1) The hamiltonian of the nuclear system: sum of a kinetic energy term and a two-body potential:

$$H = \sum_{ij} < i|T|j > a_i^+ a_j + \frac{1}{4} \sum_{ijkl} < ij|V|kl > a_i^+ a_j^+ a_l a_k$$

the expectation value in the Slater determinant state $|\Phi>$:

$$E[\rho] =< \Phi|H|\Phi >= \sum_{ij} < i|T|j > \rho_{ji} + \frac{1}{2} \sum_{ijkl} < ij|V|kl > \rho_{ki}\rho_{lj}$$

defines the energy E as a functional of the single-particle density matrix ρ associated with the state $|\Phi>$:

$$\rho_{ij} =< i|\rho|j >=< \Phi|a_j^+ a_i|\Phi >$$

From the variational condition: $\delta\{E[\rho] - \mathrm{tr}\Lambda(\rho^2 - \rho)\} = 0$

derive the Hartree-Fock equation: $[h, \rho] \equiv h\rho - \rho h = 0$

where the HF hamiltonian is defined by: $h_{ij} \equiv< i|h|j >= \dfrac{\partial E[\rho]}{\partial \rho_{ji}}$

2) For the Hamiltonian:

$$\begin{aligned}\hat{H} &= H - \mu N \\ &= \sum_{ij} < i|T-\mu|j > a_i^+ a_j + \frac{1}{4}\sum_{ijkl} < ij|V|kl > a_i^+ a_j^+ a_l a_k\end{aligned}$$

minimize the expectation value: $E[\rho,\kappa] \equiv < \Phi|\hat{H}|\Phi > \equiv E[\mathcal{R}]$

(where the ground state is a quasiparticle vacuum) with respect to the single-particle density and the pair tensor:

$$\begin{aligned}\rho_{ij} &= \langle\Phi|a_j^+ a_i|\Phi\rangle = (V^* V^T)_{ij} = \rho_{ji}^* \\ \kappa_{ij} &= \langle\Phi|a_j a_i|\Phi\rangle = (V^* U^T)_{ij} = -\kappa_{ji}\end{aligned}$$

to derive the Hartree-Fock-Bogoliubov equation: $[\mathcal{H},\mathcal{R}] = 0$

where the generalized density matrix is defined: $\mathcal{R} = \begin{pmatrix} \rho & \kappa \\ -\kappa^* & 1-\rho^* \end{pmatrix}$

3) Starting from the definition of the BCS ground state:

$$|\tilde{0}\rangle = \prod_{\nu>0} (u_\nu + v_\nu a_\nu^+ a_{\bar{\nu}}^+)|0\rangle$$

with the normalization: $$\langle\tilde{0}|\tilde{0}\rangle = \prod_{\nu>0} (u_\nu^2 + v_\nu^2)$$

show that the particle number expectation value equals:

$$\langle\tilde{0}|\hat{n}|\tilde{0}\rangle = \sum_{\nu>0} 2v_\nu^2$$

and the particle number uncertainty:

$$\Delta n^2 = \langle\tilde{0}|\hat{n}^2|\tilde{0}\rangle - \langle\tilde{0}|\hat{n}|\tilde{0}\rangle^2 = 4\sum_{\nu>0} u_\nu^2 v_\nu^2$$

4) For the pairing Hamiltonian:

$$\mathcal{H} = H - \lambda\hat{n} = \sum_{\nu>0}(\epsilon_\nu - \lambda)(a^+_\nu a_\nu + a^+_{\bar{\nu}} a_{\bar{\nu}}) - G \sum_{\mu,\nu>0} a^+_\mu a^+_{\bar{\mu}} a_{\bar{\nu}} a_\nu$$

and with the Lagrange multiplier chosen such that the average particle number equals the actual number of particles:

$$\langle\tilde{0}|\hat{n}|\tilde{0}\rangle = n$$

where:

$$|\tilde{0}\rangle = \prod_{\nu>0}(u_\nu + v_\nu a^+_\nu a^+_{\bar{\nu}})|0\rangle$$

is the BCS vacuum for quasiparticle operators: $c_\nu|\tilde{0}\rangle = 0 \quad \forall\nu$

$$\begin{aligned} c^+_\nu &= u_\nu a^+_\nu - v_\nu a_{\bar{\nu}} \\ c_\nu &= u_\nu a_\nu - v_\nu a^+_{\bar{\nu}} \end{aligned}$$

show that a constrained variational calculation (with respect to u_υ and v_υ)

$$\delta\langle\tilde{0}|\mathcal{H}|\tilde{0}\rangle = 0$$

$$\langle\tilde{0}|\mathcal{H}|\tilde{0}\rangle = U_0 = \sum_{\nu>0}\left[2(\epsilon_\nu - \lambda)v_\nu^2 - Gv_\nu^4\right] - G\left[\sum_{\nu>0} u_\nu v_\nu\right]^2$$

leads to the gap equation:

$$\frac{d}{dv_\nu}U_0 = 0 \quad \Longrightarrow \quad 2(\epsilon'_\nu - \lambda)u_\nu v_\nu = \Delta(u_\nu^2 - v_\nu^2)$$

where:

$$\Delta \equiv G\sum_{\nu>0} u_\nu v_\nu \qquad \epsilon'_\nu \equiv \epsilon_\nu - Gv_\nu^2$$

with the solution for
the occupation probabilities:

$$u_\nu^2 = \frac{1}{2}\left[1 + \frac{(\epsilon'_\nu - \lambda)}{\sqrt{(\epsilon'_\nu - \lambda)^2 + \Delta^2}}\right]$$

$$v_\nu^2 = \frac{1}{2}\left[1 - \frac{(\epsilon'_\nu - \lambda)}{\sqrt{(\epsilon'_\nu - \lambda)^2 + \Delta^2}}\right]$$

