Thanksgiving Day Anyons

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ALF, Kachru, Kaplan, Katz, Wacker

Outline

1) Anyons Review

2) EFT and the Fermi Surface

3) Coupling a Fermion to a massless z=2 ($\omega \sim k^2$) scalar

4) A Semi-classical Theory of Anyons

5) A Quantum Theory of Anyons

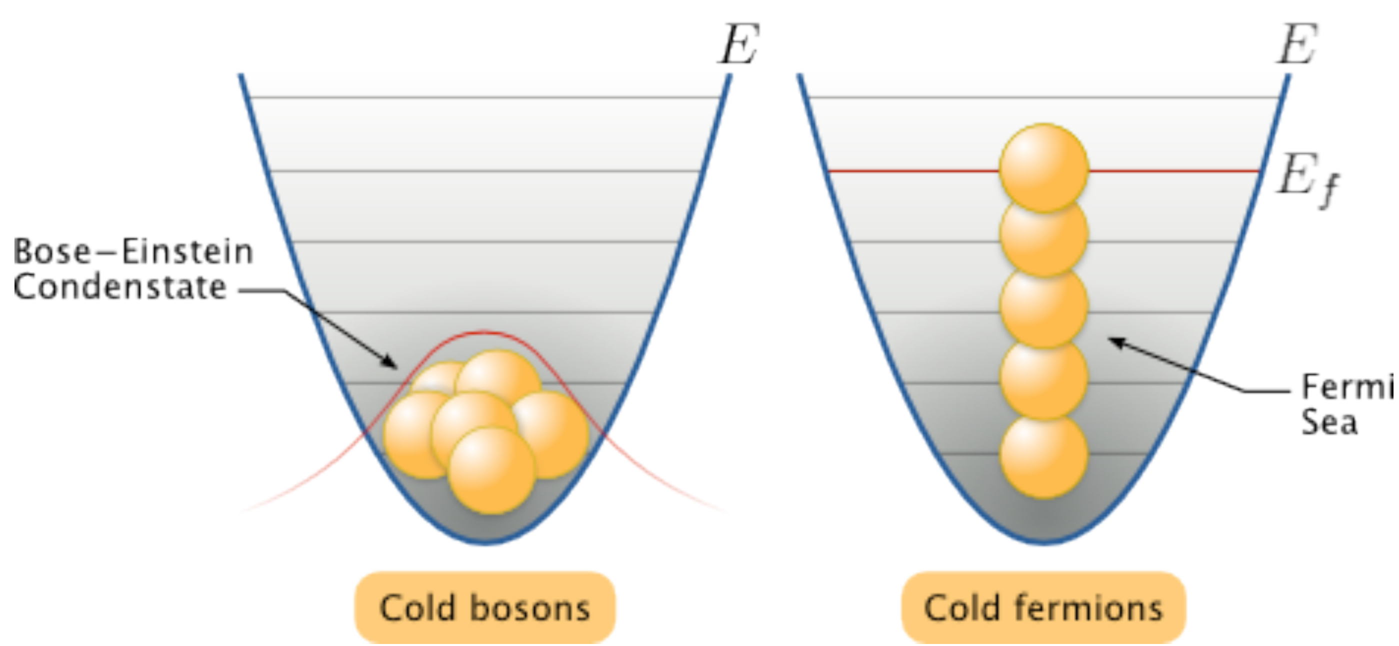
6) Relation to Lifshitz-Chern-Simons theory

7) Conclusions

Statistics of Particles

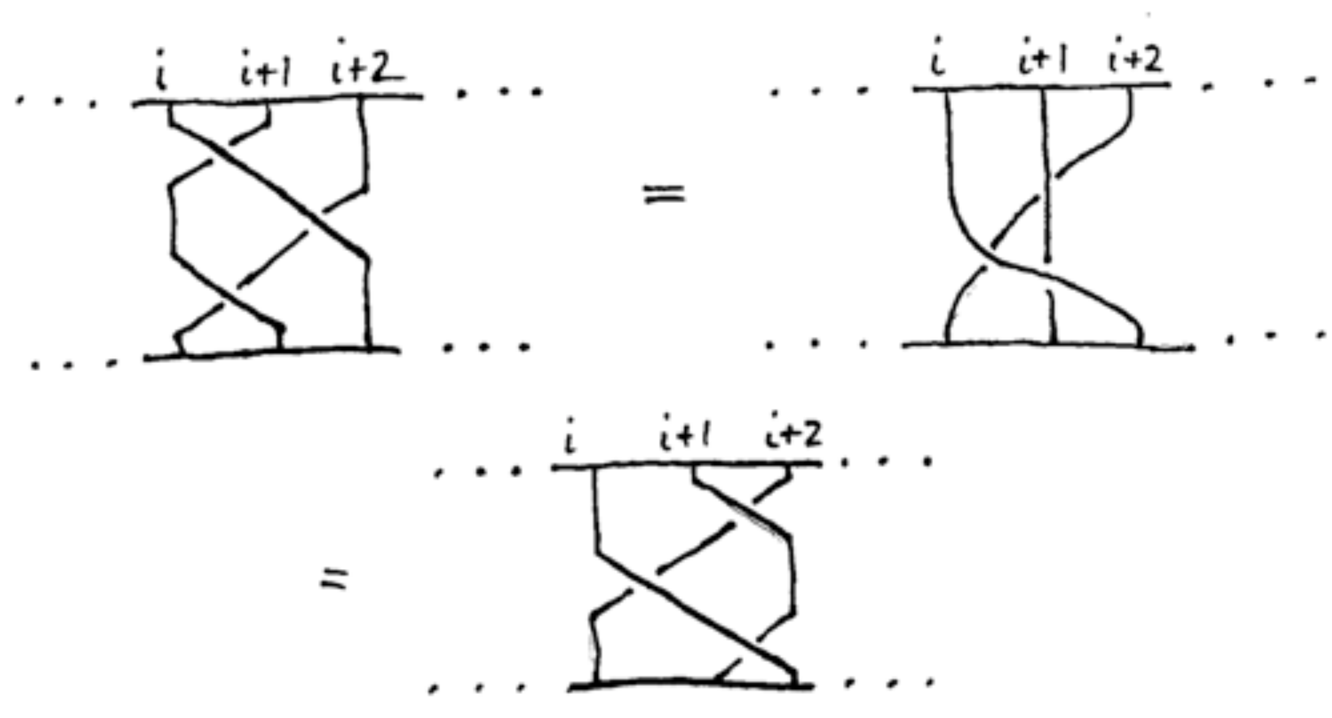
The statistics of particles is defined by the change in their wavefunction when we move them around each other

