Boundary Current Algebra and Multiparticle HS Symmetry

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Higher Spins, Strings and Holography

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Plan

I Higher-spin algebra

II Free fields and currents

III Twistor current operator algebra as multiparticle symmetry

IV Multiparticle symmetry as a string-like HS symmetry

V Butterfly formulae for n-point functions

VI Conclusion

Higher-Spin Theory versus String Theory

HS theories: $\Lambda \neq 0$, $m = 0$

symmetric fields $s = 0, 1, 2, \dots \infty$

String Theory: $\Lambda = 0$, $m \neq 0$ **except for a few zero modes**

mixed symmetry fields $\vec{s} = 0, 1, 2, \dots \infty$

String theory has much larger spectrum:

HS Theory: first Regge trajectory

Pattern of HS gauge theory is determined by HS symmetry

What is a string-like extension of a global HS symmetry underlying a string-like extension of HS theory?

Global Higher-Spin Symmetry

HS symmetry in AdS_{d+1}:

Maximal symmetry of a d-dimensional free conformal field(s)=singletons

usually, scalar (Rac) and/or spinor (Di)

Admissibility condition: a set of fields resulting from gauging a global HS symmetry should match some its unitary representation.

Example: SUSY algebra admits a UIRREP $(2, N \times 3/2, \frac{1}{2}N(N-1) \times 1, \ldots)$

There should be a HS-module containing the AdS_{d+1} module associated with gravity: $D(2, E_0(2))$

$D(s, E_0(s))$ is a massless module of spin s. $E_0(s)$ for $s \geq 1$ is the boundary of the unitarity region

Oscillator realization

$$P^a = P^a_{AB}\{Y^A, Y^B\}, \qquad M^{ab} = M^{ab}_{AB}\{Y^A, Y^B\}, \qquad [Y_A, Y_B] = C_{AB}$$

Tensoring modules: $Y^A \to Y^A_i$, $[Y^A_i, Y^B_j] = \delta_{ij} C^{AB}$, $i, j = 1, \dots N$

$$P^a = P^a_{AB} \sum_i \{Y^A_i, Y^B_i\}, \qquad M^{ab} = M^{ab}_{AB} \sum_i \{Y^A_i, Y^B_i\}$$

If $|E_0(2)\rangle$ vacuum was a Fock vacuum for Y^A E_0 increases as NE_0.

If there was gravity at $N = 1$: no gravity at $N > 1$.

Incompatibility of AdS extension of Minkowski first quantized string

$$M^{ab} = \sum_{n \neq 0} \frac{1}{n} x^{[a}_{-n} x^{b]}_n + p^{[a} x^{b]}, \qquad P^a = p^a$$

since $[P^a, P^b] = -\lambda^2 M^{ab}$ implies that P^a should involve all modes and hence lead to the infinite vacuum energy: no graviton

What is a symmetry that is able to unify HS gauge theory with String?

Current operator algebra

$3d$ conformal equations and HS symmetry

Conformal invariant massless equations in $d=3$

$$(\frac{\partial}{\partial x^{\alpha\beta}}\pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta})C^\pm_j(y|x)=0\,,\qquad \alpha,\beta=1,2\,,\quad j=1,\ldots\mathcal{N}$$ Shaynkman, MV (2001)

Generalization to matrix space: $\alpha,\beta=1,2,\ldots M$.

Bosons and fermions are even (odd) functions of y: $C_i(-y|x)=(-1)^{p_i}C_i(y|x)$

"Classical" field

$$\Phi_j(y|x)=C^+_j(y|x)+i^{p_j}C^-_j(iy|x)\,,\qquad \overline{\Phi}_j(y|x)=C^-_j(y|x)+i^{p_j}C^+_j(iy|x)$$

$$\left(\frac{\partial}{\partial x^{\alpha\beta}}+i\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\right)\Phi_j(y|x)=0\,,\qquad \left(\frac{\partial}{\partial x^{\alpha\beta}}-i\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\right)\overline{\Phi}_j(y|x)=0$$

Initial data: $C^\pm_j(y|0)$: **Maximal symmetry: all operators on the space of functions of** y.

$$A(Y^A)\;:\quad Y^A=(y^\alpha,\frac{\partial}{\partial y^\beta})\quad A=1,2,3,4\,,\qquad [Y^A,Y^B]=C^{AB}\,.$$

Algebra of oscillators: $3d$ **conformal HS algebra**$=AdS_4$ **HS algebra**

$sp(4)$ **subalgebra is spanned by bilinears** $T^{AB}=\{Y^A,Y^B\}\,.$

Currents

Rank-two equations: conserved currents

$$\left\{ \frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}} \right\} J(u,\, y|x) = 0$$

$J(u,\, y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$J(u,\, y\,|x) = \sum_{i=1}^{\mathcal{N}} \overline{\Phi}_i(u+y|x)\,\Phi_i(y-u|x)$$

Rank-two fields: bilocal fields in the twistor space.

