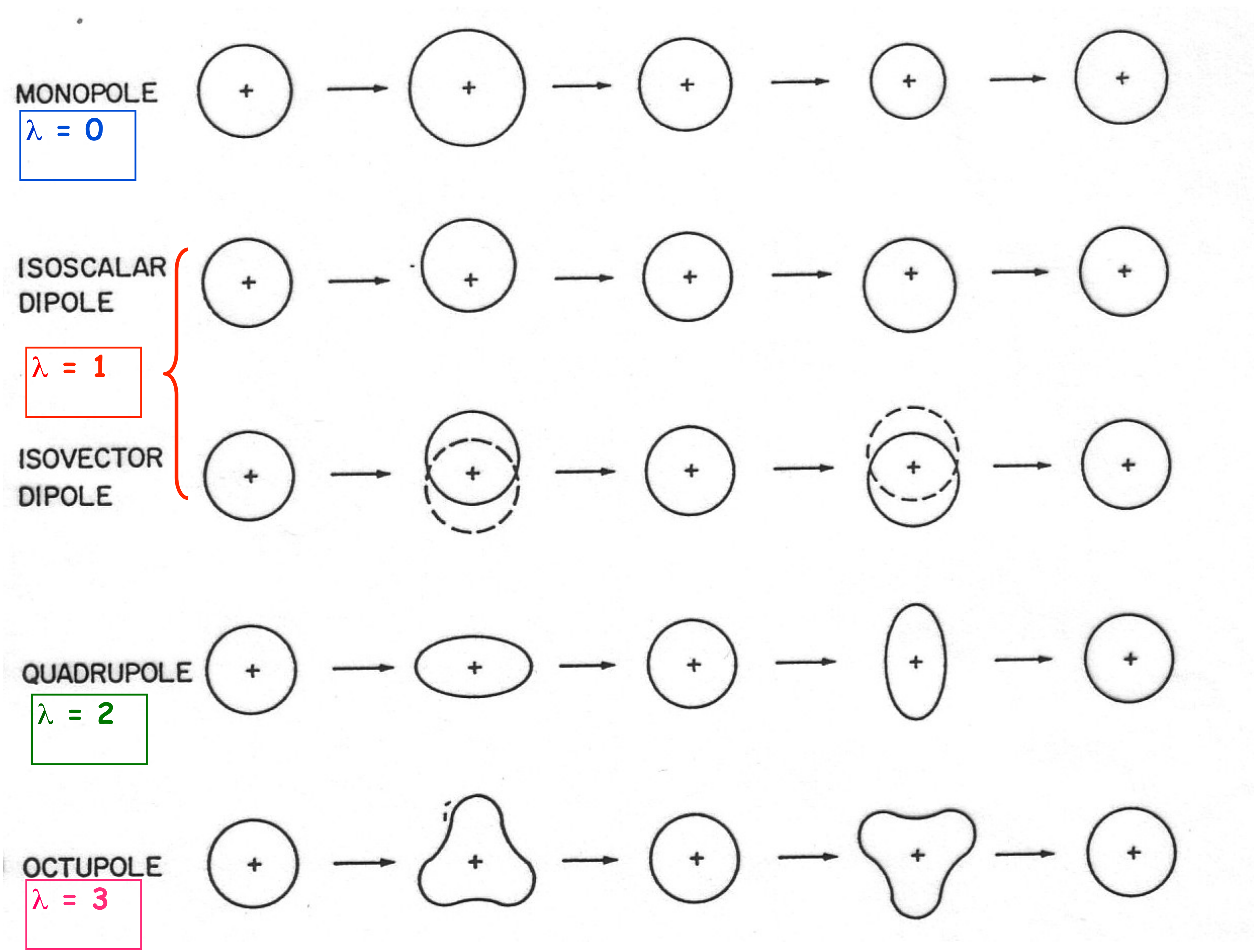
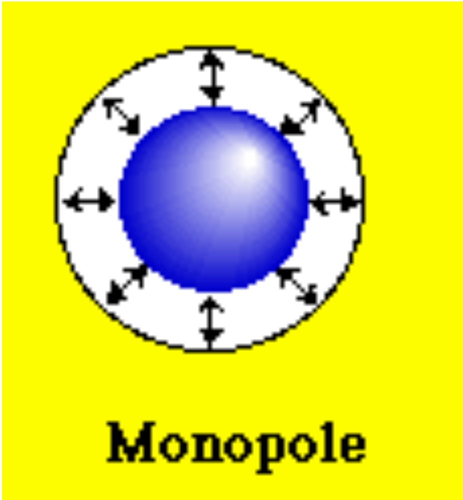
Harmonic Vibrations

