First steps in Derived Symplectic Geometry

Gabriele Vezzosi
(Institut de Mathématiques de Jussieu, Paris)

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joint work with
T. Pantev, B. Toën, and M. Vaquié
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Plan of the talk

1. Motivation : quantizing moduli spaces
2. The Derived Algebraic Geometry we'll need below
3. Examples of derived stacks
4. Derived symplectic structures I - Definition
5. Derived symplectic structures II - Three existence theorems
   - MAP(CY, Sympl)
   - Lagrangian intersections
   - $\mathbb{R}$**Perf**
6. From derived to underived symplectic structures
7. (-1)-shifted symplectic structures and symmetric obstruction theories

Motivation : quantizing moduli spaces

X - derived stack, $D_{qcoh}(X)$ - dg-category of quasi-coherent complexes on X.
$D_{qcoh}(X)$ is a symmetric monoidal i.e. $E_\infty - \otimes$-dg-category $\Rightarrow$ in particular: a dg-category ($\equiv E_0 - \otimes$-dg-cat), a monoidal dg-category ($\equiv E_1 - \otimes$-dg-cat), a braided monoidal dg-category ($\equiv E_2 - \otimes$-dg-cat), ... $E_n - \otimes$-dg-cat (for any $n \geq 0$).
(Rmk - For ordinary categories $E_n - \otimes \equiv E_3 - \otimes$, for any $n \geq 3$; for ∞-categories, like dg-categories, all different, a priori !)

n-quantization of a derived moduli space

- An n-quantization of a derived moduli space X is a (formal) deformation of $D_{qcoh}(X)$ as an $E_n - \otimes$-dg-category.
- Main Theorem - An n-shifted syplectic form on X determines an n-quantization of X.

Motivation : quantizing moduli spaces

– Main line of the proof –

- Step 1. Show that an n-shifted symplectic form on X induces a n-shifted Poisson structure on X.
- Step 2. A derived extension of Kontsevich formality (plus a fully developed deformation theory for $E_n - \otimes$-dg-category) gives a map

$$\{n\text{-shifted Poisson structures on X}\} \to \{\text{n-quantizations of X}\}.$$

□

We aren't there yet ! We have established Step 2 for all n (using also a recent result by N. Rozenblyum), and Step 1 for X a derived DM stack (all n) ; the Artin case is harder...

Perspective applications - quantum geometric Langlands, higher categorical TQFT's, higher representation theory, non-abelian Hodge theory, Poisson and symplectic structures on classical moduli spaces, etc.

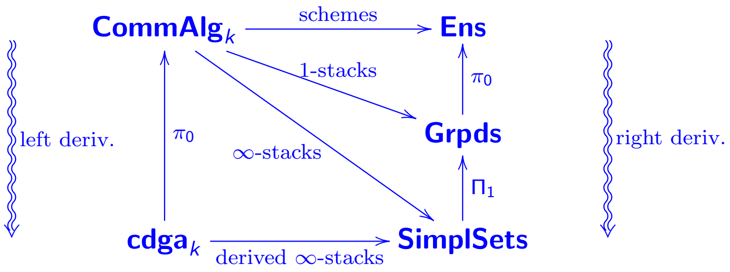
In this talk I will concentrate on derived a.k.a **shifted symplectic structures**.

Derived Algebraic Geometry (DAG)

Derived Algebraic Geometry (say over a base commutative $\mathbb{Q}$-algebra k) is a kind of algebraic geometry whose affine objects are k-cdga's i.e. commutative differential nonpositively graded algebras

$$\ldots \qquad \xrightarrow{d} A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^{0}$$

The *functor of points* point of view is



Both source and target categories are homotopy theories $\Rightarrow$ derived spaces are obtained by gluing cdga's up to homotopy (roughly).