Usually, this just generates the permutation group: represented by + or -



Statistics of Particles

In 2+1 dimensions, particle interchange generates the Braid Group, with richer structure



$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Leinaas, Myrheim, 1977

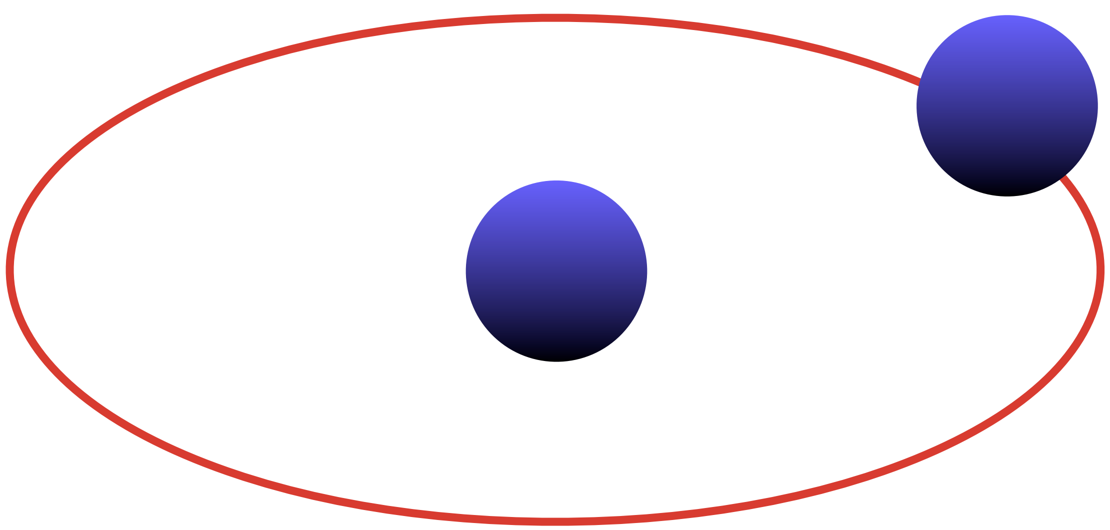
Anyons

This is related to non-trivial phases that can be generated when one particle encircles another

$$\psi(r_1, r_2) \rightarrow e^{i\vartheta_a}\psi(r_1, r_2)$$

Boson

$\vartheta_a = 0$



Fermion

$\vartheta_a = \pi$

Anyons in Field Theory

Anyons have a well-known elegant implementation in QFT through Chern-Simons theory

$$\mathcal{L} = kA \wedge dA + A^\mu j_\mu$$

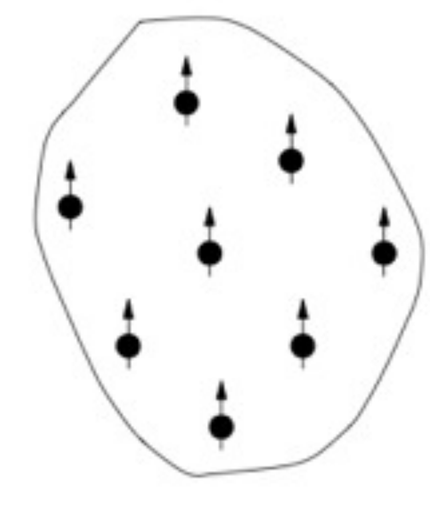
Density from a particle

$$j^0 = \psi^\dagger \psi$$

$$\Rightarrow$$

Particles carry Flux!

$$B = \frac{1}{k} \psi^\dagger \psi$$



Anyons in Field Theory

$$B = \frac{1}{k}\psi^\dagger\psi$$

The result is that after particle interchange, there is an anyonic phase (think of Aharonov-Bohm):

$$\vartheta_a = \frac{\pi}{k}$$

Relevant for fractional quantum hall effect

Wilzek, Zee

Another Theory of Anyons?

The standard Chern-Simons description is very elegant, but it is not clear that it is the only way to make a theory of anyons. Can we build others?