5) In the Hartree-Fock-Bogoliubov (HFB) approximation the quasiparticle vacuum is characterized by the generalized density matrix:

$$\mathcal{R} = \begin{pmatrix} \rho & \kappa \\ -\kappa^* & \mathbb{1} - \rho^* \end{pmatrix}$$

For a unitary transformation: $|\overline{\Phi}\rangle = U|\Phi\rangle$

show that:

$$\begin{aligned} \bar{\rho}_{ij} &= \langle\overline{\Phi}|a_j^+ a_i|\overline{\Phi}\rangle = (U\rho U^+)_{ij} \\ \bar{\kappa}_{ij} &= \langle\overline{\Phi}|a_j a_i|\overline{\Phi}\rangle = (U\kappa\tilde{U})_{ij} \end{aligned}$$

$$\bar{\mathcal{R}} = \mathcal{U}\mathcal{R}\mathcal{U}^+ \qquad \mathcal{U} = \begin{pmatrix} U & 0 \\ 0 & U^+ \end{pmatrix}$$

6) For the symmetry group of the Hamiltonian:

$$[H, \mathcal{R}] = 0 \quad \Longrightarrow \quad |\Phi\Omega\rangle = \mathcal{R}(\Omega)|\Phi\rangle$$

degenerate deformed states (states of broken symmetry).

To restore the symmetry use the new trial function:

$$|\Psi\rangle = \int d\Omega f(\Omega)|\Phi\Omega\rangle \equiv \int d\Omega f(\Omega)\mathcal{R}(\Omega)|\Phi\rangle$$

By requiring that the energy expectation:

$$E = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

is stationary with respect to variations of f* and f, derive the Hill-Wheeler equation:

$$\int d\Omega' \langle\Phi\Omega|H - E|\Phi\Omega'\rangle f(\Omega') = 0$$

7) The RPA ground-state is defined by: $Q_\nu|RPA\rangle = 0$

$$Q_\nu^+ = \sum_{mi} \left(X_{mi}^\nu a_m^+ a_i - Y_{mi}^\nu a_i^+ a_m \right)$$

Starting from the equation of motion:

$$\langle 0| \left[\delta Q, \left[H, Q_\nu^+\right]\right] |0\rangle = (E_\nu - E_0) \, \langle 0| \left[\delta Q, Q_\nu^+\right] |0\rangle$$

and considering two type of variations: $\delta Q|0\rangle = a_m^+ a_i|0\rangle$ $\qquad \delta Q|0\rangle = a_i^+ a_m|0\rangle$

in the quasi-boson approximation:

$$\langle RPA| \left[a_i^+ a_m, a_n^+ a_j\right] |RPA\rangle \approx \langle HF| \left[a_i^+ a_m, a_n^+ a_j\right] |HF\rangle = \delta_{ij}\delta_{mn}$$

derive the RPA equations:

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} = \hbar\Omega_\nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix}$$

8) Assume that the exact two-body Hamiltonian H is invariant under a continuous symmetry operation generated by a one-body hermitian operator:

$$[H, \hat{P}] = 0$$

Assume that the HF ground-state violates this symmetry (obvious for space translations)

$$[\rho^{(0)}, \hat{P}] \neq 0$$

Show that P is an exact but spurious solution of the RPA equation:

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} P \\ -P^* \end{pmatrix} = 0$$

$$|P\rangle = \sum_{mi} \left(P_{mi} a_m^+ a_i + P_{mi}^* a_i^+ a_m \right) |RPA\rangle$$

9) Consider the response of a nuclear system to an external time-dependent field:

$$F(t) = Fe^{-i\omega t} + F^{+}e^{i\omega t} \qquad\qquad F(t) = \sum_{kl} f_{kl}(t) a_k^+ a_l$$

Assume that the field is weak, that it causes only small changes of the nuclear density:

$$\rho_{kl}(t) = \langle \Phi(t) | a_l^+ a_k | \Phi(t) \rangle$$

Starting from the equation of motion: $i\hbar\dot{\rho} = [\ h[\rho] + f(t), \rho\ ]$

and $\rho(t) = \rho^{(0)} + \delta\rho(t) \qquad \delta\rho(t) = \rho^{(1)} e^{-i\omega t} + \rho^{(1)+} e^{i\omega t}$

with the additional assumption: $\rho^2 = \rho \quad \Longrightarrow \quad \rho^{(0)}\delta\rho + \delta\rho\rho^{(0)} = \delta\rho$

Derive the linear response equation:

$$\left\{ \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \rho^{(1)ph} \\ \rho^{(1)hp} \end{pmatrix} = - \begin{pmatrix} f^{ph} \\ f^{hp} \end{pmatrix}$$

for the ph and hp matrix elements in linear order in the external field.