Derived stacks

This gives us a category $\mathbf{dSt}_k$ of derived stacks over k, which admits, in particular

- $\mathbb{R}\mathrm{Spec}(A)$ as affine objects (A being a cdga)
- fiber products (up to homotopy)
- internal HOM's (up to homotopy)
- an adjunction $\mathbf{dSt}_k \underset{j}{\overset{\mathrm{t}_0}{\rightleftarrows}} \mathbf{St}_k$, where
  - The truncation functor t_0 is right adjoint, and $\mathrm{t}_0(\mathbb{R}\mathrm{Spec}(A)) \simeq \mathrm{Spec}(H^0 A)$
  - j is fully faithful (up to homotopy) but does **not** preserve fiber products nor internal HOM's $\rightsquigarrow$ tgt space of a scheme Y is different from tgt space of $j(Y)$!
    (and, in fact, the derived tangent stack
    $$\mathbb{R}TX := \mathrm{HOM}_{\mathbf{dSt}_k}(Spec\, k[\varepsilon], X) \simeq Spec_X(Sym_X(\mathbb{L}_X))$$
    for any X).
- deformation theory (e.g. the cotangent complex) is natural in DAG (i.e. satisfies universal properties in $\mathbf{dSt}_k$).

Some examples of derived stacks

- [Derived affines] $A \in \mathbf{cdga}_k^{\leq 0}$ $\mathbb{R}Spec\, A : \mathbf{cdga}_k^{\leq 0} \to SSets$ $B \mapsto Map_{\mathbf{cdga}_k^{\leq 0}}(A, B) = (Hom_{\mathbf{cdga}_k^{\leq 0}}(QA, B \otimes_k \Omega_n))_{n \geq 0}$ where Ω_n is the cdga of differential forms on the algebraic n-simplex $Spec(k[t_0, ..., t_n]/(\sum_i t_1 - 1))$
- [Local systems] M topological space of the homotopy type of a CW-complex, $Sing(M)$ singular simplicial set of M. Denote as $Sing(M)$ the constant functor $\mathbf{cdga}_k^{\leq 0} \to SSets : A \mapsto Sing(M)$. G group scheme over $k \Rightarrow \mathbb{R}Loc(M; G) := MAP_{dSt_k}(Sing(M), BG)$ - derived stack of G-local systems on M. Its truncation is the classical stack $Loc(M; G)$. Note that $\mathbb{R}Loc(M; G)$ might be nontrivial even if M is simply connected (e.g. $T_E \mathbb{R}Loc(M; GL_n) \simeq \mathbb{R}\Gamma(X, E \otimes E^\vee)[1]$).
- [Derived tangent stack] X scheme $\Rightarrow TX := MAP_{dSt_k}(Spec\, k[\varepsilon], X)$ derived tangent stack of X. $TX \simeq \mathbb{R}Spec(Sym_{\mathcal{O}_X}(\mathbb{L}_X))$, $\mathbb{L}_X$ cotangent complex of X/k.

Some examples of derived stacks

- [Derived loop stack] X derived stack, $S^1 := B\mathbb{Z} \Rightarrow$ $LX : MAP_{dSt_k}(S^1, X)$ - derived (free) loop stack of X. Its truncation is the inertia stack of $\mathrm{t}_0(X)$ (i.e. X itself, if X is a scheme). Functions on LX give the Hochschild homology of X. S^1-invariant functions on LX give negative cyclic homology of X.
- [Perfect complexes]
  $\mathbb{R}\mathbf{Perf} : \mathbf{cdga}_k^{\leq 0} \to SSets : A \mapsto Nerve(Perf(A)^{cof}, q - iso)$
  where $Perf(A)$ is the subcategory of all A-dg-modules consisting of dualizable ($=$ homotopically finitely presented) A-dg-modules. Its truncation is the stack $\mathbf{Perf}$. The tangent complex at $E \in \mathbb{R}\mathbf{Perf}(k)$ is $\mathbb{T}_E \mathbb{R}\mathbf{Perf} \simeq \mathbb{R}End(E)[1]$. $\mathbb{R}\mathbf{Perf}$ is locally Artin of finite presentation. Note also that for any derived stack X, we define the derived stack of perfect complexes on X as
  $\mathbb{R}\mathbf{Perf}(X) := MAP_{dSt_k}(X, \mathbb{R}\mathbf{Perf})$. Its truncation is the classical stack $\mathbf{Perf}(X)$. The tangent complex, at $\mathcal{E}$ perfect over X, is $\mathbb{R}\Gamma(X, \underline{End}(\mathcal{E}))[1]$.