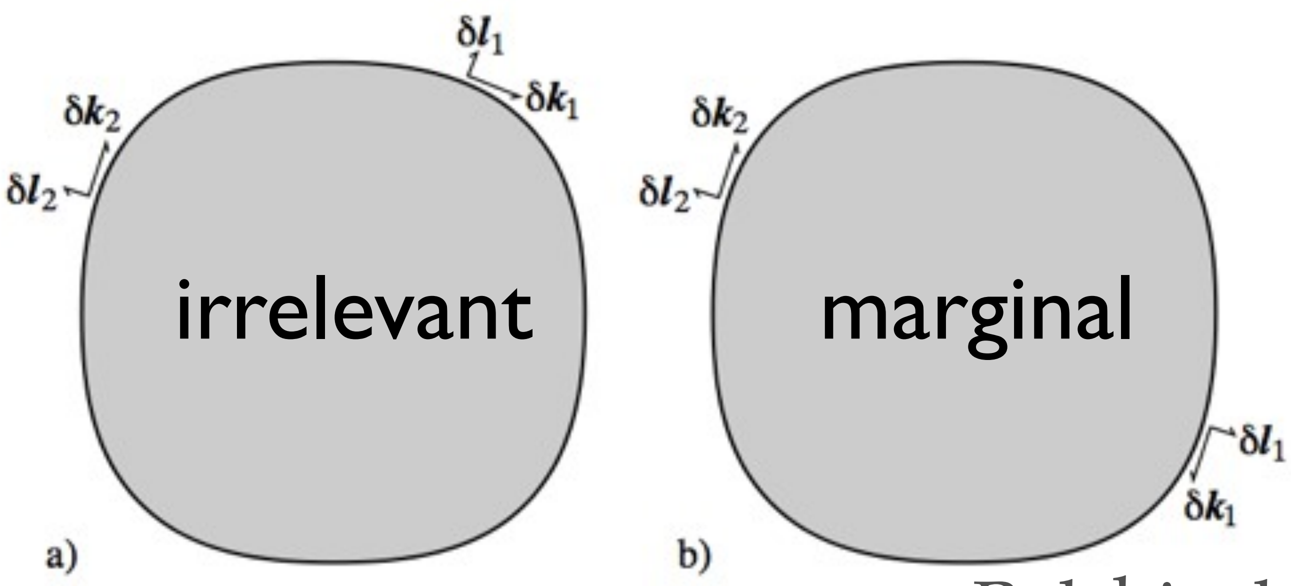
Controllable IR Modifications

One way of understanding new phases of materials is as the modified IR behavior of a Fermi surface due to interactions

In the IR, normal Fermi liquid theory can break down when the interaction becomes strong

e.g.

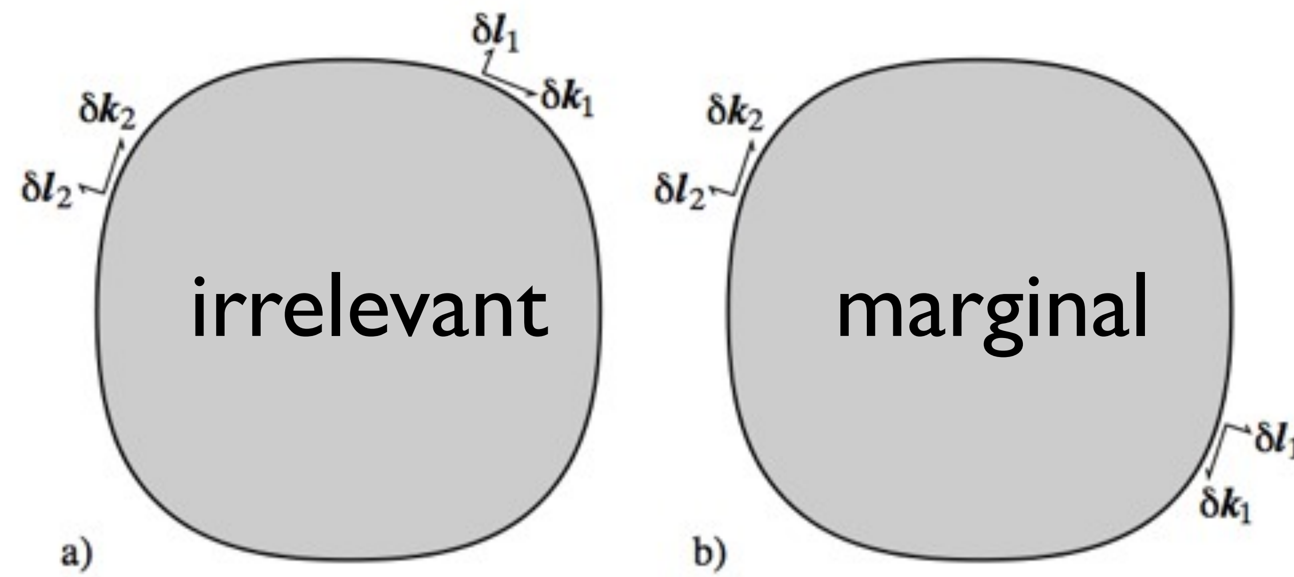
$$\mathcal{L} \supset \psi^\dagger_\sigma \psi_\sigma \psi^\dagger_{\sigma'} \psi_{\sigma'}$$



Polchinski '92

Controllable IR Modifications

$$\mathcal{L} \supset \psi^\dagger_\sigma \psi_\sigma \psi^\dagger_{\sigma'} \psi_{\sigma'}$$



$$\delta^d(\underbrace{\delta k_3 + \delta k_4}_{\sim s^0} + \underbrace{\delta \ell_3 + \delta \ell_4}_{\sim s^1})$$

