3d&5d holomorphic blocks and q-CFT correlators

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based on: arXiv:1303.2626 with F. Nieri and F. Passerini,
work in progress with F. Nieri, F. Passerini, A. Torrielli

In recent years many exact results for gauge theories on compact manifolds have been obtained by the method of SUSY localisation initiated by Pestun.

The idea is that by adding a Q-exact term to the action it is possible to reduce the path integral to a finite dimensional integral:

Localisation: $Z_{\mathcal{M}} = \int D\psi e^{-S[\Psi]} = \int D\Psi_0 e^{-S[\Psi_0]} Z_{\text{1-loop}}[\Psi_0]$

- Ψ_0: field configurations satisfying localising (saddle point) equations
- with a clever localisation scheme, Ψ_0 is a finite dimensional set
- $Z_{\text{1-loop}}[\Psi_0]$ is due to the quadratic fluctuation around Ψ_0

$\Rightarrow$ useful to study holography

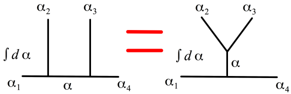
$\Rightarrow$ connect to exactly solvable models such as 2d CFTs and TQFTs

The AGT correspondence [Alday-Gaiotto-Tachikawa],[Wyllard] maps S^4 partition functions of 4d $\mathcal{N}=2$ theories $\mathcal{T}_{g,n}$ obtained wrapping M5 branes on $\Sigma_{g,n}$ (class S-theories [Gaiotto]) to Liouville correlators:

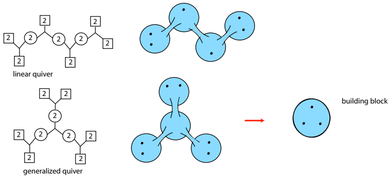
$$\boxed{Z_{S^4}[\mathcal{T}_{g,n}] = \int [\mathrm{d}a]\; Z_{cl}\; Z_{1loop}\left|Z_{inst}\right|^2 = \int \mathrm{d}\alpha\; C\cdots C\; |\mathcal{F}^{\alpha_i}_{\alpha}(\zeta)|^2 = \langle \textstyle\prod_i^n V_{\alpha_i}\rangle^{\mathrm{Liouville}}_{C_{g,n}}}$$

generalised $\mathcal{N}=2$ S-duality $\Leftrightarrow$ CFT modular invariance

- Associativity of the operator algebra requires crossing symmetry

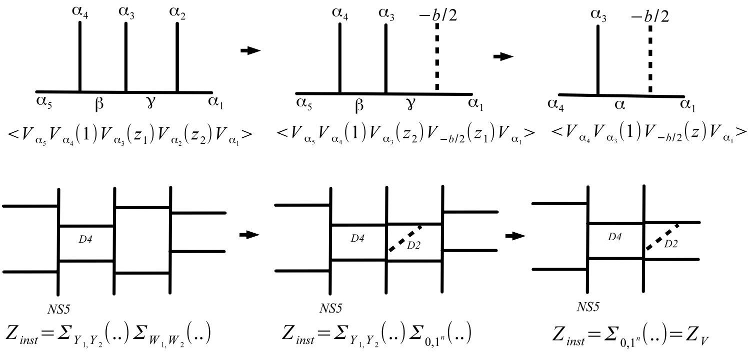


- Partition functions are invariant under generalised $\mathcal{N}=2$ S-dualities (different pant-decompositions of $\Sigma_{g,n}$)



Simple surface operators⇔degenerate primaries $\left(L_{-2}+\frac{1}{b^2}L_{-1}^2\right)V_{-b/2}=0$

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]



- Several results: degenerate conformal blocks ↔ vortex counting [Dimofte-Gukov-Hollands],[Kozcaz-Pasquetti-Wyllard],[Bonelli-Tanzini-Zhao].
- Recent proposal [Droud-Gomis-LeFloch-Lee] (see also [Benini-Cremonesi])

$$\boxed{\langle V_{\alpha_4} V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1}\rangle = Z^{SQED}_{S_2}}$$

flop symmetry ⇔ crossing symmetry

Liouville theory can be completely solved by the conformal bootstrap approach which only uses Virasoro symmetry & crossing symmetry.

Now considering that:

- there is an action of the W-algebra on the equivariant cohomology of the moduli space of instantons, [Maulik-Okounkov],[Schiffmann-Vasserot]
- $\mathcal{N} = 2$ S-duality, flop symmetry are gauge theory avatars of crossing symmetry,

we could say that S^2 and S^4 gauge theory partition functions and CFT correlators are constrained by the same bootstrap equations!

Today I will argue that a similar story holds in 3d and 5d:

- 3d partition functions $\Leftrightarrow$ degenerate q-CFT correlators
- 5d partition functions $\Leftrightarrow$ non-degenerate q-CFT correlators.

Plan of the talk

- Block-factorisation of 3d & 5d partition functions
- q-CFT correlators via the bootstrap approach
- 3d and 5d partition functions as q-CFT correlators
- Conclusions and open issues

$\mathcal{N} = 2$ theory on S_b^3

$$S_b^3 : \quad b^2|z_1|^2 + \frac{1}{b^2}|z_2|^2 = 1$$

Coulomb branch localization scheme [Hama-Hosomichi-Lee].