Derived symplectic structures I - Definition

To generalize the notion of symplectic form in the derived world, we need to generalize the notion of 2-form, of closedness , and of nondegeneracy. In the derived setting, it is closedness the trickier one: it is no more a property but a list of coherent data on the underlying 2-form !

Why? Let A be a (cofibrant) cdga, then $\Omega^{\bullet}_{A/k}$ is a bicomplex : vertical d coming from the differential on A, horizontal d is de Rham differential d_{DR}. So you don't really want $d_{DR}\omega = 0$ but $d_{DR}\omega \sim 0$ with a specified 'homotopy'; but such a homotopy is still a form ω_1

$$d_{DR}\omega = \pm d\omega_1$$

And we further require that $d_{DR}\omega_1 \sim 0$ with a specified homotopy

$$d_{DR}\omega_1 = \pm d(\omega_2),$$

and so on.

This $(\omega, \omega_1, \omega_2, \cdots)$ is an infinite set of higher coherencies data not properties!

Derived symplectic structures I - Definition

More precisely: the guiding paradigm comes from negative cyclic homology: if $X = Spec\, R$ is smooth over k ($\mathrm{char}(k) = 0$) then the HKR theorem tells us that

$$HC_p^-(X/k) = \Omega_{X/k}^{p,cl} \oplus \prod_{i \geq 0} H_{DR}^{p+2i}(X/k)$$

and the summand $\Omega_{X/k}^{p,cl}$ is the weight (grading) p part.

So, a fancy (but homotopy invariant) way of defining classical closed p-forms on X is to say that they are elements in $HC_p^-(X/k)^{(p)}$ (weight p part).

How do we see the weights appearing geometrically?

Through derived loop stacks.

Derived symplectic structures I - Definition

Derived loop stacks

X derived Artin stack locally of finite presentation
• $LX := \mathrm{MAP}_{dSt_k}(S^1 := B\mathbb{Z}, X)$ - derived free loop stack of X
• $\widehat{LX}$ - formal derived free loop stack of X (formal completion of LX along constant loops $X \to LX$)
• $\mathcal{H} := \mathbb{G}_m \ltimes B\mathbb{G}_a$ acts on $\widehat{LX}$

Rmk - If X is a derived **scheme**, the canonical map $\widehat{LX} \to LX$ is an equivalence.
- $\mathcal{H}$-action on $\widehat{LX}$: $\widehat{LX} \simeq \widehat{L^{aff}X}$, where $L^{aff}X := MAP_{dSt_k}(B\mathbb{G}_a, X)$, and the obvious action of $\mathbb{G}_m \ltimes B\mathbb{G}_a$ on $L^{aff}X$ descends to the formal completion. The S^1-action factors through this $\mathcal{H}$-action:

$$\mathbb{G}_m \circlearrowright S^1 \to B\mathbb{G}_a \circlearrowleft \mathbb{G}_m.$$

Derived symplectic structures I - Definition

$$\begin{array}{ccc} [\widehat{LX}/\mathbb{G}_m] & \longrightarrow & [\widehat{LX}/\mathcal{H}] = [\widehat{LX}/S^1] \\ & \searrow^{\pi} & \downarrow q \\ & & B\mathbb{G}_m \end{array}$$

$q_* \mathcal{O}_{[\widehat{LX}/\mathcal{H}]} =: NC^w(X/k)$: (weighted) negative cyclic homology of X/k
($\mathbb{G}_m$-equivariance $\rightsquigarrow$ grading by weights);
$\pi_* \mathcal{O}_{[\widehat{LX}/\mathbb{G}_m]} =: DR(X/k) \simeq \mathbb{R}\Gamma(X, Sym^{\bullet}_X(\mathbb{L}_X[1])) \simeq \mathbb{R}\Gamma(X, \oplus_p (\wedge^p \mathbb{L}_X)[p])$:
(weighted) derived de Rham complex (Hochschild homology) of X/k
($(\wedge^p \mathbb{L}_X)[p]$: weight-p part)
So, the diagram above gives a weight-preserving map

$$NC^w(X/k) \longrightarrow DR(X/k)$$

(classically: $HC^- \to HH$: negative-cyclic to Hochschild)

Derived symplectic structures I - Definition

We use the map $NC^w(X/k) \longrightarrow DR(X/k)$ to define

n-shifted (closed) p-forms

X derived Artin stack locally of finite presentation ($\rightsquigarrow \mathbb{L}_X$ is perfect).