Primaries: $3d$ **currents of all integer and half-integer spins**

$$J(u,0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} J_{\alpha_1\dots\alpha_{2s}}(x)\,, \qquad \tilde{J}(0,y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1}\dots y^{\alpha_{2s}} \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x)$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1\dots\alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Differential equations: conservation condition

$$\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u,0|x) = 0\,, \qquad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} \tilde{J}(0,y|x) = 0$$

D-functions

Unfolded dynamics leads to quantization:

Particles and antiparticles: definite frequencies

$$C^{\pm}(y|x) = (2\pi)^{-M/2} \int d\xi^M c^{\pm}(\xi) \exp \pm i[\xi_\alpha \xi_\beta x^{\alpha\beta} + y^\alpha \xi_\alpha]$$

Time: $x^{\alpha\beta} = tT^{\alpha\beta}$ with a positive definite $T^{\alpha\beta}$.

Solutions with $c^{\pm}(\xi) = const$

$$\mathcal{D}^{\pm}(y|x) = \mp i(2\pi)^{-M} \int d\xi^M \exp \pm i[\xi_\alpha \xi_\beta x^{\alpha\beta} + y^\alpha \xi_\alpha]\,.$$

$$\mathcal{D}^{\pm}(y|x) = \mathcal{D}^{\pm}(x) \exp[-\frac{i}{4} x^{-1}_{\alpha\beta} y^\alpha y^\beta]$$

$$\mathcal{D}^{\pm}(x) = \pm \frac{i}{2^M \pi^{M/2}} \exp \pm \frac{i\pi I_x}{4} |det|x||^{-1/2}$$

Normalization is such that

$$\mathcal{D}^{\pm}(y|0) = \mp i \delta^M(y)$$

Rank-one twistor to boundary evolution

$$C^{\pm}(y|x) = \mp i \int d^M y' \mathcal{D}^{\mp}(y' - y|x' - x) C^{\pm}(y'|x')\,.$$

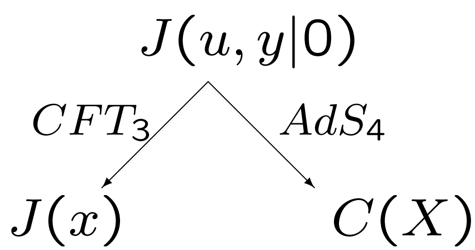
AdS/CFT from twistors

Bulk extesion is trivially achieved by means of twistor-to-bulk $\mathcal{D}$-function

$$\mathcal{D}(y|X)\,, \qquad X=(x,z)$$

$$D_0\mathcal{D}(y|X)=0\,, \qquad \mathcal{D}^{\pm}(y|0)=\mp i\delta^M(y)$$

Twistor-like transforms make the correspondence tautological



Being simple in terms of unfolded dynamics and twistor space holographic duality in terms of usual space-time may be obscure

Quantization

Operator fields obey

$$[\widehat{C}_j^-(y|x)\,,\widehat{C}_k^+(y'|x')] = \frac{1}{2i}\Big(\mathcal{D}^-(y-y'|x-x') + (-1)^{p_j p_k}\mathcal{D}^-(y+y'|x-x')\Big)$$

Commutation relations make sense at $x = x'$

$$[\widehat{C}_j^-(y|x)\,,\widehat{C}_k^+(y'|x)] = \frac{1}{2}\delta_{jk}\Big(\delta(y-y') + (-1)^{p_j p_k}\delta(y+y')\Big)$$

Singularity at $(y,x) = (y',x')$ **does not imply singularity at** $x = x'$.

Space-time operator algebra is reconstructed by twistor-to-boundary $\mathcal{D}$-functions from the operator algebra in the twistor space.