- Random Phase Approximation
- Linear Response Theory
- Beyond the Mean-Field Approximation

NUCLEAR VIBRATIONAL MODES:



1. DENSITY VIBRATIONS

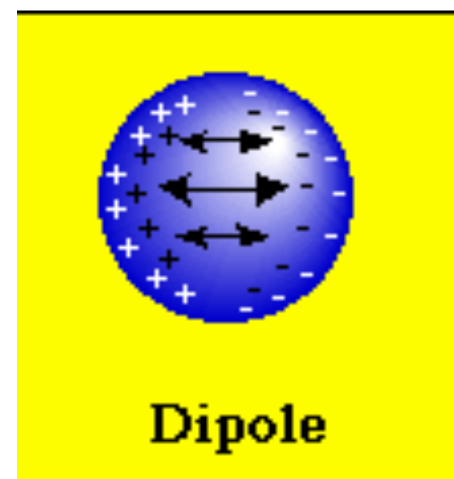


λ=0 the volume of the nucleus changes, but not its shape. Nuclear matter has a high compression modulus ***K≈250 MeV***, and the excitation energies of this mode of vibrations are relatively high ***≈80 $A^{1/3}$***.

2. SHAPE VIBRATIONS

…if there are no changes in density, the excitation energies can be much lower.

λ=1 ***T=0 isoscalar dipole mode:*** oscillations around a fixed point in the laboratory system. All nucleons move together and there are no changes in the internal structure of the nucleus: ***center of mass oscillations.***



λ=1 ***T=1 isovector dipole mode:*** protons and neutrons oscillate with opposite phases ⇒ ***GIANT DIPOLE RESONANCE***

GIANT DIPOLE RESONANCE

$\gamma + {}^{197}\mathrm{Au}$ ® ${}^{197}\mathrm{Au}^*$ ® ${}^{196}\mathrm{Au} + n$