antipodal points have a marginal interaction

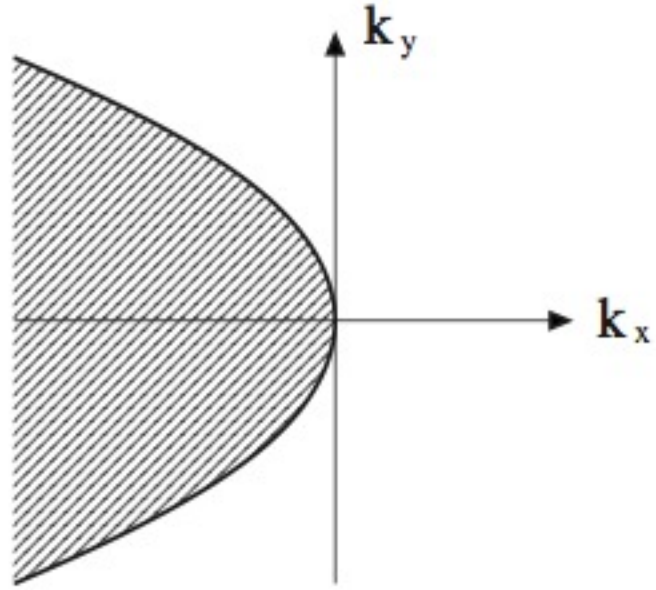
Polchinski '92

Fermi Surface and EFT

One of the simplest things one can try is coupling the Fermi surface to a massless boson through a Yukawa

$$\mathcal{L} = \psi_j^\dagger(\partial_t - i\partial_x - \partial_y^2)\psi_j + \frac{e}{\sqrt{N}}\phi\psi_j^\dagger\psi_j + (\partial\phi)^2$$

Fermi Surface:



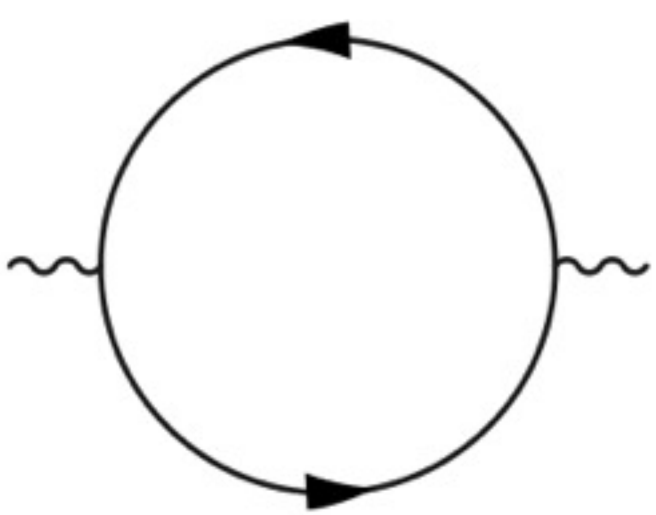
Studied by Sung-Sik Lee, Millis, Metlitski and Sachdev

Fermi Surface and EFT

But not controlled in IR even at large N!

$$\mathcal{L} = \psi_j^\dagger(\partial_t - i\partial_x - \partial_y^2)\psi_j + \frac{e}{\sqrt{N}}\phi\psi_j^\dagger\psi_j \quad +(\partial\phi)^2$$

One-loop renormalization changes ϕ scaling



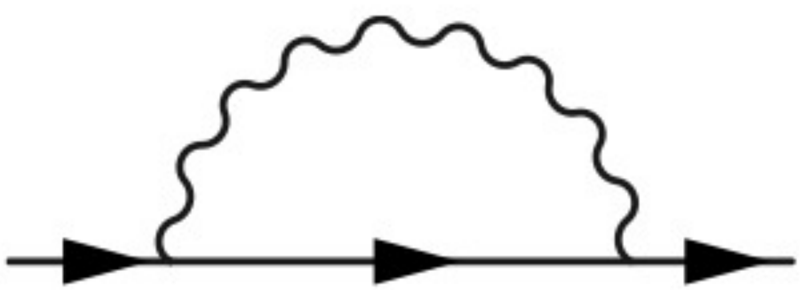
$$\sim \frac{|k_0|}{|k_y|} + k_0^2 + k_x^2 + k_y^2$$

S.S. Lee, '09

Fermi Surface and EFT

But not controlled in IR even at large N!

Substituting back into fermion 1-loop propagator gives bizarre ψ scaling



$$\sim \frac{i}{N}\mathrm{sgn}(k_0)|k_0|^{2/3} + ik_0 + k_x + k_y^2$$

S.S. Lee, '09

As shown by Lee, and Metlitski and Sachdev, the Large N expansion breaks down (due to IR divergences in 1/N subleading terms)

Fermi Surface and Scalars

General goal: explore systems of bosons coupled to fermions with only marginal interactions so they are calculable over a wide of scales.
Look for interesting IR phases.

Simple boson generalization: z=2

$$\mathcal{L} \supset \dot{\phi}^2 - \kappa(\nabla^2\phi)^2$$

Arises in physical critical points where $(\nabla\phi)^2$ term has essentially be tuned to zero

Coupling to a Gapless Scalar

For simplicity, we will couple this to a free non-relativistic fermion (not to a Fermi surface).

$$\mathcal{L} \supset -\psi^\dagger i\partial_t \psi - \gamma|\vec{\nabla}\psi|^2$$

How shall we couple these? The z=2 scalar has a large symmetry group

$$\mathcal{L} \supset \dot{\phi}^2 - \kappa(\nabla^2\phi)^2$$

$$\phi \to \phi + ct$$

$$\phi \to \phi - v(x,y)$$

for *any* harmonic function $\nabla^2 v = 0$

Fermi Surface and EFT

What interaction can we add that preserves these symmetries?

$$\phi \to \phi + ct$$
$$\phi \to \phi - v(x,y) \qquad \nabla^2 v = 0$$

We can add

$$g(\nabla^2 \phi)\psi^\dagger \psi$$

Fermi Surface and EFT

$$\phi \to \phi + ct$$
$$\phi \to \phi - v(x,y) \qquad \nabla^2 v = 0$$

Make ψ transform like

$$\psi \to e^{i\alpha u(x,y)}\psi \quad \text{with} \quad \nabla_i u = \epsilon_{ij}\nabla^j v$$

