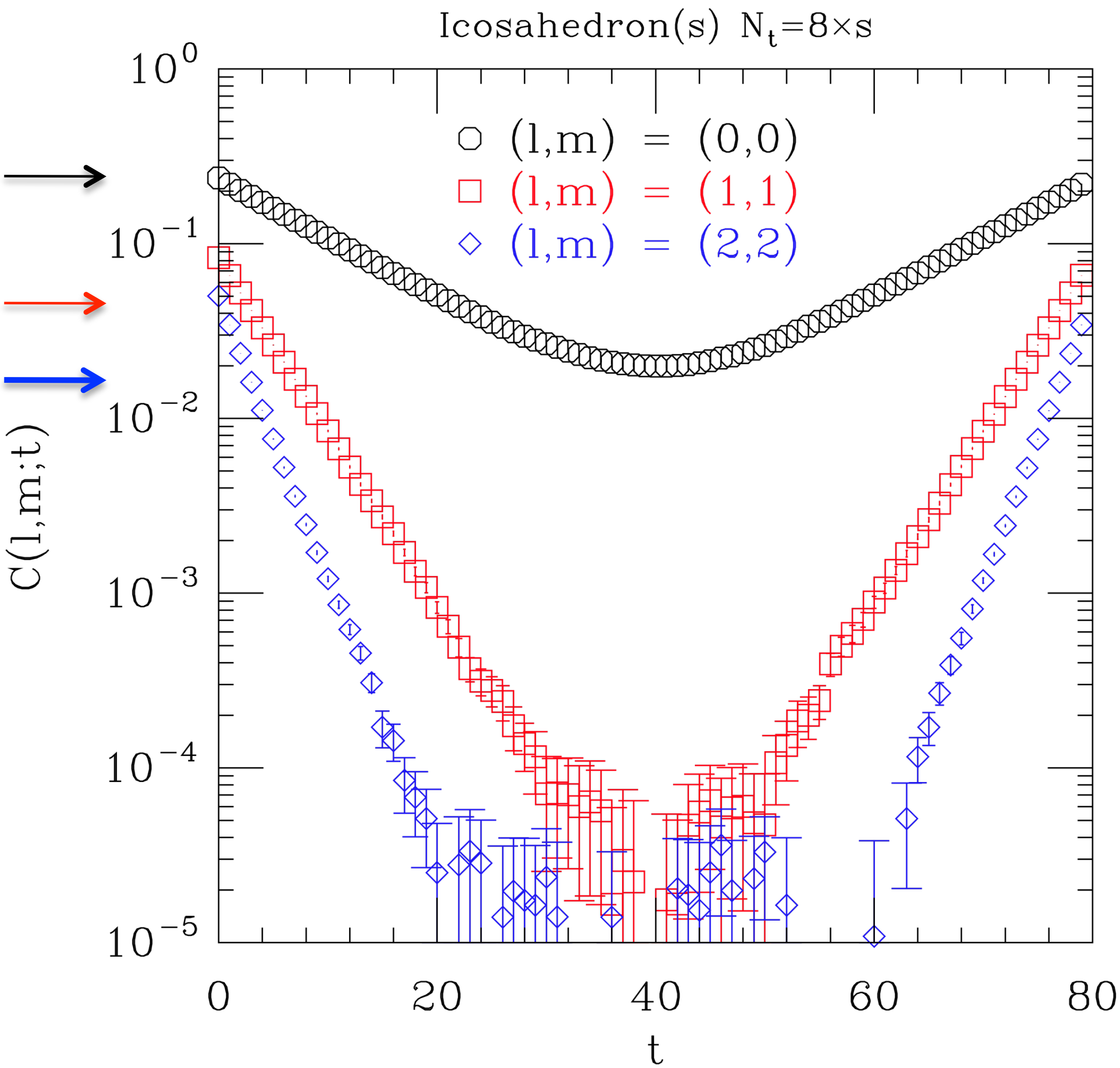
C. Ruge, P. Zhu and F. Wagner Physica A (1994) 431