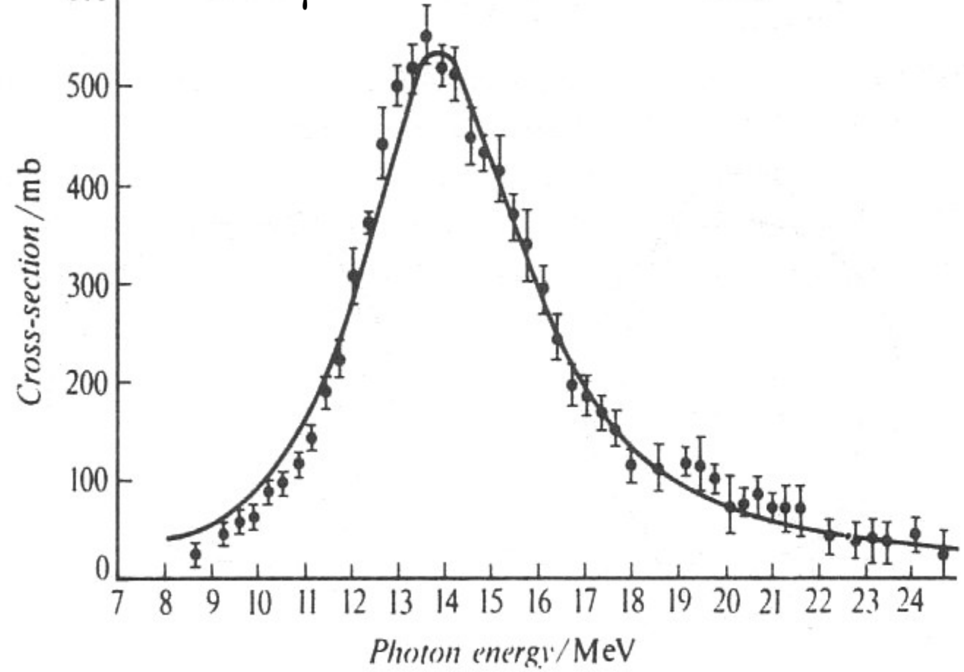
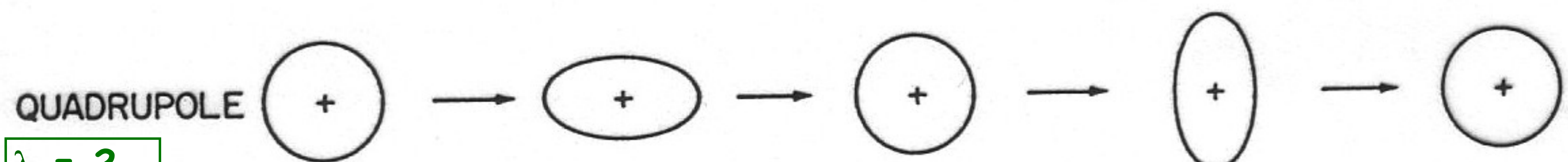


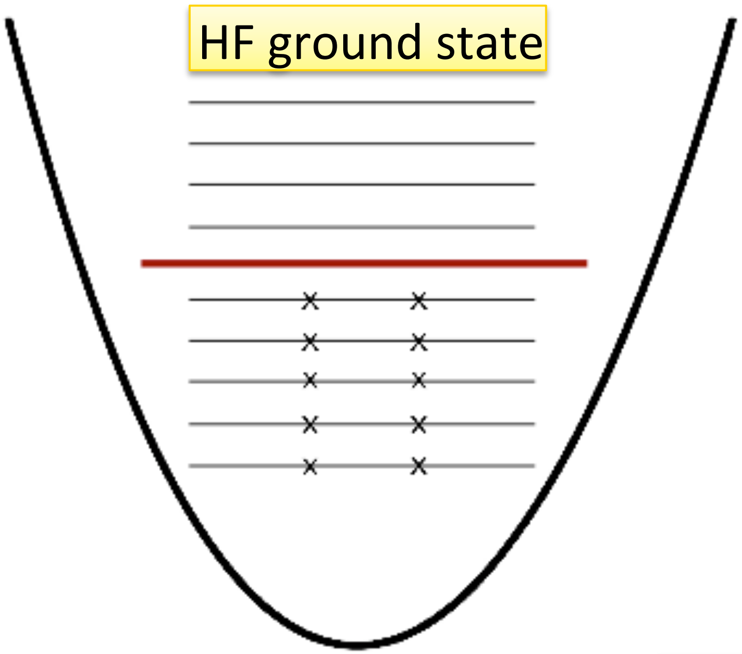
Fig. 8.10 Giant resonance of photodisintegration in ^{197}Au. The yield of neutrons is shown as a function of the energy of the monochromatic photons used to produce the reaction (Fultz, S. C. *et al.*, *Phys. Rev.*, **127**, 1273, 1963).