SQED: $U(1)$ gauge group, N_f chirals m_i, N_f anti-chirals $\tilde{m}_k$, with FI ξ.

$$Z_{S^3}^{SQED} = \int dx\ G_{cl} \cdot G_{\text{1-loop}} = \int dx\ e^{2\pi i x \xi} \prod_{j,k}^{N_f} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

The 1-loop contribution of a chiral multiplet is:

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}\,, \qquad Q = b + 1/b\,.$$

Higgs-branch-like factorized form: [S.P.]

$$Z_{S^3}^{SQED} = \sum_{i}^{N_f} G_{cl}^{(i)} G_{1\text{-loop}}^{(i)} \left|\left| \mathcal{Z}_V^{(i)} \right|\right|_S^2$$

- $G_{cl}^{(i)}, G_{1loop}^{(i)}$ evaluated on the i-th SUSY vacuum of the effective $(2,2)$ theory:

$$G_{cl}^{(i)} = e^{-2\pi i \xi m_i}, \qquad G_{1\text{-loop}}^{(i)} = \prod_{j,k}^{N_f} \frac{s_b(m_j - m_i + iQ/2)}{s_b(\tilde{m}_k - m_i - iQ/2)},$$

- Vortices on $\mathbb{R}^2 \times S^1$ satisfy basic hypergeometric equations:

$$\mathcal{Z}_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = {}_{N_f}\Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z).$$

- S-pairing: $\left|\left| f(x;q) \right|\right|_S^2 = f(x;q) f(\tilde{x};\tilde{q})$

$$x_i = e^{2\pi m_i/\omega_1}, \quad y_i = e^{2\pi \tilde{m}_i/\omega_1}, \quad z = e^{2\pi\xi/\omega_1}, \quad q = e^{2\pi i \frac{\omega_2}{\omega_1}},$$
$$\tilde{x}_i = e^{2\pi m_i/\omega_2}, \quad \tilde{y}_i = e^{2\pi \tilde{m}_i/\omega_2}, \quad \tilde{z} = e^{2\pi\xi/\omega_2}, \quad \tilde{q} = e^{2\pi i \frac{\omega_1}{\omega_2}},$$
$$\omega_1 = b, \qquad \omega_2 = 1/b$$

Classical (mixed Chern-Simons) terms can be factorized:

$$G_{cl}^{(i)} = \left|\left|\mathcal{G}_{cl}^{(i)}\right|\right|_S^2$$

using that:

$$e^{-\frac{(\log x)^2}{2\log q}+\frac{\log q}{24}-\frac{\pi^2}{6\log q}} = \left|\left|\theta(x;q)\right|\right|_S^2, \qquad \theta(x;q) := (-q^{\frac{1}{2}}x)_\infty(-q^{\frac{1}{2}}x^{-1})_\infty$$

to obtain

$$\boxed{Z_{S^3}^{SQED} = \sum_i^{N_f} G_{\text{1-loop}}^{(i)}\left|\left|\mathcal{G}_{cl}^{(i)}\mathcal{Z}_V^{(i)}\right|\right|_S^2}$$

Finally we can factorize the 1-loop part too using that

$$e^{\frac{i\pi}{2}(iQ/2+z)^2}s_b(iQ/2+z) = \left|\left|(qe^{2\pi z/\omega_1};q)_\infty\right|\right|_S^2,$$

and obtain the block factorized form:

$$\boxed{Z_{S^3}^{SQED} = \sum_i^{N_f}\left|\left|\mathcal{B}_{(i)}^{3d}\right|\right|_S^2, \qquad \mathcal{B}_{(i)}^{3d} := \mathcal{G}_{cl}^{(i)}\mathcal{G}_{\text{1-loop}}^{(i)}\mathcal{Z}_V^{(i)}}$$

Blocks are expressed in terms of periodic variables $e^{2\pi z/\omega_1}$, $e^{2\pi z/\omega_2}$, (invariant under shift $z \to z + k\omega_i$).

- In the semiclassical limit, $q = e^{\beta\epsilon}$, $\epsilon \to 0$, finite β, we find:

$$\mathcal{B}^{(i)} \overset{\epsilon \to 0}{\sim} \exp\left[\frac{1}{\epsilon}\widetilde{W}|_{s^{(i)}(x)}\right]$$

  where $\widetilde{W}|_{s^{(i)}(x)}$ is the twisted superpotential evaluated on the i-th SUSY vacuum.

- Blocks form a basis of solutions to a system of difference equations, in this case basic hypergeometric operator.

- The factorization is not unique, blocks are defined up to q-constants $c(x;q)$ satisfying:

$$c(qx;q) = c(x;q)\,, \qquad \left|\left|c(x;q)\right|\right|_S^2 = 1$$

  Notice that multiplication by $c(x;q)$ does not change the semiclassical limit (asymptotics of solutions).

$\mathcal{N}=2$ theory on $S^2 \times_q S^1$

Computes the (generalised) super-conformal-index [Imamura-Yokoyama],[Kapustin-Willet],[Dimofte-Gukov-Gaiotto].

SQED with fugacities:

$$
\begin{aligned}
&(\phi_i, r_i), \qquad i = 1, \cdots N_f, \qquad (+)\ \text{flavor} \quad U(1)^{N_f}\,,\\
&(\xi_i, l_i), \qquad i = 1, \cdots N_f, \qquad (-)\ \text{flavor} \quad U(1)^{N_f}\,,\\
&(\omega, n), \qquad \text{topological} \quad U(1)\,,\\
&(t, s), \qquad \text{gauged} \quad U(1)\,.
\end{aligned}
$$

$$Z_{S^2\times S^1} = \sum_{s\in\mathbb{Z}} \int \frac{dt}{2\pi i t} t^n \omega^s \prod_{j=1}^{N_f} \chi(t\phi_j, s+r_j) \prod_{k=1}^{N_f} \chi(t^{-1}\xi_k^{-1}, -s-l_k)\,.$$

The 1-loop contribution of a chiral multiplet is:

$$\chi(\zeta, m) = (q^{1/2}\zeta^{-1})^{-m/2} \prod_{k=0}^{\infty} \frac{(1-q^{l+1}\zeta^{-1}q^{-m/2})}{(1-q^{l}\zeta q^{-m/2})}$$

Higgs-branch-like factorized form [Beem-Dimofte-S.P.],[Dimofte-Gaiotto-Gukov]

$$Z^{SQED}_{S^2\times S^1} = \sum_{i=1}^{N_f} G^{(i)}_{cl}\; G^{(i)}_{\text{1-loop}} \left|\left|\mathcal{Z}^i_V\right|\right|^2_{id}$$

