Symmetries of black holes and index theory

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J. Grover, J. B. Gutowski, GP, and W. A. Sabra arXiv:1303.0853;
J. B. Gutowski, GP, arXiv:1303.0869;
U. Gran, J. B. Gutowski, GP, arXiv:1306.5765

Horizon symmetry enhancement

- Conjecture 1: The number of Killing (parallel) spinors N of smooth horizons is

$$N = 2N_- + \mathrm{Index}(D_E)$$

where $N_- \geq 0$, D_E is a Dirac operator twisted by E defined on the horizon sections $\mathcal{S}$. E depends on the gauge symmetries of supergravity.

- Conjecture 2: Smooth horizons with non-trivial fluxes and $N_- \neq 0$ admit a $\mathfrak{sl}(2, \mathbb{R})$ symmetry subalgebra

The conjectures have been proved in the following cases.

- D=5 minimal gauged, D=11, IIB, heterotic and IIA (in progress) supergravities.

Remarks

- If the index vanishes, which is the case for non-chiral theories, then N is even. In particular for all odd dimensional horizons, N is even.
- The horizons of all non-chiral theories have a $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra
- If $N_- = 0$, then $N = \mathrm{index}(D_E)$ and so the number of Killing spinors is determined by the topology of horizons.

Symmetry enhancement: Examples and puzzles

Extreme black holes and branes may exhibit symmetry enhancement near the horizons [Gibbons, Townsend]. For example

- RN black hole has symmetry $\mathbb{R} \oplus \mathfrak{so}(3)$ which near the horizon enhances to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(3)$, [Carter]
- M2-brane: Symmetry enhances from $\mathfrak{so}(2,1) \oplus_s \mathbb{R}^3 \oplus \mathfrak{so}(8)$ to $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)$, [Duff, Stelle]
- M5-brane: Symmetry enhances from $\mathfrak{so}(5,1) \oplus_s \mathbb{R}^6 \oplus \mathfrak{so}(5)$ to $\mathfrak{so}(6,2) \oplus \mathfrak{so}(5)$, [Güven]
- Similarly for three or more intersecting M-branes [Townsend, GP].
- NS5-brane: Symmetry does NOT enhance

So why does symmetry enhance in some backgrounds?

- Claim: For black holes (super)symmetry enhancement near a horizon is a consequence of smoothness

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Consequences and Applications

These results can be applied in a variety of problems

- The existence of higher dimensional black holes with exotic topologies and geometries
  Asymptotically AdS_5 rings in minimal 5d gauged supergravity have been ruled out! [Grover, Gutowski, GP, Sabra; Grover, Gutowski, Sabra]
- Microscopic counting of entropy for black holes
  The presence of $\mathfrak{sl}(2,\mathbb{R})$ justifies the use of conformal mechanics in entropy counting.
- AdS/CFT: Provides a new method to classify all AdS backgrounds in supergravity.
- Geometry: A generalization of Lichnerowicz theorem for connections with GL holonomy.

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Parallel spinors and topology

The number of parallel spinors N_p of 8-d manifolds with holonomy strictly $Spin(7)$, $SU(4)$, $Sp(2)$ and $\times^2 Sp(1)$ is

$$N_p = \mathrm{index}(D) = \frac{1}{5760}(-4p_2 + 7p_1^2)$$

for $N_p = 1, 2, 3, 4$, respectively.

Proof: Use the identity $D^2 = \nabla^2 - \frac{1}{4}R$ to establish the Lichnerowicz formula

$$\int \parallel D\epsilon \parallel^2 = \int \parallel \nabla\epsilon \parallel^2 + \frac{1}{4}\int R \parallel \epsilon \parallel^2$$

Since for $Spin(7)$, $SU(4)$, $Sp(2)$ and $\times^2 Sp(1)$ manifolds, $R = 0$, and $\ker D^\dagger = \{0\}$, then all zero modes of the Dirac operator D are ∇-parallel and

$$N_p = \dim \mathrm{Ker}(D) - 0 = \dim \mathrm{Ker}(D) - \dim \mathrm{Ker}(D^\dagger) = \mathrm{index}(D)$$

- it is possible to test whether manifolds with given Pontryagin classes admit a given number of parallel (Killing) spinors!