Define Covariant derivative

$$\mathcal{D}^i = \nabla^i + i\alpha\epsilon^{ij}\nabla_j\phi$$
$$\mathcal{D}^i\psi \to e^{i\alpha u(x,y)}\mathcal{D}^i\psi$$

So we can add $\quad -\gamma|\mathcal{D}\psi|^2$

Scaling

$$\mathcal{L} \supset \gamma|(\nabla_i + i\alpha\epsilon_{ij}\nabla^j\phi)\psi|^2 + g(\nabla^2\phi)\psi^\dagger\psi$$

This theory is (classically) scale-invariant under the following scaling

$$dt \to s^{-1}dt, \quad dx \to s^{-\frac{1}{2}}dx, \quad \phi \to s^0\phi, \quad \psi \to s^{\frac{1}{2}}\psi$$

All couplings are marginal, so weakly coupled theory remains under control over a wide range of scales

Symmetry

$$\mathcal{L} \supset \gamma|(\nabla_i + i\alpha\epsilon_{ij}\nabla^j\phi)\psi|^2 + g(\nabla^2\phi)\psi^\dagger\psi$$

We can group the space-dependent shift boson and fermion shift symmetry together:

$$f(x,y) = u(x,y) + iv(x,y)$$

$$f(z = x + iy) \leftrightarrow \nabla_i u = \epsilon_{ij}\nabla^j v$$

"Cauchy-Riemann (gauge) Symmetry"

Semi-classical Anyons

The theory has the global symmetry:

$$\psi \to e^{i\alpha u}\psi \qquad \text{constant } u$$

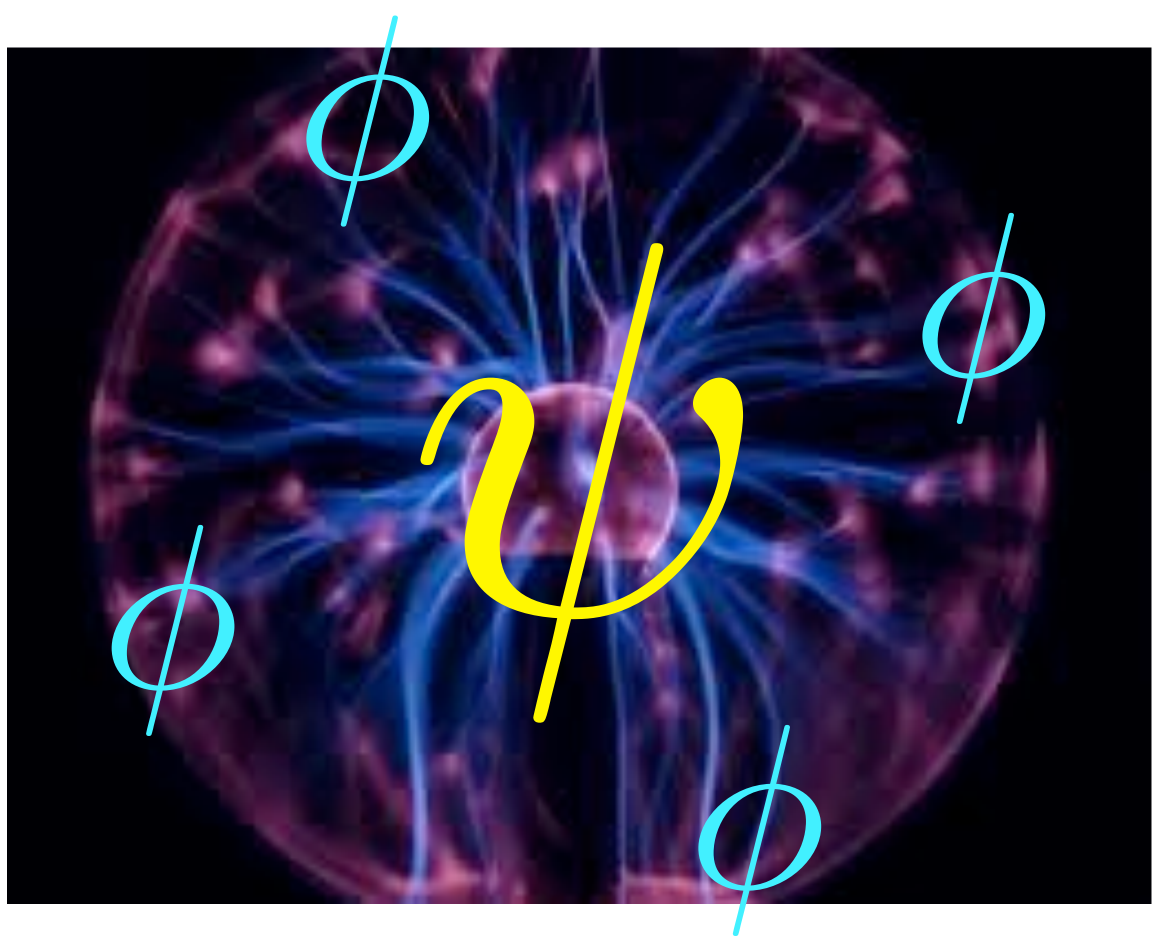
ψ number $\qquad$ ψ momentum

Noether current: $J_N^0 = \psi^\dagger \psi \qquad J_N^i = -i\gamma\psi^\dagger \overleftrightarrow{D}^i \psi$

Semi-classical Anyons

Each ψ will source the ϕ field

$$\ddot{\phi} + \nabla^4 \phi = g \nabla^2 J_N^0 - \alpha \epsilon_{ij} \nabla^i J_N^j$$