- $G^{(i)}_{cl} G^{(i)}_{\text{1-loop}}$ are evaluated on the i-th SUSY vacuum.
- The *id*-pairing is defined by $\left|\left|f(x;q)\right|\right|^2_{id} := f(x;q)f(\tilde{x};\tilde{q})$ with:

$$x_i = \phi_i q^{r_i/2}, \qquad \tilde{x}_i = \phi_i^{-1} q^{r_i/2}, \qquad y_i = \xi_i q^{l_i/2}, \qquad \tilde{y}_i = \xi_i^{-1} q^{l_i/2},$$
$$z = \omega q^{n/2}, \qquad \tilde{z} = \omega^{-1} q^{n/2}, \qquad \tilde{q} = q^{-1}$$

As before we can factorize the classical and 1-loop term and obtain:

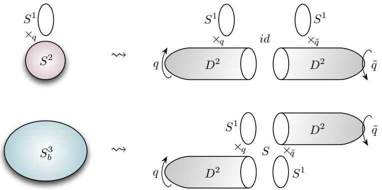
$$\boxed{Z^{SQED}_{S^2\times S^1} = \sum_{i=1}^{N_f} G^{(i)}_{\text{1-loop}} \left|\left|\mathcal{G}^{(i)}_{cl}\, \mathcal{Z}^i_V\right|\right|^2_{id} = \sum_i^{N_f} \left|\left|\mathcal{B}^{3d}_{(i)}\right|\right|^2_{id}}$$

to summarize:

$$Z_{S^3} = \sum_i^{N_f} G_{\text{1-loop}}^{S^3,(i)} \left|\left| \mathcal{G}_{cl}^{(i)} \mathcal{Z}_V^{(i)} \right|\right|_S^2 = \sum_i^{N_f} ||\mathcal{B}_{(i)}^{3d}||_S^2$$

$$Z_{S^2\times S^1} = \sum_i^{N_f} G_{\text{1-loop}}^{S^2\times S^1,(i)} \left|\left| \mathcal{G}_{cl}^{(i)} \mathcal{Z}_V^{(i)} \right|\right|_{id}^2 = \sum_i^{N_f} ||\mathcal{B}_{(i)}^{3d}||_{id}^2$$

Same blocks with different pairing gives $Z_{S^3}, Z_{S^2\times S^1}$ "like" $S^3, S^2\times S^1$ are obtained by gluing solid tori with $S, id \in SL(2,\mathbb{Z})$.



Holomorphic blocks $\mathcal{B}^{3d}$ are Melvin cigar $D\times_q S^1$ partition functions.

[Beem-Dimofte-S.P.]

Observe the following flop symmetry of SQED partition functions:

$$Z_{S^3}^{SQED} = \int dx\ e^{2\pi i x \xi} \prod_{j,k}^{N_f} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

$$\text{is invariant under}: \ m_i \leftrightarrow -\tilde{m}_k \text{ and } \xi \leftrightarrow -\xi$$

exchanges phase I and phase II

$$Z_{S^2\times S^1} = \sum_{s\in\mathbb{Z}} \int \frac{dt}{2\pi i t} t^n \omega^s \prod_{j=1}^{N_f} \chi(t\phi_j, s + r_j) \prod_{k=1}^{N_f} \chi(t^{-1}\xi_k^{-1}, -s - l_k)$$

$$\text{is invariant under}: \ \omega \leftrightarrow \omega^{-1},\ n \leftrightarrow -n,\ \phi_j \leftrightarrow \xi_j^{-1},\ r_j \leftrightarrow -l_j$$

exchanges phase I and phase II

FLOP SYMMETRY is rather trivial on the Coulomb branch; but on the Higgs branch it implies non-trivial relations between blocks (analytic continuation $z \to z^{-1}$ from phases I to phase II):

$$Z^{I}_{S^2\times S^1} = \sum_{i}^{N_f} G^{(i),I}_{\text{1-loop}} \left|\left| \mathcal{G}^{(i),I}_{cl} \, \mathcal{Z}^{(i),I}_{V} \right|\right|^2_{id} =$$

$$= \sum_{i}^{N_f} G^{(i),II}_{\text{1-loop}} \left|\left| \mathcal{G}^{(i),II}_{cl} \, \mathcal{Z}^{(i),II}_{V} \right|\right|^2_{id} = Z^{II}_{S^2\times S^1}$$

$$Z^{I}_{S^3} = \sum_{i}^{N_f} G^{(i),I}_{\text{1-loop}} \left|\left| \mathcal{G}^{(i),I}_{cl} \, \mathcal{Z}^{(i),I}_{V} \right|\right|^2_{S} =$$

$$= \sum_{i}^{N_f} G^{(i),II}_{\text{1-loop}} \left|\left| \mathcal{G}^{(i),II}_{cl} \, \mathcal{Z}^{(i),II}_{V} \right|\right|^2_{S} = Z^{II}_{S^3}$$

this structure is reminiscent of crossing symmetry in 2d CFT correlators.

$\mathcal{N}=1$ theories on S^5

Localisation on $\omega_1^2|z_1|^2+\omega_2^2|z_2|^2+\omega_3^2|z_3|^2=1$ yields:

$$Z_{S^5}=\int d\sigma\ Z_{cl}(\sigma,\tau)\ Z_{\text{1-loop}}(\sigma,\vec{m})\ \left|\left|\mathcal{Z}^{5d}_{inst}\right|\right|^3_{SL(3,Z)}$$

[Kallen-Zabzine],[Kallen-Qui-Zabzine],[Hosomichi-Seong-Terashima],[Imamura],
[Lockhart-Vafa],[Kim-Kim-Kim],[Haghighat-Iqbal-Kozcaz-Lockhart-Vafa]

- $\mathbb{R}^4\times S^1$ instantons $\mathcal{Z}^{5d}_{inst}(e^{2\pi\sigma/e_3},e^{2\pi\vec{m}/e_3};q,t)$ are localized at fixed points of the Hopf fibration and are paired as:

$$\left|\left|f(e^{2\pi z/e_3};q,t)\right|\right|^3_{SL(3,Z)}:=\prod_{k=1}^{3}f(e^{2\pi z/e_3};q,t)_k\,,\quad q=e^{2\pi i e_1/e_3},t=e^{2\pi i e_2/e_3}$$

$$k=\quad 1:(e_1,e_2,e_3)=(\omega_3,\omega_2,\omega_1)\,,\quad 2:(e_1,e_2,e_3)=(\omega_1,\omega_3,\omega_2)\,,$$
$$3:(e_1,e_2,e_3)=(\omega_1,\omega_2,\omega_3)$$