Correlators for C(t)

$l = 0$

$l = 1$

$l = 2$



Checking the Descendant Spacing

β	s	N_t	Ratio	err
160987	10	80	0.987755	0.0125021
160990	6	48	0.979986	0.00143025
160990	9	72	0.989285	0.00180139
160990	12	96	0.995924	0.00147443
160990	15	120	0.9954	0.00147322
160995	6	48	0.975761	0.00140368
160995	9	72	0.990249	0.00145843
160995	12	96	0.990219	0.00143569
160995	15	120	0.996529	0.00144666

Ratio $(m_2 - m_1)/(m_1 - m_0) = 1$ corresponds to uniform descendant spacing.

Numerical Test (Preliminary)

- Equal spacing (s=15) test of descendants:

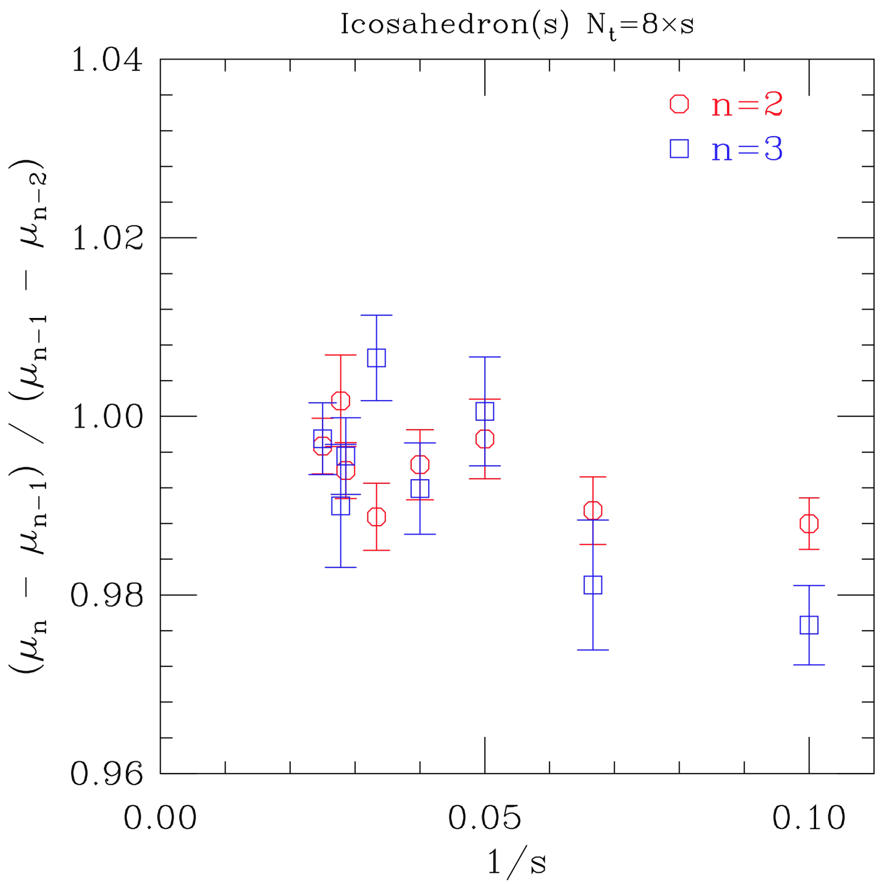
$$\frac{m_2 - m_1}{m_1 - m_0} = 0.996(1) \qquad \text{at} \quad \beta = 0.160995$$