- The space of n-shifted p-forms on X/k is $\mathcal{A}^p(X; n) := |DR(X/k)[n-p](p)| \simeq |\mathbb{R}\Gamma(X, (\wedge^p \mathbb{L}_X)[n])|$
- The space of closed n-shifted p-forms on X/k is $\mathcal{A}^{p,cl}(X; n) := |NC^w(X/k)[n-p](p)|$
- The homotopy fiber of the map $\mathcal{A}^{p,cl}(X; n) \to \mathcal{A}^p(X; n)$ is the space of keys of a given n-shifted p-form on X/k.

Rmks - $|-|$ is the geometric realization; for an n-shifted p-form, being closed is not a condition; any n-shifted closed p-form has an underlying n-shifted p-form (via the map above); for $n = 0$, and X a smooth underived scheme, we recover the usual notions.

Derived symplectic structures I - Definition

n-shifted symplectic forms

X derived Artin stack locally of finite presentation (so that $\mathbb{L}_X$ is perfect).

- A n-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X; n))$ - is nondegenerate if its adjoint $\omega^\flat : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism (in $D_{qcoh}(X)$). The subspace of $\mathcal{A}^2(X, n)$ of connected components of nondegenerate 2-forms is denoted by $\mathcal{A}^2(X, n)^{nd}$.
- The space of n-shifted symplectic forms $Sympl(X; n)$ on X/k is the subspace of $\mathcal{A}^{2,cl}(X; n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

$$\begin{array}{ccc} Sympl(X, n) & \longrightarrow & \mathcal{A}^{2,cl}(X, n) \\ \downarrow & & \downarrow \\ \mathcal{A}^2(X, n)^{nd} & \longrightarrow & \mathcal{A}^2(X, n) \end{array}$$

Derived symplectic structures I - Definition

- Nondegeneracy involves a kind of duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$
- In particular: X smooth underived scheme $\rightsquigarrow$ may only admit 0-shifted symplectic structures, and these are then just usual symplectic structures.
- $G = GL_n \rightsquigarrow BG$ has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

$$k \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^{\vee}[-1] \wedge \mathfrak{g}^{\vee}[-1])[2] = Sym^2 \mathfrak{g}^{\vee}$$

  given by the dual of the trace map $(A, B) \mapsto tr(AB)$.
- Same as above (with a choice of G-invariant symm bil form on $\mathfrak{g}$) for G reductive over k. Rmk - The induced quantization is the "quantum group" (i.e. quantization is the $\mathbb{C}[[t]]$-braided mon cat given by completion at $q = 1$ of $Rep(G(g))$ $\mathbb{C}[q, q^{-1}]$-braided mon cat).
- The n-shifted cotangent bundle $T^*X[n] := Spec_X(Sym(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Derived symplectic structures on mapping stacks

Derived version of a result by Alexandrov-Kontsevich-Schwarz-Zaboronsky:

Existence Theorem 1 - Derived mapping stacks

Let F be a derived Artin stack equipped with an n-shifted symplectic form $\omega \in Symp(F, n)$. Let X be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation $[X] : \mathbb{R}\underline{End}(\mathcal{O}_X) \longrightarrow k[-d]$ of degree d. If the derived mapping stack $MAP(X, F)$ is a derived Artin stack locally of finite presentation over k, then, $MAP(X, F)$ carries a canonical $(n-d)$-shifted symplectic structure.