- 1-loop contributions are:

$$Z^{\text{vect}}_{\text{1-loop}}(\sigma)=\prod_{\alpha>0}S_3(i\alpha(\sigma))S_3(-i\alpha(\sigma))\qquad Z^{\text{hyper}}_{\text{1loop}}(\sigma,m)=\prod_{\rho\in R}S_3\left(i(\rho(\sigma)+m)+\frac{E}{2}\right)^{-1}$$

$$S_3(x)=\prod_{i,j,k}(i\omega_1+j\omega_2+k\omega_3+x)(i\omega_1+j\omega_2+k\omega_3+E-x)\,,\qquad E=\omega_1+\omega_2+\omega_3$$

We factorize the classical part (Yang-Mills and Chern-Simons terms):

$$Z_{cl}(\sigma,\tau) = \left|\left|\mathcal{Z}_{cl}\right|\right|^3_{SL(3,Z)}$$

using that [Felder-Varchenko]:

$$e^{-\frac{2\pi i}{3!}B_{33}(x,\vec{\omega})} = \left|\left|\Gamma_{q,t}(x/e_3)\right|\right|^3_{SL(3,Z)}, \qquad \Gamma_{q,t}(z) = \frac{(e^{-2\pi iz}\,q\,t\,;q,t)}{(e^{2\pi iz};q,t)}$$

and obtain:

$$\boxed{Z_{S^5} = \int\ d\sigma\ Z_{\text{1-loop}}(\sigma,\vec{m})\ \left|\left|\mathcal{F}\right|\right|^3_{SL(3,Z)}, \qquad \mathcal{F} := \mathcal{Z}_{\text{cl}}\mathcal{Z}^{5d}_{\text{inst}}}$$

we can factorize the 1-loop part as well:

$$Z_{\text{1-loop}}(\sigma, \vec{m}) = \left|\left|\mathcal{Z}_{\text{1-loop}}\right|\right|^3_{SL(3,Z)}$$

using that:

$$S_3(iz) = e^{-\frac{\pi i}{3!}B_{33}(iz)}\left|\left|(e^{-\frac{2\pi}{e_3}z}; q, t)\right|\right|^3_{SL(3,Z)}$$

and obtain the block factorized form which respects periodicity (invariance under shift $z \to z + ik\omega_i$) in each sector:

$$\boxed{Z_{S^5} = \int \, d\sigma \, \left|\left|\mathcal{B}^{5d}\right|\right|^3_{SL(3,Z)}, \qquad \mathcal{B}^{5d} := \mathcal{Z}_{\text{cl}} \, \mathcal{Z}_{\text{1-loop}} \, \mathcal{Z}^{5d}_{\text{inst}}}$$

For example blocks of the $SU(2)$, $N_f = 4$ theory are:

$$\mathcal{B}^{5\text{d}} = \frac{\Gamma_{q,t}\left(\frac{\pm i\sigma + 1/g^2 - \sum_f im_f/2 + \kappa}{e_3}\right)}{\Gamma_{q,t}\left(\frac{\pm i\sigma + \kappa}{e_3}\right)} \cdot \frac{(e^{\frac{2\pi i}{e_3}[\pm 2i\sigma]}; q, t)}{\prod_f (e^{\frac{2\pi i}{e_3}[\pm i\sigma + im_f]}; q, t)} \cdot \mathcal{Z}^{5d}_{inst}$$

where κ keeps track of the ambiguity of the factorization.

$\mathcal{N}=1$ theories on $S^4 \times S^1$

Coulomb branch localization yields: [Kim-Kim-Lee],[Terashima],[Iqbal-Vafa]

$$Z_{S^4\times S^1} = \int d\sigma \; Z_{\text{1-loop}}(\sigma, \vec{m}) \; \left|\left|\mathcal{Z}^{5d}_{inst}\right|\right|^2_{id}$$

- $\mathbb{R}^4 \times S^1$ instantons $\mathcal{Z}^{5d}_{inst}(e^{2\pi\sigma/e_3}, e^{2\pi\vec{m}/e_3}; q, t)$ are localized at N and S poles and are paired as:

$$\left|\left|f(e^{2\pi z/e_3}; q, t)\right|\right|^2_{id} := \prod_{k=1}^{2} f(e^{2\pi z/e_3}; q, t)_k \,, \quad q = e^{2\pi i e_1/e_3}, t = e^{2\pi i e_2/e_3}$$

$$k = \;\; 1 : (e_1, e_2, e_3) = (1/b_0, b_0, 2\pi i/\beta)\,, \quad 2 : (e_1, e_2, e_3) = (1/b_0, b_0, -2\pi i/\beta)$$

- 1-loop contributions can be re-written as:

$$Z^{\text{vect}}_{\text{1-loop}} = \prod_{\alpha>0} \Upsilon^\beta\left(i\alpha(\sigma)\right)\Upsilon^\beta\left(-i\alpha(\sigma)\right)\,, \quad Z^{\text{hyper}}_{\text{1-loop}} = \prod_{\rho\in R} \Upsilon^\beta\left(i(\rho(\sigma)+m) + \frac{Q_0}{2}\right)^{-1}$$

with $Q_0 = b_0 + 1/b_0$ and

$\Upsilon^\beta(X) \propto \prod_{n_1,n_2} \sinh\frac{\beta}{2}\left(X + n_1 b_0 + \frac{n_2}{b_0}\right)\sinh\frac{\beta}{2}\left(-X + (n_1+1)b_0 + \frac{(n_2+1)}{b_0}\right).$

now since $\left|\left|\Gamma_{q,t}(z)\right|\right|^2_{id} = 1$ we have $\left|\left|\mathcal{Z}_{cl}\right|\right|^2_{id} = 1$ we can write

$$\boxed{Z_{S^4\times S^1} = \int da\; Z_{\text{1-loop}}(a,\vec{m})\; \left|\left|\mathcal{F}\right|\right|^2_{id}}$$

where $\mathcal{F}$ is the same block appearing in Z_{S^5}.