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Horizon metric

Near a smooth Killing horizon a coordinate system can be adapted such that the metric is [Isenberg, Moncrief; Friedrich, et al]

$$ds^2 = 2du[dr + r\,h_I(r,y)dy^I + r f(r,y)du] + \gamma_{IJ}(y,r)dy^I dy^J$$

Assuming analyticity in r, and for an extreme black hole,

$$f(0,y) = 0$$

a near horizon limit can be defined leading to a near horizon metric

$$ds^2 = 2du[dr + r\,h_I dy^I + r^2\,\Delta du] + \gamma_{IJ} dy^I dy^J$$

where

$$h_I = h_I(0,y)\ , \quad \Delta = \partial_r f|_{r=0}\ , \quad \gamma_{IJ} = \gamma_{IJ}(0,y)$$

- The near horizon metric has two isometries generated by translations in u and the scale transformation

$$u \to \ell^{-1} u \ , \quad r \to \ell r$$

- The two Killing vectors

$$\partial_u \ , \quad -u\partial_u + r\partial_r$$

  do not commute. The algebra of isometries is NOT $\mathfrak{sl}(2, \mathbb{R})$
- The Gaussian null coordinate system can be adapted in the presence of other fields like Maxwell and k-form gauge potentials
- The co-dimension 2 space given by $u = r = 0$ is the horizon section, $\mathcal{S}$, and it is required to be compact without boundary.

M-horizons

The near horizon fields of D=11 supergravity are

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + d\tilde{s}^2(\mathcal{S}) \ , \\ F &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r\mathbf{e}^+ \wedge d_h Y + X \ , \quad d_h Y = dY - h \wedge Y \ , \end{aligned}$$

where

$$\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du \ , \quad \mathbf{e}^i = e^i{}_J dy^J$$

The steps in the proof are as follows.

- Integration of KSEs along the lightcone directions r, u
- Independent KSEs on $\mathcal{S}$
- Horizon Dirac equations
- Two Lichnerowicz type of theorems
- Index and number of Killing spinors

Integrability of KSEs along the lightcone

The KSEs are

$$\mathcal{D}_M\epsilon = \nabla_M\epsilon - \Big(\frac{1}{288}{\Gamma_M}^{L_1L_2L_3L_4}F_{L_1L_2L_3L_4} - \frac{1}{36}F_{ML_1L_2L_3}\Gamma^{L_1L_2L_3}\Big)\epsilon = 0$$

These can be integrated along to lightcone directions to give

$$\epsilon = \epsilon_+ + \epsilon_- \ , \quad \Gamma_\pm\epsilon_\pm = 0 \ ,$$

with

$$\epsilon_+ = \eta_+, \qquad \epsilon_- = \eta_- + r\Gamma_-\Theta_+\eta_+ \ ,$$

and

$$\eta_+ = \phi_+ + u\Gamma_+\Theta_-\phi_-, \qquad \eta_- = \phi_- \ ,$$

where

$$\Theta_\pm = \left(\frac{1}{4}h_i\Gamma^i + \frac{1}{288}X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{12}Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2}\right) \ ,$$

and $\phi_\pm = \phi_\pm(y)$ do not depend on r or u.

Independent KSEs

The integration along the lightcone directions has two consequences. First after using the field equations and Bianchi identities, the remaining independent KSEs are

$$\nabla_i^{(\pm)}\phi_\pm \equiv \tilde{\nabla}_i\phi_\pm + \Psi_i^{(\pm)}\phi_\pm = 0\ ,$$

where

$$\begin{aligned}\Psi_i^{(\pm)} \quad &= \quad \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4}X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36}X_{i\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} \\ &\quad \pm\frac{1}{24}\Gamma_i{}^{\ell_1\ell_2}Y_{\ell_1\ell_2} \mp \frac{1}{6}Y_{ij}\Gamma^j\ ,\end{aligned}$$

and $\tilde{\nabla}$ the Levi-Civita connection of $\mathcal{S}$.