Important Rmk - A degree d $\mathcal{O}$-orientation on X is a kind of Calabi-Yau structure of dimension d , in particular any smooth and proper Calabi-Yau scheme (or Deligne-Mumford stack) $f : X \to Spec\, k$ of dim d admits a degree d $\mathcal{O}$-orientation. Indeed, any $\omega_X = \wedge^d \Omega^1_X \simeq \mathcal{O}_X$ gives $\mathbb{R}Hom(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{R}Hom(\mathcal{O}_X, \omega_X) \simeq \mathbb{R}f_* \omega_X \simeq \mathbb{R}f_* f^! k[-d] \to k[-d]$ (where last map is trace map in coherent duality).

Idea of the proof of Theorem 1 – We can mimick the following well-known construction (hat-product) in differential geometry.
Let M^m compact C^∞, N C^∞

$$N \xleftarrow{\;ev\;} M \times Map_{C^\infty}(M,N) \xrightarrow{\;pr_M\;} M$$

$$\Omega^p_M \times \Omega^q_N \to \Omega^{p+q-m}_{Map(M,N)} : (\alpha, \beta) \mapsto \int_M pr_M^* \alpha \wedge ev^* \beta := \widehat{\alpha\beta}$$

($\int_M$: integration along the fiber).
If (N, ω) is symplectic, η volume form on M, then $\widehat{\eta\omega} \in \Omega^2_{Map(M,N)}$ is symplectic.
Note that in this case there is no shift ($n = 0$). □

Derived symplectic structures on mapping stacks

Some Corollaries of Theorem 1

Let (F, ω) be n-shifted symplectic derived Artin stack.

- Betti - If $X = M^d$ compact, connected, topological manifold. The choice of fund class $[X]$ yields a canonical $(n-d)$-shifted sympl structure on $MAP(X, F)$.
- Calabi-Yau - X Calabi-Yau smooth and proper k-scheme (or k-DM stack), with geometrically connected fibres of dim d. The choice of a trivialization of the canonical sheaf ω_X yields a canonical $(n-d)$-shifted sympl structure on $MAP(X, F)$.
- de Rham - Y smooth proper DM stack with geometrically connected fibres of dim d. The choice of a fundamental class $[Y] \in H^{2d}_{DR}(Y, \mathcal{O})$ yields a canonical $(n-2d)$-shifted symplectic structure on $MAP(X := Y_{DR}, F)$.

Example of Betti: X n-symplectic $\Rightarrow$ its derived loop space LX is $(n-1)$-symplectic.

Derived symplectic structures on mapping stacks

Corollaries of the previous corollaries - E.g. one could take $F = BG$, G reductive affine group scheme, with a chosen G-invariant symm bil form on $Lie(G)$. The corollaries give $(2-d)$-shifted (resp. $(2-2d)$-shifted) symplectic structures on the derived stack of G-local systems and G-bundles (resp. of de Rham G-local systems = flat G-bundles on Y) on Y.

Derived symplectic structures on lagrangian intersections

Existence Theorem 2 - Derived lagrangian intersections

Let (F, ω) be n-shifted symplectic derived Artin stack, and $L_i \to F$ a map of derived stacks equipped with a Lagrangian structure, $i = 1, 2$. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.

In particular, if $F = Y$ is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and $L_i \hookrightarrow Y$ is a smooth closed lagrangian substack, $i = 1, 2$, then the derived intersection $L_1 \times_F L_2$ is canonically (-1)-shifted symplectic.

Rmk - An interesting case is the derived critical locus $\mathbb{R}Crit(f)$ for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here

$$\begin{array}{ccc} \mathbb{R}Crit(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle df} \\ Y & \xrightarrow[0]{} & T^*Y \end{array}$$

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Then one checks that such an h actually comes from a closed (-1)-shifted symplectic form on L_{12}. $\square$

Derived symplectic structure on $\mathbb{R}\mathbf{Perf}$

Recall the derived stack

$$\mathbb{R}\mathbf{Perf} : \mathbf{cdga}_k^{\leq 0} \to \mathbf{SSets} \, : A \mapsto Nerve(Perf(A)^{cof}).$$

Existence theorem 3 - $\mathbb{R}\mathbf{Perf}$ is 2-shifted symplectic

The derived stack $\mathbb{R}\mathbf{Perf}$ is 2-shifted symplectic.