Again we can factorize the 1-loop term too and obtain:

$$\boxed{Z_{S^4\times S^1} = \int d\sigma\; \left|\left|\mathcal{B}^{5d}\right|\right|^2_{id}}$$

with the same holomorphic blocks $\mathcal{B}^{5d}$ appearing in Z_{S^5}.

to summarize:

$$\boxed{Z_{S^5} = \int \; d\sigma \; Z^{S^5}_{\text{1-loop}} \; \Big|\Big|\mathcal{F}\Big|\Big|^3_{SL(3,Z)} = \int \; d\sigma \; \Big|\Big|\mathcal{B}^{5d}\Big|\Big|^3_{SL(3,Z)}}$$

$$\boxed{Z_{S^4\times S^1} = \int \; d\sigma \; Z^{S^4\times S^1}_{\text{1-loop}} \; \Big|\Big|\mathcal{F}\Big|\Big|^2_{id} = \int d\sigma \; \Big|\Big|\mathcal{B}^{5d}\Big|\Big|^2_{id}}$$

- Respecting periodicity we find universal blocks $\mathcal{B}^{5d}$.
- The intermediate factorization in terms of $\mathcal{F}$ will be more convenient for the q-CFT interpretation.

Degeneration of 5d partition function

For special values of mass parameters integrals defining partition functions *localize* to discrete sums and satisfy difference equations.

Poles in $Z^{S^5}_{\text{1-loop}}$ and $Z^{S^4\times S^1}_{\text{1-loop}}$ move and pinch the integration contour; the (meromorphic) continuation of partition functions requires taking residues of poles crossing the integration path.

Comments:

- A similar mechanisms reduces non-degenerate Liouville correlators to degenerate ones, which satisfy differential equations.
- Analogy with the AGT set-up suggests that the degenerate sector of the CFT corresponds to codimension two defects on the gauge theory side. This is the case also for the superconformal 4d index. [Gaiotto-Rastelli-Razamat]

Consider the $SU(2)$, $N_f = 4$ theory on S^5. The poles structure of $Z^{S^5}_{\text{1-loop}}$ is such that:

$$\text{for} \quad m_1 + m_2 = -i\omega_3 \qquad \text{the integral localizes} \qquad \int d\sigma \Rightarrow \sum_{\{\sigma_1,\sigma_2\}}$$

When evaluated on $\sigma = \{\sigma_1, \sigma_2\}$, instantons degenerate to vortices:

$$\mathcal{Z}^{5d}_{inst,1} = \sum_{Y_1,Y_2}(\cdots) \to \sum_{0,1^n}(\cdots) = \mathcal{Z}^{(i)}_V\,, \qquad \mathcal{Z}^{5d}_{inst,2} = \sum_{W_1,W_2}(\cdots) \to \sum_{0,n}(\cdots) = \tilde{\mathcal{Z}}^{(i)}_V\,,$$

$$\mathcal{Z}^{5d,III}_{inst,3} = \sum_{X_1,X_2}(\cdots) \to \sum_{0,0}(\cdots) = 1$$

and:

$$Z^{SCQCD}_{S^5} = \int d\sigma\, Z^{S^5}_{\text{1-loop}} \left|\left|\mathcal{Z}_{cl}\mathcal{Z}^{5d}_{inst}\right|\right|^3_{SL(3,Z)} \Rightarrow \sum_i^2 G^{S^3,(i)}_{1loop} \left|\left|\mathcal{G}^{(i)}_{cl}\,\mathcal{Z}^{(i)}_V\right|\right|^2_S = Z^{SQED}_{S^3}$$

An identical degeneration works for permutations of $\omega_1, \omega_2, \omega_3$, corresponding to the three big S^3 inside S^5.

A similar mechanisms for $m_1 + m_2 = -ib_0$ leads to

$$Z^{SCQCD}_{S^4\times S^1} \Rightarrow Z^{SQED}_{S^2\times S^1}$$

Towards a q-CFT

so far we have seen that

- 3d gauge theory flop symmetry $\Leftrightarrow$ crossing symmetry of CFT correlators.
- 5d $\to$ 3d degeneration $\Leftrightarrow$ analytical continuation of momenta of primary operators to degenerate values in CFT correlators.
- 5d instantons $\Leftrightarrow$ deformed Virasoro $\mathcal{V}ir_{q,t}$ blocks (numerous "5d-AGT" results). [Awata-Yamada],[many others]

We will now construct correlation functions with underlying deformed Virasoro symmetry and try to map them to 3d&5d partition functions.

q-deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t and generators T_n with $n \in \mathbb{Z}$

[Shiraishi-Kubo-Awata-Odake],[Lukyanov-Pugai],[Frenkel-Reshetikhin],[Jimbo-Miwa]

$$\begin{aligned}[T_n\,,\,T_m] \;&=\; -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}\,T_{m+l} - T_{m-l}\,T_{n+l}) \\ &\quad - \frac{(1-q)(1-t^{-1})}{1-p}((q/t)^n - (q/t)^{-n})\delta_{m+n,0}\end{aligned}$$

where $f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right]$

- For $t = q^{-b_0^2}$ and $q \to 1$, $\mathcal{V}ir_{q,t}$ reduces to Virasoro.
- chiral blocks with degenerate primaries (singular states in the Verma module) satisfy difference equations.