Quantum currents: $\quad J_{jk}(y_1,y_2|x) =: \hat{\Phi}_j(y_1|x)\overline{\hat{\Phi}}_k(y_2|x):$

Generating function J_g^2 **with test-function** g

$$J_g^2 = \int dw_1 dw_2\, g^{mn}(w_1,w_2) J_{mn}(w_1,w_2|0)\,,$$

$$J_g^2(x) = \int dw_1 dw_2\, g^{mn}_{ab}(w_1,w_2) J^{ab}_{mn}(w_1,w_2|x) = J^2_{g(x)}$$

x**-dependence of** $g^{mn}_{ab}(x)$ $(a,b=\pm)$ **is reconstructed by** $\mathcal{D}$**-functions**

Twistor current algebra

Elementary computation gives

$$J^2_g J^2_{g'} = J^4_{g\times g'} + J^2{}_{[g\,,g']_\star} + \mathcal{N} tr_\star(g\star g')J^0$$

Convolution product $\star$ is related to HS star-product via half-Fourier transform

$$\tilde{g}(w,v) = (2\pi)^{-M/2}\int d^M u \exp[iw_\alpha u^\alpha] g(v+u, v-u)$$

Star product of AdS_4 HS theory results from OPE of boundary currents

Full set of operators

$$J^{2m}_g =: \underbrace{J^2_g \dots J^2_g}_{m} : \qquad J^0_g = Id$$

What is the associative twistor operator algebra?!

Since

$$J^2_{g_1} J^2_{g_2} - J^2_{g_2} J^2_{g_1} = J^2_{[g_1\,,g_2]_\star}$$

This is universal enveloping algebra $U(h)$ of the HS algebra h

Explicit construction of multiparticle algebra

Universal enveloping algebra $U(l(A))$ of a Lie algebra $l(A)$ associated with an associative algebra A has remarkable properties allowing to obtain very explicit description of the operator product algebra

Let $\{t_i\}$ be some basis of A

$$a \in A: \qquad a = a^i t_i\,, \qquad\qquad t_i \star t_j = f_{ij}^k t_k$$

$$t_i \sim J^2\,, \qquad\qquad a^i \sim g(w_1, w_2)$$

$U(l(A))$ is algebra of functions of α_i (commutative analogue of t_i)

Explicit composition law of $M(A)$

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp\left(\frac{\overleftarrow{\partial}}{\partial \alpha_i} f_{ij}^n \alpha_n \frac{\overrightarrow{\partial}}{\partial \alpha_j}\right) G(\alpha)$$

where derivatives $\frac{\overleftarrow{\partial}}{\partial \alpha_i}$ and $\frac{\overrightarrow{\partial}}{\partial \alpha_j}$ act on F and G, respectively.

Associativity of $\star$ of A implies associativity of $\circ$ of $M(A)$

As a linear space, A is represented in $M(A)$ by linear functions

$F(\alpha) = a^i \alpha_i \qquad\qquad a^i \alpha_i \Leftrightarrow a^i t_i$

Operator product algebra

Composition law for linear functions

$$F(\alpha)\circ G(\alpha) = F(\alpha)G(\alpha) + F(\alpha)\star G(\alpha)$$

differs from current operator algebra

$$F(\alpha)\diamond G(\alpha) = F(\alpha)G(\alpha) + \frac{1}{2}[F(\alpha)\,,G(\alpha)]_\star + \mathcal{N} tr_\star(F(\alpha)G(\alpha))$$

Uniqueness of the Universal enveloping algebra implies that the two composition laws are related by a basis change

Generating function $G_\nu = \exp\nu \quad \nu = \nu^i\alpha_i \in A$ **is replaced by**

$$\tilde{G}_\nu = \exp[-\frac{\mathcal{N}}{4} tr_\star ln_\star(e_\star - \frac{1}{4}\nu\star\nu)\exp[\nu\star(e_\star - \frac{1}{2}v)_\star^{-1}]$$

$$\tilde{T}^u_{i_1\ldots i_n} = \frac{\partial^n}{\partial\nu^{i_1}\ldots\partial\nu^{i_n}}\tilde{G}_{(\nu)}\Big|_{\nu=0}$$

The resulting composition law is

$$\tilde{G}_\nu\diamond\tilde{G}_\mu = \left(\frac{det_\star|e_\star - \frac{1}{4}\nu\star\nu|\,det_\star|e_\star - \frac{1}{4}\mu\star\mu|}{det_\star|e_\star - \frac{1}{4}\sigma_{1,-\frac{1}{2}}(\nu,\mu)\star\sigma_{1,-\frac{1}{2}}(\nu,\mu)|}\right)^{\frac{\mathcal{N}}{4}}\tilde{G}_{\sigma_{1,-\frac{1}{2}}(\nu,\mu)}$$

$$\sigma_{1,-\frac{1}{2}}(\nu,\mu) = 2(e_\star - (e_\star - \frac{1}{2}\mu)\star(e_\star + \frac{1}{4}\nu\star\mu)_\star^{-1}\star(e_\star - \frac{1}{2}\mu)$$