Second, if ϕ_- is a solution, $\nabla_i^{(-)}\phi_- = 0$, then

$$\nabla_i^{(+)}\phi'_+ = 0\ , \quad \phi'_+ = \Gamma_+\Theta_-\phi_-$$

Horizon Dirac operators

The associated horizon Dirac operators are

$$\mathcal{D}^{(\pm)}\phi_{\pm} = \Gamma^i \tilde{\nabla}_i \phi_{\pm} + \Psi^{(\pm)}\phi_{\pm} = 0 \, ,$$

where

$$\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_\ell \Gamma^\ell + \frac{1}{96} X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{8} Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2} \, .$$

Clearly,

$$\nabla_i^{(\pm)}\phi_{\pm} = 0 \Longrightarrow \mathcal{D}^{(\pm)}\phi_{\pm} = 0$$

The converse is also true, ie

$$\nabla_i^{(\pm)}\phi_{\pm} \Longleftrightarrow \mathcal{D}^{(\pm)}\phi_{\pm} = 0$$

A maximum principle

The proof of converse for the $\mathcal{D}^{(+)}$ operator relies on the formula that if $\mathcal{D}^{(+)}\phi_+ = 0$, then

$$\tilde{\nabla}^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 - h^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2\langle \tilde{\nabla}^{(+)i}\phi_+, \tilde{\nabla}^{(+)}_i\phi_+\rangle \ .$$

Using the maximum principle for the function $\parallel \phi_+ \parallel^2$ based on the compactness of $\mathcal{S}$, one concludes that

$$\tilde{\nabla}^{(+)}_i\phi_+ = 0 \ , \quad \parallel \phi_+ \parallel^2 = \text{const} \ .$$

which gives the proof of a Lichnerowicz type of theorem for $\mathcal{D}^{(+)}$

A Lichnerowicz Theorem for $\mathcal{D}^{(-)}$

This is based on a partial integration formula,

$$\begin{aligned} \int_{\mathcal{S}} \parallel \mathcal{D}^{(-)} \phi_- \parallel^2 &= \int_{\mathcal{S}} \parallel \tilde{\nabla}^{(-)} \phi_- \parallel^2 + \int_{\mathcal{S}} \langle \mathcal{B} \phi_-, \mathcal{D}^{(-)} \phi_- \rangle \\ &+ \text{FEs}, \ \text{BI}, \ \text{surf. terms} \end{aligned}$$

where $\mathcal{B}$ depends on the fluxes and one of the FEs is

$$\begin{aligned} \tilde{R}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j &= -\frac{1}{2} Y_{i\ell} Y_j{}^{\ell} + \frac{1}{12} X_{i\ell_1\ell_2\ell_3} X_j{}^{\ell_1\ell_2\ell_3} \\ &+ \delta_{ij} \left(\frac{1}{12} Y_{\ell_1\ell_2} Y^{\ell_1\ell_2} - \frac{1}{144} X_{\ell_1\ell_2\ell_3\ell_4} X^{\ell_1\ell_2\ell_3\ell_4} \right), \end{aligned}$$

The surface terms vanish because $\mathcal{S}$ is compact without boundary. So if the field equations and Bianchi identities are satisfied, then all zero modes of $\mathcal{D}^{(-)}$ are $\tilde{\nabla}^{(-)}$-parallel.

Index and supersymmetry

The spin bundle splits $S = S_+ \oplus S_-$ on $\mathcal{S}$ with respect to $\Gamma_\pm$, and $\mathcal{D}^{(+)} : \Gamma(S_+) \to \Gamma(S_+)$ and its adjoint $(\mathcal{D}^{(+)})^\dagger : \Gamma(S_+) \to \Gamma(S_+)$.
$\mathcal{D}^{(+)}$ has the same principal symbol as the Dirac operator and $\mathrm{Index}(\mathcal{D}^{(+)}) = 0$ as $\dim\mathcal{S} = 9$. Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker(\mathcal{D}^{(+)})^\dagger \,.$$

Then $(\mathcal{D}^{(+)})^\dagger\Gamma_+ = \Gamma_+\mathcal{D}^{(-)}$ and so

$$\dim\ker(\mathcal{D}^{(+)})^\dagger = \dim\ker\mathcal{D}^{(-)}$$

Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker\mathcal{D}^{(-)} \,.$$

The number of supersymmetries of a near horizon geometry is the number of parallel spinors of $\nabla^{(\pm)}$ and so from the Lichnerowicz theorems and the index

$$N = \dim\ker\mathcal{D}^{(+)} + \dim\ker\mathcal{D}^{(-)} = 2\dim\ker\mathcal{D}^{(-)} = 2N_-\,.$$

This proves that the number of supersymmetries preserved by M-horizon geometries is even.