  [Awata-Kubo-Morita-Odake-Shiraishi], [Awata-Yamada],[Schiappa-Wyllard]

q-deformed Bootstrap Approach:

We will construct q-correlators using the conformal bootstrap approach:

3-point function is derived exploiting symmetries, without using the Lagrangian. [Belavin-Polyakov-Zamolodchikov],[Teschner]

Consider 4-point function with a degenerate insertion

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z, \tilde{z}) V_{\alpha_1}(0) \rangle \sim G(z, \tilde{z})$$

take $V_{\alpha_2}(z, \tilde{z})$ to have a null state at level 2, then

$$D(A, B; C; q; z) G(z, z) = 0 \,, \qquad D(\tilde{A}, \tilde{B}; \tilde{C}; \tilde{q}; \tilde{z}) G(z, \tilde{z}) = 0 \,,$$

where $D(A, B; C; q; z)$ is the q-hypergeometric operator.

$G(z, \tilde{z})$ is a bilinear combination of solutions of the q-hypergeometric eq.

Around $z = 0$

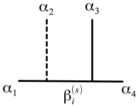
$$I_1^{(s)} = {}_2\Phi_1(A, B; C; z)\,, \qquad I_2^{(s)} = \frac{\theta(q^2 C^{-1} z^{-1}; q)}{\theta(qC^{-1}; q)\theta(qz^{-1}; q)}\, {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2 C^{-1}; z)$$

For $q \to 1$ becomes the undeformed s-channel basis.

s-channel correlator:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(s)} K_{ij}^{(s)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_*^2 = \sum_i$$



$K_{ij}^{(s)}$ is diagonal with elements related to 3-point functions

$$K_{ii}^{(s)} = C(\alpha_4, \alpha_3, \beta_i^{(s)})\, C(Q_0 - \beta_i^{(s)}, -b_0/2, \alpha_1)\,, \quad \beta_i^{(s)} = \alpha_1 \pm \frac{b_0}{2}\,, \;\; i = 1, 2$$

For the moment assume generic pairing $\left|\left|(\cdots)\right|\right|_*^2$.

Around $z = \infty$

$$I_1^{(u)} = \frac{\theta(qA^{-1}z^{-1};q)}{\theta(A^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2z^{-1})\,,$$

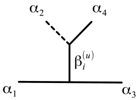
$$I_2^{(u)} = \frac{\theta(qB^{-1}z^{-1};q)}{\theta(B^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2z^{-1})$$

For $q \to 1$ limit becomes the undeformed u-channel basis.

u-channel correlator:

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(r)V_{\alpha_2}(z)V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(u)} K_{ij}^{(u)} I_j^{(s)}$$

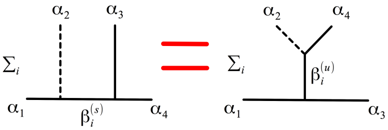
$$= \sum_{i=1}^{2} K_{ii}^{(u)} \left|\left| I_i^{(u)} \right|\right|_*^2 = \sum_i$$



$K_{ij}^{(u)}$ is diagonal with elements related to 3-point functions

$$K_{ii}^{(u)} = C(\alpha_1, \alpha_3, \beta_i^{(u)})\, C(Q_0 - \beta_i^{(u)}, -b_0/2, \alpha_4)\,, \quad \beta_i^{(u)} = \alpha_4 \pm \frac{b_0}{2}\,, \quad i = 1, 2$$

impose crossing symmetry



$$\boxed{K_{11}^{(s)} \left|\left| I_1^{(s)} \right|\right|_*^2 + K_{22}^{(s)} \left|\left| I_2^{(s)} \right|\right|_*^2 = K_{11}^{(u)} \left|\left| I_1^{(u)} \right|\right|_*^2 + K_{22}^{(u)} \left|\left| I_2^{(u)} \right|\right|_*^2}$$

analytic continuation $I_i^{(s)} = \sum_{j=1}^{2} M_{ij} I_j^{(u)}$, $\tilde{I}_i^{(s)} = \sum_{j=1}^{2} \tilde{M}_{ij} \tilde{I}_j^{(u)}$ yields:

$$\boxed{\sum_{k,l=1}^{2} K_{kl}^{(s)} \tilde{M}_{ki} M_{lj} = K_{ij}^{(u)}}$$

Solving these equations we can determine 3-point functions. But we need to specify the pairing $\left|\left|(\cdots)\right|\right|_*^2$ $\rightarrow$ use 3d gauge theory pairings!

id-pairing *q*-CFT

Now assume that chiral blocks are paired as:

$$\Big|\Big| f(x;q) \Big|\Big|^2_{id} = f(x;q) f(\tilde{x};\tilde{q}) \,.$$

with:

$$x = e^{\beta X}\,, \quad \tilde{x} = e^{-\beta X}\,, \quad \tilde{q} = q^{-1}$$

The bootstrap equations are solved by:

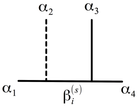
$$C_{id}(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{\Upsilon^\beta(2\alpha_T - Q_0)} \prod_{i=1}^{3} \frac{\Upsilon^\beta(2\alpha_i)}{\Upsilon^\beta(2\alpha_T - 2\alpha_i)}$$

where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$, $Q_0 = b_0 + 1/b_0$ and

$$\Upsilon^\beta(X) \propto \prod_{n_1,n_2=0}^{\infty} \sinh\left[\frac{\beta}{2}\left(X + n_1 b_0 + \frac{n_2}{b_0}\right)\right] \sinh\left[\frac{\beta}{2}\left(-X + (n_1+1)b_0 + \frac{(n_2+1)}{b_0}\right)\right]$$

SQED $N_f = 2$ on $S^2 \times S^1 \Leftrightarrow$ *id*-pairing 4-point degenerate correlator

$$Z^{SQED}_{S^2\times S^1} = \sum_{i=1}^{2} G^{(i),I}_{1loop} \left|\left| \mathcal{G}^{(i),I}_{cl} \mathcal{Z}^{(i),I}_{V} \right|\right|^2_{id} \sim \sum_{i=1}^{2} K^{(s)}_{ii} \left|\left| I^{(s)}_{i} \right|\right|^2_{id} = \sum_{i}$$



dictionary:

$$z_{CFT} \sim z_{gauge}\,, \quad q = e^{\beta/b_0}\,, \quad \alpha_2 = -b_0/2$$

$$\alpha_1 = \tfrac{Q_0}{2} + i\tfrac{\Phi_1-\Phi_2}{2}\,, \quad \alpha_3 = \tfrac{b_0}{2} - i\tfrac{\Xi_1+\Xi_2-\Phi_1-\Phi_2}{2}\,, \quad \alpha_4 = \tfrac{Q_0}{2} - i\tfrac{\Xi_1-\Xi_2}{2}\,,$$

where $\phi_i = e^{i\beta\,\Phi_i}$, $\xi_i = e^{i\beta\,\Xi_i}$.

- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- $\beta \to 0$ limit recovers [Doroud-Gomis-LeFloch-Lee]
  - CFT: $\mathcal{V}ir_{q,t} \to$ Virasoro, we recover Liouville theory results
  - gauge: $S^2 \times S^1$ partition function reduces to S^2 partition function

S-pairing q-CFT

Now assume that chiral blocks are paired as:

$$\Big|\Big|f(x;q)\Big|\Big|_S^2 = f(x;q)f(\tilde{x};\tilde{q})\,.$$

where

$$x = e^{2\pi i X/\omega_2}\,, \quad \tilde{x} = e^{2\pi i X/\omega_1}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}\,, \quad \tilde{q} = e^{2\pi i \frac{\omega_2}{\omega_1}}$$

The bootstrap equations are solved by:

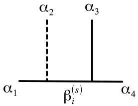
$$C_S(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{S_3(2\alpha_T - E)} \prod_{i=1}^{3} \frac{S_3(2\alpha_i)}{S_3(2\alpha_T - 2\alpha_i)}$$

where $E = \omega_1 + \omega_2 + \omega_3$ and

$$S_3(X) \propto \prod_{n_1,n_2,n_3=0} (\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + X)\,(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + E - X)$$

$\boxed{\text{SQED } N_f = 2 \text{ on } S^3_b \Leftrightarrow S\text{-pairing 4-point degenerate correlator}}$

$$Z_{S^3}^{SQED} = \sum_{i=1}^{2} G_{1loop}^{(i),I} \left|\left| \mathcal{G}_{cl}^{(i),I} \mathcal{Z}_V^{(i),I} \right|\right|_S^2 \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_S^2 = \sum_i$$



dictionary:

$$\alpha_2 = -\omega_3/2, \qquad \omega_1 = b, \qquad \omega_2 = \frac{1}{b}, \qquad z_{CFT} \sim z_{gauge}$$

$$\alpha_1 = \frac{E}{2} + i\frac{m_1 - m_2}{2}\,, \quad \alpha_3 = \frac{\omega_3}{2} - i\frac{\tilde{m}_1 + \tilde{m}_2 - m_1 - m_2}{2}\,, \quad \alpha_4 = \frac{E}{2} - i\frac{\tilde{m}_1 - \tilde{m}_2}{2}\,,$$

- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- three possibilities:

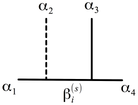
$$\alpha_2 = -\omega_k/2, \qquad b = \omega_i, \qquad \frac{1}{b} = \omega_j\,, \quad i \neq j \neq k = 1, 2, 3\,.$$

corresponding to the three big deformed S^3 inside a deformed S^5.

so far:

3d gauge theory partition functions $\Leftrightarrow$ q-CFT degenerate correlators

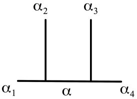
$$Z^{SQED}_{S,id} = \sum_{i=1}^{2} G^{(i),I}_{1loop} \left|\left| \mathcal{G}^{(i),I}_{cl} \, \mathcal{Z}^{(i),I}_{V} \right|\right|^2_{S,id} \sim \sum_{i=1}^{2} K^{(s)}_{ii} \left|\left| I^{(s)}_i \right|\right|^2_{S,id} = \sum_i$$



Let's now consider non-degenerate correlators

Example:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{S,id} = \int d\alpha$$



$$= \int d\alpha \; C_{S,id} \, C_{S,id} \, (\mathrm{Conf.Blocks})$$

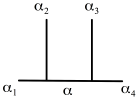
the degeneration mechanism suggests that

5d gauge theory partition functions $\Leftrightarrow$ q-CFT non-degenerate correlators

$Z_{S^4\times S^1}$ is captured by non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and *id*-pairing 3-point function.

Example: SCQCD, $SU(2)$, $N_f = 4 \Leftrightarrow$ 4-point correlator

$$Z^{SCQCD}_{S^4\times S^1} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{id} = \int d\alpha$$



- 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:
  [Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$\mathcal{Z}^{5d,SCQCD}_{inst} = \mathrm{F}^{qt}_{\alpha_1\alpha_2\alpha\alpha_3\alpha_4}(z)$$

- 1-loop vs 3-point function:

$$Z^{\text{vect}}_{\text{1loop}}(\sigma) \prod_{i=1}^{4} Z^{\text{hyper}}_{\text{1loop}}(\sigma, m_i) = C_{id}(\alpha_1, \alpha_2, \alpha) C_{id}(Q_0 - \alpha, \alpha_3, \alpha_4)$$

dictionary:

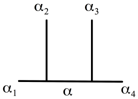
$$\alpha = i\sigma + \frac{Q_0}{2}\,,\ \ \alpha_1 \pm \alpha_2 = im_{1,2} + Q_0\,, \quad \alpha_3 \pm \alpha_4 = im_{3,4} + Q_0$$

$\rightarrow$ use C_{id} since $S^2 \times S^1$ is a codim-2 defect in $S^4 \times S^1$ (cf.[Iqbal-Vafa])

Z_{S^5} is captured by non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and S-pairing 3-point function.