Idea of proof – By definition, there is a universal perfect complex $\mathcal{P}$ on $\mathbb{R}\mathbf{Perf}$, and it is easy to prove that

$$\mathbb{T}_{\mathbb{R}\mathbf{Perf}} \simeq \mathbb{R}\mathcal{E}nd(\mathcal{P})[1]$$

Use the Chern character for derived stacks ([Toën -V, 2011]) to put

$$\omega^{Perf} := \mathrm{Ch}(\mathcal{P})^{(2)}$$

(weight 2 part). Using Atiyah classes, the underlying 2-form is non-degenerate. □

Derived symplectic structure on $\mathbb{R}\mathbf{Perf}$

Some corollaries of Thms. 1 (MAP) and 3 (RPerf)

- Betti - If $X = M^d$ compact, connected, topological manifold. The choice of fundamental class $[X]$ yields a canonical $(2-d)$-shifted sympl structure on $MAP(M, \mathbb{R}\mathbf{Perf}) = \mathbb{R}\mathbf{Perf}(M)$.
- Calabi-Yau - X Calabi-Yau smooth and proper k-scheme (or k-DM stack), with geometrically connected fibres of dim d. The choice of a trivialization of the canonical sheaf ω_X yields a canonical $(2-d)$-shifted sympl structure on $MAP(X, \mathbb{R}\mathbf{Perf}) = \mathbb{R}\mathbf{Perf}(X)$.
- de Rham - Y smooth proper DM stack with geometrically connected fibres of dim d. The choice of a fundamental class $[Y] \in H^{2d}_{DR}(Y, \mathcal{O})$ yields a canonical $(2-2d)$-shifted sympl structure on $MAP(Y_{DR}, \mathbb{R}\mathbf{Perf}) =: \mathbb{R}\mathbf{Perf}_{DR}(Y)$.

From derived to underived symplectic structures

Using Theorems 1 (MAP) and 3 ($\mathbb{R}\mathbf{Perf}$) we may recover some (underived) symplectic structures on smooth moduli spaces. E.g. :

• Simple local systems on curves – C a smooth, proper, geom connected curve over k, G simple algebraic group over k. Consider the underived stacks $\mathbf{Loc}_{DR}(C;G)^s$, $\mathbf{Loc}(C^{top};G)^s$ of simple de Rham and simple topological G-local systems on C. By using

$$\mathbf{Loc}_{DR}(C;G)^s \overset{j}{\hookrightarrow} \mathbb{R}\mathbf{Loc}_{DR}(C;G) \qquad \mathbf{Loc}(C^{top};G)^s \overset{j}{\hookrightarrow} \mathbb{R}\mathbf{Loc}(C^{top};G)$$

we recover, with a uniform proof, the symplectic structures of **Goldman**, **Weinstein-Jeffreys**, **Inaba-Iwasaki-Saito** (the original proofs are very different from each other).

• Perfect complexes on CY surfaces – S a CY surface over k (i.e. $K3$ or abelian), fix $K_S \simeq \mathcal{O}_S$. Let $\mathbb{R}\mathbf{Perf}(S)^s \hookrightarrow \mathbb{R}\mathbf{Perf}(S)$ the open derived substack cassifying simple complexes (i.e. $Ext^i_S(E,E) = 0$ for $i < 0$, $Ext^0_S(E,E) \simeq k$). Consider $t_0(\mathbb{R}\mathbf{Perf}(S)^s) := \mathbf{Perf}(S)^s$ and its coarse moduli space $Perf(S)^s$. We recover the results of **Mukai** and **Inaba** (2011) that $Perf(S)^s$ is a smooth and symplectic algebraic space.

(-1)-shifted symplectic structures and symmetric obstruction theories

X derived stack (locally finitely presented), $j : \mathrm{t}_0(X) \hookrightarrow X \Rightarrow$

$$j^*\mathbb{L}_X \to \mathbb{L}_{\mathrm{t}_0(X)}$$

is a perfect obstruction theory in the sense of Behrend-Fantechi (a $[-1, 0]$-perfect obstruction theory, if X is *quasi-smooth*).
So if $\mathcal{X}$ is a given stack ther is a map

$$\{\text{lfp dstacks with truncation } \simeq \mathcal{X}\} \to \{\text{perfect obstruction theories on } \mathcal{X}\}$$

What do we gain if X is moreover (-1)-shifted symplectic?
ω: (-1)-shifted symplectic form on $X \Rightarrow$ underlying 2- form

$$\omega : \mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[-1]$$

and its adjoint $\Theta_\omega : \mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[-1]$.