QUADRUPOLE OSCILLATIONS (λ = 2):



$\lambda = 2$

Microscopic picture:



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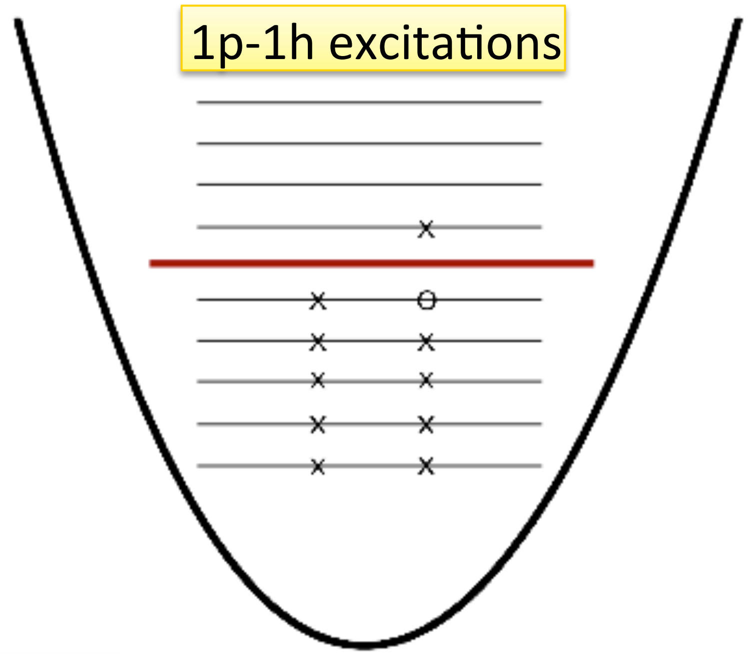

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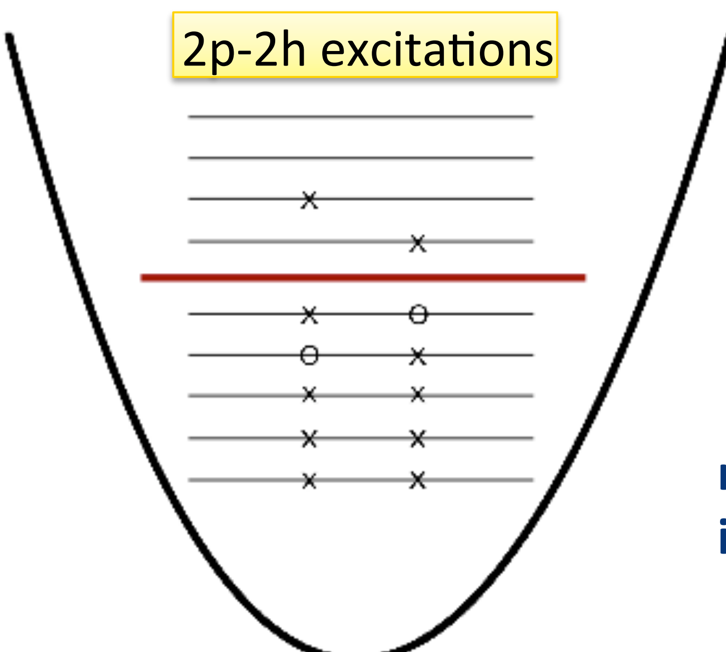


+ . . .

m,n: states above the Fermi level

i,j: states below the Fermi level

Vibrations in QM:

$$|\Psi(t)\rangle = |0\rangle + \sum_{\nu} c_\nu |\nu\rangle e^{-iE_\nu t/\hbar}$$

excited states

→ the corresponding density (to first order in c_ν):

$$\rho(\mathbf{r},t) = \langle\Psi(t)|\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)|\Psi(t)\rangle = \rho^{(0)}(\mathbf{r}) + \delta\rho(\mathbf{r},t)$$

with:

$$\delta\rho(\mathbf{r},t) = \sum_{\nu} c_\nu \ \langle 0|\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)|\nu\rangle \ e^{-iE_\nu t/\hbar} + c.c.$$

A Fourier transform of this density gives the contribution of the different excited states:

$$\rho^{(1)\nu}(\mathbf{r}) = \langle 0|\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)|\nu\rangle$$ Transition density