Generating function for correlators $\langle J^{2n}J^{2m}\rangle$ of all currents

$$\langle \tilde{G}_\nu \tilde{G}_\mu \rangle = \left(\frac{det_\star|e_\star - \frac{1}{4}\nu\star\nu|\, det_\star|e_\star - \frac{1}{4}\mu\star\mu|}{det_\star|e_\star - \frac{1}{4}\sigma_{1,-\frac{1}{2}}(\nu,\mu)\star\sigma_{1,-\frac{1}{2}}(\nu,\mu)|} \right)^{\frac{\mathcal{N}}{4}}$$

$$J^{2n}_{g_1\dots g_n} = g^{i_1}\dots g^{i_n}\frac{\partial^n}{\partial\nu^{i_1}\dots\partial\nu^{i_n}}\tilde{G}_\nu\Big|_{\nu=0}$$

Theories with different $\mathcal{N}$: different frames of the same algebra!

$U(h)$ **possesses different invariants (traces) generating different (inequivalent) systems of n-point functions**

What are models associated with different frame choices?!

Infinitely many (conformal?) nonlinear models not respecting Wick theorem!?

Multiparticle algebra as a symmetry of a multiparticle theory

$l(U(h))$

- contains h as a subalgebra
- admits quotients containing up to k^{th} tensor products of h:

  k Regge trajectories?!
- Acts on all multiparticle states of HS theory
- Obey admissibility condition

  Oscillator realization: $[Y_i^A, Y_j^B] = \delta_{ij} C^{AB} \mathbf{E_i}$

Promising candidate for a HS symmetry algebra of HS theory with mixed symmetry fields like String Theory

Agrees with the ideas of Singleton String

Engquist, Sundell (2005, 2007)

String Theory as a theory of bound states of HS theory

Chang, Minwalla, Sharma and Yin (2012)

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Part II

Butterfly product

OPE of $J^2_{g_1}\dots J^2_{g_n}$ is described by ordered sets of g_i by butterfly product

$$g_{j_1,\dots,j_k} \bowtie g_{i_1,\dots,i_m} = g_{j_1,\dots,j_k,i_1,\dots,i_m} = \begin{cases} g_{j_1,\dots,j_k} \triangleleft g_{i_1,\dots,i_m} & \text{if } j_k < i_1\,, \\ g_{j_1,\dots,j_k} \triangleright g_{i_1,\dots,i_m} & \text{if } j_k > i_1\,, \\ 0 & \text{if } j_k = i_1\,. \end{cases}$$

$$(g \triangleleft g')^{mn}_{ab}(w_1,w_2) = 2\delta_{kj}\tau^{\mathbf{dc}}\int dp\; g^{mk}_{a\,c}(w_1,p)\, g'^{jn}_{d\,b}(p,w_2)\,,$$

$$(g \triangleright g')^{mn}_{ab}(w_1,w_2) = 2\delta_{kj}\tau^{\mathbf{c\,d}}\int dp\; g^{mk}_{a\,c}(w_1,-p)\, g'^{jn}_{d\,b}(p,w_2)$$

$$\tau^{ab} = \delta^a_+\delta^b_-$$

$$tr_{\triangleleft}(g) = \delta_{mn}\tau^{ab}\int dp\, g^{mn}_{a\,b}(-p,p)\,, \qquad tr_{\triangleright}(g) = \delta_{mn}\tau^{ba}\int dp\, g^{mn}_{a\,b}(p,p)$$

$\triangleleft$ **and** $\triangleright$ **are mutually associative:** $\alpha\triangleleft + \beta\triangleright \;\forall\alpha,\beta\in\mathbb{C}$ **is associative**

$A_{\bowtie}$ **possesses a trace**

$$tr_{\bowtie}\left(g_{j_1,\dots,j_k}\right) = \begin{cases} tr_{\triangleright}\left(g_{j_1,\dots,j_k}\right) & \text{if } j_1 < j_k\,, \\ tr_{\triangleleft}\left(g_{j_1,\dots,j_k}\right) & \text{if } j_1 > j_k\,, \\ 0 & \text{if } j_1 = j_k \quad \text{or } k < 2 \end{cases}$$