Construction of ϕ_+ spinors from ϕ_- spinors

Recall that if $\nabla^{(-)}\phi_- = 0$, then

$$\nabla^{(+)}\phi_+ = 0\,, \quad \phi_+ = \Gamma_+\Theta_-\phi_-\,.$$

To find a second supersymmetry, $\phi_+ \neq 0$. Indeed after a partial integration argument and some use of the maximum principle

$$\mathrm{Ker}\,\Theta_- \neq \{0\} \Longleftrightarrow F = 0, h = \Delta = 0$$

So if $\mathrm{Ker}\,\Theta_- \neq \{0\}$, the near horizon geometries have vanishing fluxes and are products $\mathbb{R}^{1,1} \times S^1 \times X^8$, where X^8 has holonomy contained in $Spin(7)$.

- For horizons with non-trivial fluxes if $\phi_- \neq 0$, then $\phi_+ = \Gamma_+\Theta_-\phi_- \neq 0$

$\mathfrak{sl}(2,\mathbb{R})$ symmetry

Every near horizon geometry with non-trivial fluxes admits at least two Killing spinors given by

$$\epsilon_1 = \epsilon(\phi_-, 0)\ , \quad \epsilon_2 = \epsilon(\phi_-, \phi_+)\ , \quad \phi_+ = \Gamma_+ \Theta_- \phi_-$$

These give rise to 3 Killing vector bi-linears given by

$$\begin{aligned} K_1 &= -2u \parallel \phi_+ \parallel^2 \partial_u + 2r \parallel \phi_+ \parallel^2 \partial_r + V^i \tilde{\partial}_i\ , \\ K_2 &= -2 \parallel \phi_+ \parallel^2 \partial_u\ , \\ K_3 &= -2u^2 \parallel \phi_+ \parallel^2 \partial_u + (2 \parallel \phi_- \parallel^2 + 4ru \parallel \phi_+ \parallel^2)\partial_r + 2uV^i \tilde{\partial}_i\ , \end{aligned}$$

where V is a Killing vector on $\mathcal{S}$ which leaves all the data invariant.
They satisfy the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[K_1, K_2] = 2 \parallel \phi_+ \parallel^2 K_2\ , \quad [K_2, K_3] = -4 \parallel \phi_+ \parallel^2 K_1\ , \quad [K_3, K_1] = 2 \parallel \phi_+ \parallel^2 K_3\ .$$

- If $V = 0$, the near horizon geometries of M-theory are $AdS_2 \times_w \mathcal{S}$

IIB horizons

There are two significant differences in the investigation of M-horizons and IIB horizons

- The IIB supergravity has an algebraic KSE, the dilatino KSE
- The index of the Dirac operator on even-dimensional manifolds may not vanish

Nevertheless, the proof of the conjecture for IIB horizons proceeds along similar lines to that of M-horizons. In particular,

- the KSEs can be integrated along the lightcone by writing $\epsilon = \epsilon_- + \epsilon_+$, $\Gamma_\pm \epsilon_\pm = 0$
- the independent KSEs are those which arise from the naive restrictions of the KSEs of IIB supergravity on $\mathcal{S}$
- there are Lichnerowicz type of theorems for the horizon Dirac operators
- the number of supersymmetries is given by an index formula

Independent KSEs

After integration along the lightcone, the independent KSEs are

$$\nabla_i^{(\pm)}\phi_\pm = \tilde{\nabla}_i\phi_\pm + \Psi_i^{(\pm)}\phi_\pm = 0\,, \quad \mathcal{A}^{(\pm)}\phi_\pm = 0$$

where

$$\begin{aligned}\Psi_i^{(\pm)} \quad = \quad & -\frac{i}{2}\Lambda_i \mp \frac{1}{4}h_i \mp \frac{i}{4}Y_{i\ell_1\ell_2}\Gamma^{\ell_1\ell_2} \mp \frac{i}{12}\Gamma_i{}^{\ell_1\ell_2\ell_3}Y_{\ell_1\ell_2\ell_3} \\ & + \left(\pm\frac{1}{16}\Gamma_i{}^j\Phi_j \mp \frac{3}{16}\Phi_i - \frac{1}{96}\Gamma_i{}^{\ell_1\ell_2\ell_3}H_{\ell_1\ell_2\ell_3} + \frac{3}{32}H_{i\ell_1\ell_2}\Gamma^{\ell_1\ell_2}\right)C*\,,\end{aligned}$$

and

$$\mathcal{A}^{(\pm)} = \mp\frac{1}{4}\Phi_i\Gamma^i + \frac{1}{24}H_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} + \xi_i\Gamma^i C*\ .$$