Example: SCQCD, $SU(2)$, $N_f = 4 \Leftrightarrow$ 4-point correlator

$$Z_{S^5}^{SCQCD} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_S = \int d\alpha$$



- 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:
  [Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$\mathcal{Z}_{inst}^{5d,SCQCD} = \mathrm{F}^{qt}_{\alpha_1 \alpha_2 \alpha \alpha_3 \alpha_4}$$

- 1-loop vs 3-point function:

$$Z_{\text{1loop}}^{\text{vect}}(\sigma) \prod_{i=1}^{4} Z_{\text{1-loop}}^{\text{hyper}}(\sigma, m_i) = C_S(\alpha_1, \alpha_2, \alpha) C_S(E - \alpha, \alpha_3, \alpha_4)$$

with dictionary: $\alpha = i\sigma + \frac{E}{2}\,, \quad \alpha_1 + \alpha_2 = im_1 + E\,, \quad \alpha_1 - \alpha_2 = im_2$
$\alpha_3 + \alpha_4 = im_3 + E\,, \quad \alpha_3 - \alpha_4 = im_4\,.$

$\rightarrow$ use C_S since S^3 is a codimension two defect in S^5 (cf.[Lockhart-Vafa])

Fusion relations for q-CFT

3-point functions define the fusion rules of two primaries for $z_1 \to z_2$:

$$V_{\alpha_2}(z_2)V_{\alpha_1}(z_1) \simeq \int \mathrm{d}\alpha\ C_S(\alpha_2, \alpha_1, \alpha) V_\alpha(z_2)$$

when we analytically continue from $\mathrm{Re}(\alpha_1) = \mathrm{Re}(\alpha_2) = E/2$ to degenerate values:

$$\alpha_2 = -\frac{n_1\omega_1 + n_2\omega_2 + n_3\omega_3}{2} = -\frac{n \cdot \omega}{2}$$

poles in $C_S(\alpha_2, \alpha_1, \alpha)$ pinch the integration contour and the OPE is defined by the sum over the residues from poles located at

$$\alpha^* = \alpha_1 - s \cdot \omega/2; \quad s_k = -n_k + 2j; \quad j \in \{0, 1, \ldots, n_k\}$$

for a total of $(n_1 + 1)(n_2 + 1)(n_3 + 1)$ contributions.

Knowing the fusion rules we can evaluate the four-point correlator:

$$\langle V_{\alpha_1}(0) V_{-\frac{n\cdot\omega}{2}}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty)\rangle = \sum_{\{\alpha^*\}\in OPE} Res[CC] \mathcal{F}_1^{n_3 n_2} \mathcal{F}_2^{n_1 n_3} \mathcal{F}_3^{n_1 n_2}$$

where $\mathcal{F}^{n_i n_j}$ contains sums over Hook tableaux (n_i, n_j).

The simplest case corresponding to $n = (0, 0, 1)$ yields:

$$\langle V_{\alpha_1}(0) V_{-\frac{\omega_3}{2}}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty)\rangle = \sum_{\alpha=\alpha_1 \pm \frac{\omega_3}{2}} Res[CC]\ \mathcal{F}_1^{10} \mathcal{F}_2^{01} \mathcal{F}_3^{00} =$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_S^2 = Z_{S^3}^{SQED}$$

Blocks $\mathcal{F}^{n_i n_j}$, corresponding to higher degenerates, should be related to non-elementary codimension-two defect operators (cf.[Dimofte-Gukov-Hollands])

Integrability

Knowing 3-point functions we can compute reflection coefficients:

$$R^{id}(\alpha_1) = \frac{C_{id}(Q-\alpha_1,\alpha_2,\alpha_3)}{C_{id}(\alpha_1,\alpha_2,\alpha_3)}\,, \qquad R^{S}(\alpha_1) = \frac{C_{S}(E-\alpha_1,\alpha_2,\alpha_3)}{C_{S}(\alpha_1,\alpha_2,\alpha_3)}$$

and try to connect them to scattering matrices of spin-chains built from Jost functions appearing in the plane-wave asymptotics of the scattering wave function.

$\cdots$ our current understanding after searching the literature [Gerasimov-.Kharchev-Marshakov-Mironov-Morozov-Olshanetsky], [Takhtajan-Faddeev],[Freund-Zabrodin],[Babujian-Tsvelik],[Kirillov-Reshetikhin], [Doikou-Nepomechie],[Freund-Zabrodin],[Davies-Foda-Jimbo-Miwa-Nakayashiki], [Freund-Zabrodin], [Faddeev-Takhtajan],[etc.]

goes as follows $\rightarrow$

XYZ

$$J(u)=\Pi_{k=0}\frac{\Gamma_q(iu+rk)\Gamma_q(iu+rk+r+1)}{\Gamma_q(iu+rk+1/2)\Gamma_q(iu+rk+r+1/2)}$$

$$q=e^{-4\gamma},\ r=\frac{-i\pi\tau}{2\gamma}$$

$q\rightarrow 1,\ r=const$

$\tau\rightarrow i\infty$

XXZ *ferro*

$J(u)\sim\Gamma_2$

affinization

$$R^S(\alpha_1)=\frac{C_S(E-\alpha_1,\alpha_2,\alpha_3)}{C_S(\alpha_1,\alpha_2,\alpha_3)}$$

XXZ *anti-ferro*

$J(u)\sim\Gamma_q$

affinization

$$R^{id}(\alpha_1)=\frac{C_{id}(Q_0-\alpha_1,\alpha_2,\alpha_3)}{C_{id}(\alpha_1,\alpha_2,\alpha_3)}$$

$\omega_1\rightarrow\infty,\ \omega_2=\frac{1}{\omega_3}=b_0$

$\beta\rightarrow 0$

$$R^{Liouville}(\alpha_1)=\frac{C^{DOZZ}(Q-\alpha_1,\alpha_2,\alpha_3)}{C^{DOZZ}(\alpha_1,\alpha_2,\alpha_3)}$$

affinization

Liouville mini-super-space: $J(u)\sim\Gamma$

XXX *ferro*

Conclusions & outlook

Hints of a q-CFT-like structure in 5d and 3d partition functions.

- Degenerate correlators/3d partition functions are crossing-symmetry/flop invariant; Is there crossing-symmetry for non-degenerate correlators what is its 5d gauge theory meaning?
- Consider other pairings $\left|\left|(\cdots)\right|\right|_*^2$ and other geometries.
- Use q-CFT to study gauge theory. For example construct q-CFT Verlinde loop operators and study their gauge theory meaning.
- Explore the integrable structure of q-CFT correlators.