Butterfly formulae

Consider a distribution with free parameters μ^j

$$\mathcal{G}(\mu) = \sum_{j=1}^{\infty} \mu^j g_j \,, \qquad \mathcal{G}_k(\mu) = \underbrace{\mathcal{G}(\mu) \bowtie \ldots \bowtie \mathcal{G}(\mu)}_{k} \,, \qquad \mathcal{G}(\mu)_0 = Id_{\bowtie}$$

Generating function

$$E(\mathcal{G}(\mu)) = det^{-1}_{\bowtie} \Big| Id_{\bowtie} - \mathcal{G}(\mu) \Big| \, \mathrm{exp}_{\times}(\mathcal{G}(\mu) \bowtie (Id_{\bowtie} - \mathcal{G}(\mu))^{-1}_{\bowtie})$$

commutative product $\times$ encodes the normal ordering $J^{2n}_g \times J^{2m}_f \sim : J^{2n}_g \mathcal{J}^{2m}_f$

$$J^2_{g_{j_1}} \ldots J^2_{g_{j_k}} = \left(\frac{\partial}{\partial \mu^{j_1}} \ldots \frac{\partial}{\partial \mu^{j_k}} E(\mathcal{G}(\mu)) \right) \Big|_{\mu=0} \quad \text{for} \quad j_1 < \cdots < j_k$$

The coefficient in front of the central element in the multiple product of bilinear currents $J^2_{g_1} \ldots J^2_{g_n}$ gives all n-point functions

$$\Phi(g_1, \ldots g_n) = \left(\frac{\partial}{\partial \mu^1} \ldots \frac{\partial}{\partial \mu^n} det^{-1}_{\bowtie} \Big| Id_{\bowtie} - \mathcal{G}(\mu) \Big| \right)_{\mu=0}$$

$$\Phi(g_1, \ldots g_n) = \langle J^2(g_1) \ldots J^2(g_n) \rangle$$

Space-time n-point functions

Space-time currents

$$J_\gamma(x) = \gamma^{mn}_{ab}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}) J^{ab}_{mn}(y_1, y_2|x)\Big|_{y_{1,2}=0} = J^2_{g(w_1,w_2,x;\gamma)}$$

$$g(w_1, w_2, x; \gamma) = -\gamma^{mn}_{ab}(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}) \mathcal{D}_a(w_1 - y_1|x)\mathcal{D}_b(w_2 - y_2|x)\Big|_{y_{1,2}=0}$$

Substitution into generating function gives all n-point functions of conserved currents in space-time, reproducing previously known results

Giombi, Prakash, Yin (2011), Colombo, Sundell (2012), Didenko, Skvortsov (2012)

extending them to supercurrents and $4d$ correlators and fixing relative coefficients Gelfond, MV (arXiv:1301.3123)

Example: boson-boson currents

$$\left\langle J_{\eta_1}(X^1)\dots J_{\eta_n}(X^n)\right\rangle^b_{con} =$$

$$2^{n-1}\eta_{(n)}\left(\partial_U\right)\sum_{\mathcal{S}_n}\frac{\left(\cos Q_{(n)}\cos P_{1,\,2}\cdots\cos P_{n-1,\,n}\cos P_{n,\,1}\right)(U)}{\mathsf{D}^{1,2}\dots\mathsf{D}^{n-1,n}\mathsf{D}^{n,1}}\Bigg|_{U=0}$$

where

$$P_{i,\,j} = -\frac{1}{2}(X^i - X^j)^{-1}_{AB}U^{iA}U^{jB}$$

$$Q_{(p)} = Q_{1,\,2,\,3} + \dots + Q_{p-2,\,p-1,\,p} + Q_{p-1,\,p,\,1} + Q_{p,\,1,\,2}$$

$$Q_{i,j,k} = \frac{1}{4}\left((X^i - X^j)^{-1}_{AB} + (X^j - X^k)^{-1}_{AB}\right)U^{jA}U^{jB}$$

$$\mathsf{D}^{j,\,k} = (4\pi)^{\frac{M}{2}}\exp\left(\mathrm{sign}(k-j)\frac{i\pi I_{X^k - X^j}}{4}\right)\sqrt{|\det(X^k - X^j)|}$$

Conclusion

HS computations are most easily done in the twistor space for all spins at once

Remarkable interplay between classical and quantum physics in HS theory

A multiparticle theory: quantum HS theory and String theory

Multiparticle algebra is a Hopf algebra.
Relation with integrable structures underlying both String theory and analysis of amplitudes?!

By virtue of Flato-Fronsdal type theorems multiparticle theory will be a theory in infinite-dimensional space where different types of fields live on delocalized branes of different dimensions