(-1)-shifted symplectic structures and symmetric obstruction theories

So, via the isomorphism $\Theta_\omega : \mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[-1]$, the underlying 2-form $\omega : \mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[-1]$, gives

$$(Sym^2 \mathbb{L}_X)[-2] \simeq \mathbb{L}_X[-1] \wedge \mathbb{L}_X[-1] \to \mathcal{O}_X[-1].$$

Therefore (by shifting by [2], and restricting along $j : \mathrm{t}_0(X) \hookrightarrow X$) we find that the obstruction theory

$$j^* \mathbb{L}_X \to \mathbb{L}_{\mathrm{t}_0(X)}$$

is a symmetric obstruction theory in the sense of Behrend-Fantechi. So we have a map

$\{$(-1)-sympl dstacks X s.t. $\mathrm{t}_0(X) \simeq \mathcal{X}\} \to \{$symm perfect obstr theories on $\mathcal{X}\}$

(-1)-shifted symplectic structures and symmetric obstruction theories

All known examples of symmetric obstruction theories actually come from (-1)-derived symplectic structures.

Some examples :

- Any derived intersections of two smooth lagrangians L_1 and L_2 inside a smooth symplectic variety M is (-1)-shifted symplectic $\Rightarrow L_1 \cap L_2$ has a canonical $[-1, 0]$-perfect symmetric obstruction theory.
- Y - elliptic curve; M - smooth symplectic variety $\Rightarrow MAP(Y, M)$ is canonically $(0 - 1 = -1)$-shifted symplectic $\Rightarrow$ the stack of maps $Y \to M$ has a canonical $[-1, 0]$-perfect symmetric obstruction theory.

(-1)-shifted symplectic structures and symmetric obstruction theories

- Y 3-dim CY smooth algebraic variety, choose $K_Y \simeq \mathcal{O}_Y \Rightarrow$ $\mathbb{R}\mathbf{Perf}(Y) := MAP(Y, \mathbb{R}\mathbf{Perf})$ is canonically $(2-3=-1)$-shifted symplectic $\Rightarrow$ the stack of perfect complexes $\mathbf{Perf}(Y)$ has a canonical symmetric obstruction theory. Same for $\mathbb{R}\mathbf{Perf}(Y)^{si}_{\mathcal{L}}$ (classifying simple objects with fixed determinant $\mathcal{L}$) $\Rightarrow$ the stack of simple perfect complexes $\mathbf{Perf}(Y)^{s}_{\mathcal{L}}$ has a canonical $[-1,0]$-perfect symmetric obstruction theory (indeed, $\mathbb{R}\mathbf{Perf}(Y)^{si}_{\mathcal{L}}$ is quasi-smooth).

(-1)-shifted symplectic structures and symmetric obstruction theories

In the comparison symm obstr theories/(-1)-shifted symplectic forms, note that:
– obstruction theories induced by derived stacks are fully functorial, therefore functoriality of (-1)-shifted symplectic forms gives full functoriality on induced symmetric obstruction theories.
– symmetric obstruction theories induced by (-1)-shifted sympletic structures are better behaved than others (note that the closure data are forgotten by symmetric obstruction theories), e.g. they give a solution to a longstanding problem in Donaldson-Thomas theory:

Corollary (Brav-Bussi-Joyce, 2013)

The Donaldson-Thomas moduli space of simple perfect complexes (with fixed determinant) on a Calabi-Yau 3-fold is locally for the Zariski topology the critical locus of a function, the *DT-potential* on a smooth complex manifold). Locally the obstruction theory on the DT moduli space is given by the (-1)-symplectic form on the derived critical locus of the potential.

Rmk. False for general symmetric obstruction theories (Pandharipande-Thomas, April 2012)

Thank you!