Semi-Classical Anyons

Take a static density of ψ

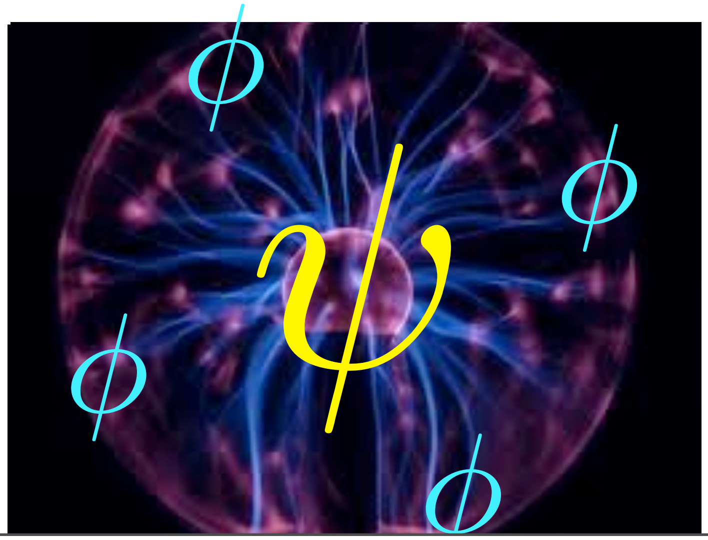
$$\rho = \psi^\dagger \psi = J_N^0$$

Each ψ will source the ϕ field

$$\nabla^2 \phi(x) = g\rho(x)$$

ψ particles produce a long range potential

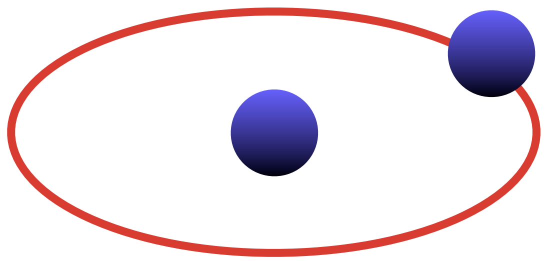
$$\phi(\vec{x}) = \frac{g}{2\pi} \log |\vec{x}|$$



Semi-Classical Anyons

Now, let's look at the phase as we move one ψ around another

The ψ one-particle action contains



$$iS_\psi \supset -i\alpha \int d\tau \left(\vec{v} \times \vec{\nabla}\phi\right)$$

Moving a ψ in a loop generates the phase

$$i\vartheta_a = -i\alpha \oint_{\partial M} d\theta(\hat{n} \times \vec{\nabla}\phi)$$

Semi-Classical Anyons

$$i\vartheta_a = -i\alpha \oint_{\partial M} d\theta(\hat{n} \times \vec{\nabla}\phi)$$

Using Stokes Theorem and the equations of motion, we find that if one ψ encircles another, it picks up an anyonic phase

$$\vartheta_a \quad = \alpha \int_M d^2x \nabla^2\phi = g\alpha \int_M d^2x \rho(x)$$

$$= g\alpha$$ Anyons!

Quantum Anyons

The phase is generated through interaction with a dynamical, massless boson

One might worry that the quantum effects of this degree of freedom significantly change the physics

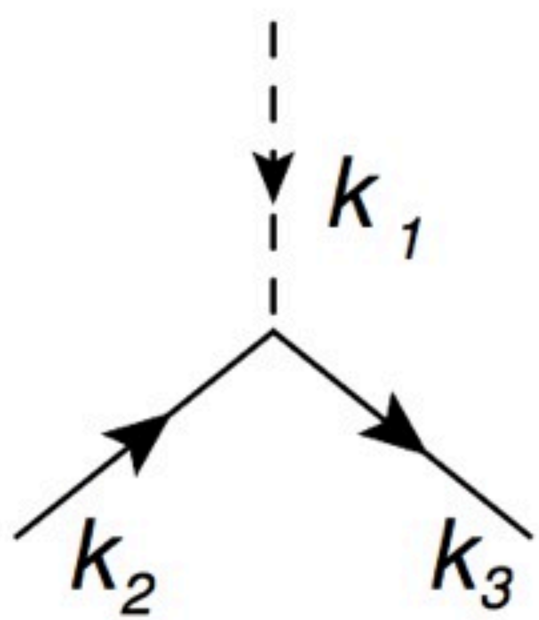
In particular, how do the marginal couplings (and in turn, ϑ_a) run in this theory?

RG Running

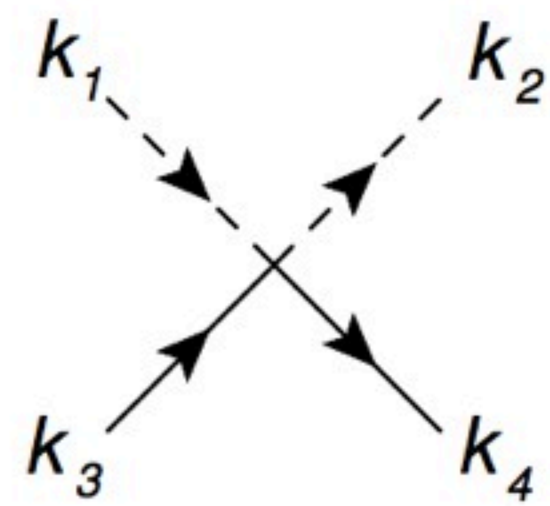
Feynman rules are simple:

$$= \frac{1}{\omega^2 + k^4}$$

$$= \frac{-1}{i\omega - \gamma k^2}$$



$$= -2i\gamma\alpha\epsilon_{ij}k_2^i k_3^j - g k_1^2$$



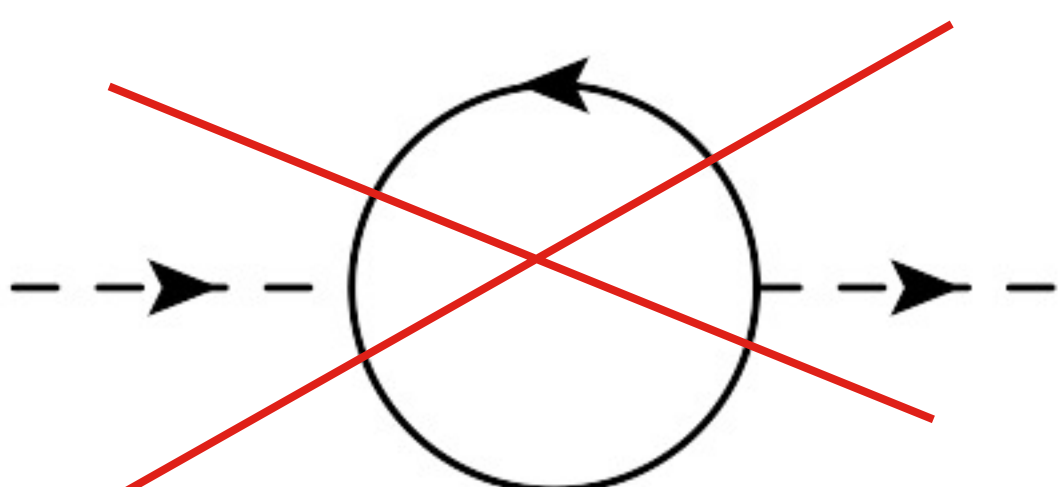
$$= -2\gamma\alpha^2 k_1 \cdot k_2$$

RG Running

The RG is greatly simplified by the fact that the fermion is non-relativistic:

$$= \frac{-1}{i\omega - \gamma k^2}$$

single pole in ω - no anti-particles!



Consequently - no ϕ propagator renormalization

RG Running

$$\mathcal{L} \supset \gamma |(\nabla_i + i\alpha\epsilon_{ij}\nabla^j\phi)\psi|^2 + g(\nabla^2\phi)\psi^\dagger\psi$$

The "Cauchy-Riemann" symmetry protects the gauge coupling α

(Since ϕ has no wavefunction renormalization)

$$\beta_\alpha = 0$$

RG Running

$$\mathcal{L} \supset \gamma |(\nabla_i + i\alpha\epsilon_{ij}\nabla^j\phi)\psi|^2 + g(\nabla^2\phi)\psi^\dagger\psi$$

Furthermore, g is protected by combination of symmetry and dynamics:

Global symmetry: $\phi \to \phi - v$ constant v

Noether current: $J^i_\phi = (-\nabla^i\nabla^2\phi - \alpha\epsilon^{ij}J^j_N + g\nabla^i J^0_N)$

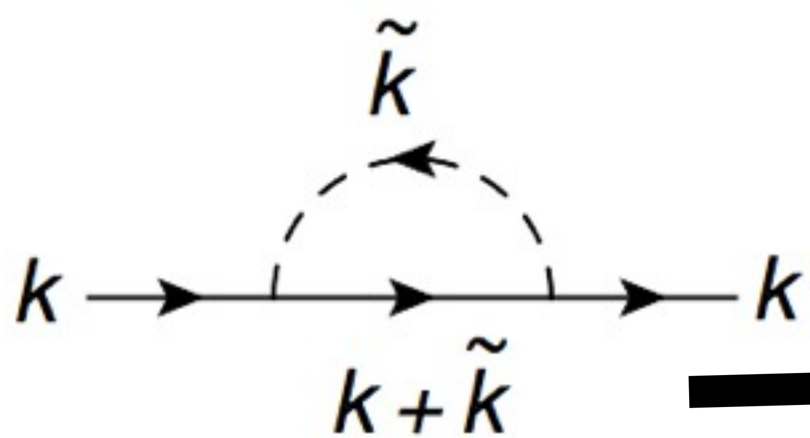
So $$g(\underbrace{\nabla^i J^0_N}_{\text{protected}}) = \underbrace{\nabla^i\nabla^2}_{\text{no w.f. renorm}}\phi + \underbrace{J^i_\phi}_{\text{protected}} + \underbrace{\alpha\epsilon^{ij}J_{jN}}_{\text{protected}}$$

RG Running

Thus g and α do not run! (We also checked explicitly that the 1-loop beta functions vanish)

$\vartheta_a = g\alpha$ so phase is RG-invariant!

However, the "mass" $\gamma = \dfrac{1}{2m_\psi}$ term runs



$$\longrightarrow \beta_\gamma > 0$$

so mass runs to be heavy in IR

Relation to Lifshitz-Chern-Simon Theory

Our theory has a massless, dynamical scalar mode, and cannot be related to Chern-Simons theory

However, it *can* be related to a critical point of z=2 Lifshitz-Chern-Simons Theory via a non-local map

$$\mathcal{L}_{\mathrm{Lif-CS}} = -\psi^\dagger i D_t \psi - \Gamma |D_i \psi|^2 + \kappa A \wedge dA$$

$$+E^i(\dot{A}_i - \nabla_i A_0) - c^2 k_0^2 E^2 - \frac{c^2}{2}(\nabla_i E_j)^2 - \frac{f^2}{2}(\nabla \times A)^2$$