One can also define the horizon Dirac operators

$$\mathcal{D}^{(\pm)} = \Gamma^i\nabla_i^{(\pm)}$$

A maximum principle

One can show

$$\nabla_i^{(+)}\phi_+ = 0\,, \quad \mathcal{A}^{(+)}\phi_+ = 0 \Longleftrightarrow \mathcal{D}^{(+)}\phi_+ = 0$$

Assuming $\mathcal{D}^{(+)}\phi_+ = 0$, one has

$$\tilde{\nabla}^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 - h^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2 \parallel \nabla^{(+)}\phi_+ \parallel^2 + \parallel \mathcal{A}^{(+)}\phi_+ \parallel^2 \quad .$$

Then the maximum principle implies that ϕ_+ is Killing and

$$\parallel \phi_+ \parallel^2 = \text{const}$$

A Lichnerowicz Theorem

Similarly,

$$\nabla_i^{(-)}\phi_- = 0\,, \quad \mathcal{A}^{(-)}\phi_- = 0 \Longleftrightarrow \mathcal{D}^{(-)}\phi_- = 0$$

Based on the formula

$$\begin{aligned}\int_{\mathcal{S}} \parallel \mathcal{D}^{(-)}\phi_- \parallel^2 \quad &= \quad \int_{\mathcal{S}} \parallel \tilde{\nabla}^{(-)}\phi_- \parallel^2 + \frac{1}{2}\int_{\mathcal{S}} \parallel \mathcal{A}^{(-)}\phi_- \parallel^2 \\ &+ \quad \int_{\mathcal{S}} \langle \mathcal{B}\phi_-, \mathcal{D}^{(-)}\phi_- \rangle + \text{FEs, BI, ST}\end{aligned}$$

where $\mathcal{B}$ depends on the fluxes.

Index and supersymmetry

Therefore the number of supersymmetries of a IIB horizon is

$$N = \dim \mathrm{Ker}(\mathcal{D}^{(+)}) + \dim \mathrm{Ker}(\mathcal{D}^{(-)})$$

On the other hand, it can be shown that

$$\dim \mathrm{Ker}(\mathcal{D}^{(+)}) - \dim \mathrm{Ker}(\mathcal{D}^{(-)}) = 2\mathrm{Index}(D_\lambda) \ ,$$

where D_λ is the Dirac operator twisted with λ the line bundle of IIB scalars.
Thus

$$N = 2\mathrm{Ker}(\mathcal{D}^{(-)}) + 2\mathrm{Index}(D_\lambda) = 2\,N_- + 2\,\mathrm{Index}(D_\lambda)$$

- All IIB horizons admit even number of supersymmetries

$\mathfrak{sl}(2,\mathbb{R})$ symmetry

If $N_- \neq 0$ for every zero mode of $\mathcal{D}^{(-)}$ there is a zero mode of $\mathcal{D}^{(+)}$ given by

$$\phi_+ = \Gamma_+ \Theta_- \phi_-$$

and $\phi_+ \neq 0$ if the background has non-trivial fluxes.

This gives rise to two linearly independent Killing spinors on IIB horizons determined by the pairs $(\phi_-, 0)$ and (ϕ_-, ϕ_+)

In turn, the two Killing spinors give rise to 3 vectors K_1, K_2 and K_3 which leave invariant all fields and satisfy a $\mathfrak{sl}(2,\mathbb{R})$ algebra

- All IIB horizons with non-trivial fluxes and $N_- \neq 0$ admit a $\mathfrak{sl}(2,\mathbb{R})$ symmetry

Summary

- Black hole horizons of non-chiral supergravity theories with non-trivial fluxes exhibit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry and preserve even number supersymmetries. This is a consequence of smoothness of black hole horizons
- For chiral supergravity theories, the number of supersymmetries of horizons can be expressed in terms of the index of a Dirac operator. For horizons with non-trivial fluxes and $N_- \neq 0$ also admit a $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra. Again this is a consequence of smoothness of horizons.
- Applications to geometry include the proof of new Lichnerowicz type of theorems for GL connections.