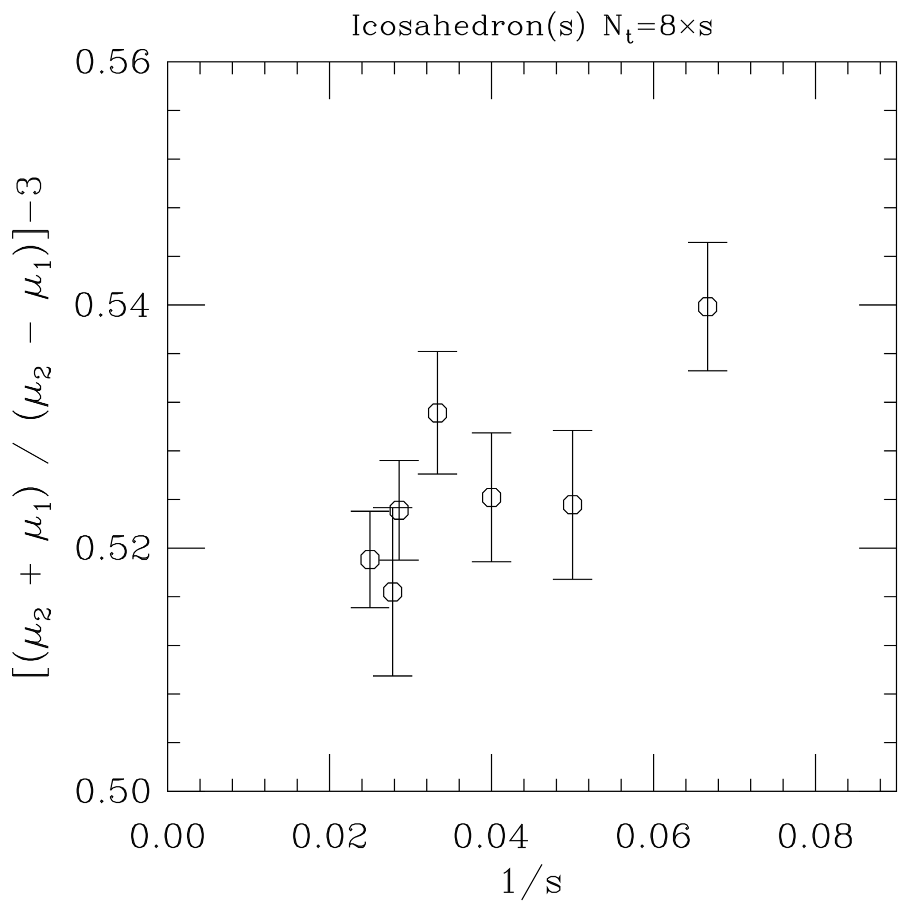
- "Speed of light"(s=8) $c = 1.486(5) \quad \text{at} \quad \beta = 0.160987$

- But critical point $\beta_{crit} = 0.16098703(3)$

- Very Rough anomalous dimensions (more soon)
  - from Binder: $\omega + 1/\nu = 2.51(11)$
  - from corr: $\Delta_\sigma = 1/2 + \eta/2 = 0.51(1)$
  - Self consistent simulations are just starting!

Hot off the Press





Some Future Directions

- Restoration of full Conformal O(d+1,1) as 1/s → 0
  - Lattice approximates only the isometries of $\mathbb{R} \times \mathbb{S}^{d-1}$
  - Check 2-pt correlator for full conformal symmetry.
  - Check 3-point and 4-point functions as well?
  - Strengthen bootstrap inequalities for 3-d Ising

- Other applications
  - O(N) model et al in 3-d
  - Add mass deformation
  - Fermions on $\mathbb{R} \times \mathbb{S}^{d-1}$ (Dirac with Vierbein)
  - Study conformal IR fixed points in Technicolor

Can you add Running coupling and mass deformations?

Callan-Symanzik Equation

$$\left[r\frac{\partial}{\partial r} + \beta(g)\frac{\partial}{\partial g} - (1+\gamma(g))m\frac{\partial}{\partial m} + 2\Delta(g) \right] C(r, \Omega, g, m, \mu) = 0$$

$$\beta(g) \simeq \omega(g^* - g)$$