Relation to Lifshitz-Chern-Simon Theory

$$\mathcal{L}_{\mathrm{Lif-CS}} \supset -c^2 k_0^2 E^2$$

$c^2 k_0^2$ determines phase of theory

$c^2 k_0^2 > 0$ theory flows in IR to Maxwell C-S

$c^2 k_0^2 < 0$ anisotropic phase $\langle E_i \rangle \neq 0$

$c^2 k_0^2 = 0$ our anyons

Relation to Lifshitz-Chern-Simon Theory

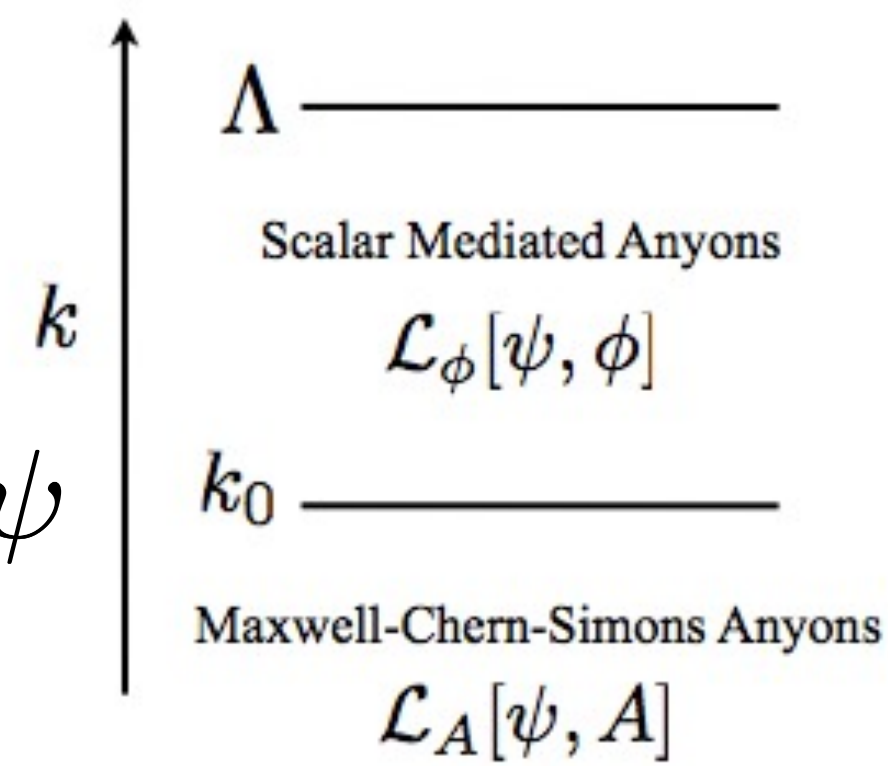
$$\mathcal{L}_{\mathrm{Lif-CS}} \supset -c^2 k_0^2 E^2 = 0$$ our anyons

Use equations of motion to perform non-local "Mulligan duality"

$$E_i = E_i[\phi, \psi; \nabla^{-2}] \quad A_i = \alpha \epsilon_{ij} \nabla_j \phi$$

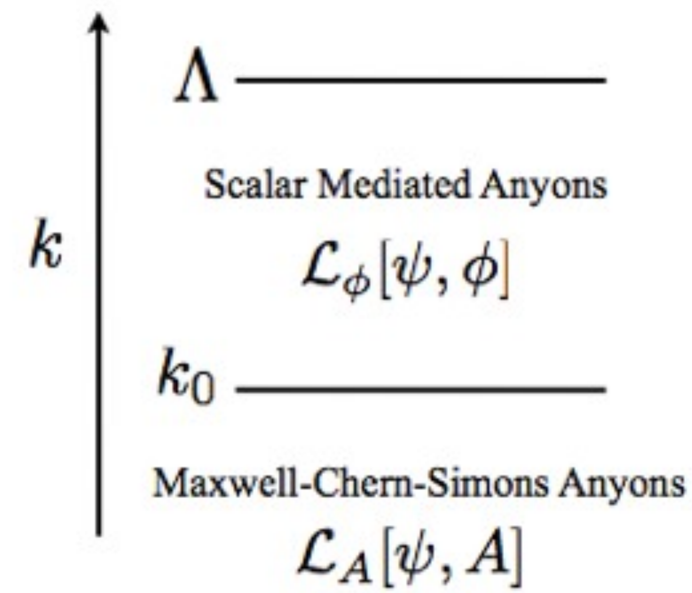
$$A_0 = A_0[\phi, \psi; \nabla^{-2}]$$

$$\lim_{k_0^2 \to 0} \mathcal{L}_{\mathrm{Lif-CS}} = \mathcal{L}_{\phi,\psi}$$



Relation to Lifshitz-Chern-Simon Theory

1) Scalar description break down in IR if we deform away from $k_0 = 0$



$$\mathcal{L}_{\rm non-critical} \sim \frac{\nabla^2}{\nabla^2 - k_0^2}\dot{\phi}^2$$

2) Phase is not given by Lif-Chern-Simons level κ

3) Local operators in Lif-CS description are non-local in ϕ description and vice versa

Conclusions

Does this scalar mode exist physically?

Some naive comments: ϕ has the symmetries

$$\phi \to \phi + c \qquad \text{(~translations)}$$

$$\phi \to \phi + vt \qquad \text{(~boosts)}$$

ϕ could be a height field of a membrane

$R\psi^\dagger\psi \sim (\nabla^2\phi)\psi^\dagger\psi$ would occur naturally

$$\mathcal{L} = \dot{\phi}^2 - \underbrace{T(\nabla\phi)^2}_{\text{tension}} - \underbrace{\kappa(\nabla^2\phi)^2}_{\text{extrinsic curvature}}$$

Conclusions

What's the ultimate goal here?

Once you discard Lorentz invariance, many new possibilities for IR behavior and phase transitions emerge

We studied a simple theory with a massless z=2 scalar coupled through marginal interactions

A systematic exploration of fermions coupled to massless modes in a controllable way is well-motivated and hopefully there are many interesting new phenomena waiting to be discovered

The End