Transition density matrix:
(in a shell model basis)

$$\rho^{(1)\nu}_{pq} = \langle 0|a_q^+ a_p|\nu\rangle$$

1. RPA: particle-hole theory with ground-state correlations

A. Derivation of the RPA equations

→ set of exact eigenstates of the hamiltonian H: $H|\nu\rangle = E_\nu|\nu\rangle$

→ define the operators Q in such a way that:

$$|\nu\rangle = Q_\nu^+|0\rangle \qquad\qquad Q_\nu|0\rangle = 0$$

⇒ equation of motion:

$$\boxed{\left[H, Q_\nu^+\right]|0\rangle = (E_\nu - E_0)\; Q_\nu^+|0\rangle}$$

→ by multiplying from the left with an arbitrary state <0|δQ:

$$\boxed{\langle 0|\left[\delta Q, \left[H, Q_\nu^+\right]\right]|0\rangle = (E_\nu - E_0)\;\langle 0|\left[\delta Q, Q_\nu^+\right]|0\rangle}$$

Tamm-Dancoff Approximation:

1) approximate the exact ground state |0> with the Hartree-Fock state |HF>
2) approximate the operator Q_ν by the collective ph-operator:

$$Q_\nu^+ = \sum_{mi} c_{mi}^\nu a_m^+ a_i$$

→ the vector space is restricted to 1p-1h excitations.

$$\boxed{\sum_{nj} \langle HF| \left[a_i^+ a_m, \left[H, a_n^+ a_j\right]\right] |HF\rangle c_{nj}^\nu = E_\nu^{TDA} c_{mi}^\nu}$$

Correlations are only taken into account for the excited states, the ground state remains a HF Slater determinant.

RPA equations:

If we assume that the ground-state contains 2p-2h correlations, then a *ph* pair can be not only created but also destroyed ⇒ more general operator Q:

$$Q^+_\nu = \sum_{mi} \left(X^\nu_{mi} a^+_m a_i - Y^\nu_{mi} a^+_i a_m \right)$$

The RPA ground state |RPA> is defined by:

$$Q_\nu |RPA\rangle = 0$$

→ two type of variations: $\delta Q|0\rangle = a^+_m a_i |0\rangle$ $\delta Q|0\rangle = a^+_i a_m |0\rangle$

⇒ from the equation of motion:

$$\begin{aligned} \langle RPA| \left[a^+_i a_m, \left[H, Q^+_\nu \right] \right] |RPA\rangle &= \hbar\Omega_\nu \, \langle RPA| \left[a^+_i a_m, Q^+_\nu \right] |RPA\rangle \\ \langle RPA| \left[a^+_m a_i, \left[H, Q^+_\nu \right] \right] |RPA\rangle &= \hbar\Omega_\nu \, \langle RPA| \left[a^+_m a_i, Q^+_\nu \right] |RPA\rangle \end{aligned}$$

Quasi-boson approximation: assume that |RPA> does not differ much from |HF>

$$\langle RPA| \left[a_i^+ a_m, a_n^+ a_j\right] |RPA\rangle \approx \langle HF| \left[a_i^+ a_m, a_n^+ a_j\right] |HF\rangle = \delta_{ij}\delta_{mn}$$

→ this relation would be exact if the *ph* operators obeyed the commutation relations for boson field operators (violates the Pauli principle!).

RPA equations:

$$\boxed{\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} = \hbar\Omega_\nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix}}$$

HF single-particle energies

$$A_{minj} = \langle HF| \left[a_i^+ a_m, \left[H, a_n^+ a_j\right]\right] |HF\rangle = (\epsilon_m - \epsilon_i)\delta_{ij}\delta_{mn} + \overline{v}_{mjin}$$

$$B_{minj} = -\langle HF| \left[a_i^+ a_m, \left[H, a_j^+ a_n\right]\right] |HF\rangle = \overline{v}_{mnij}$$

antisymmetrized two-body residual interaction

…*ph* and *hp* matrix elements of the transition density:

$$
\begin{aligned}
\rho^{\nu}_{mi} &= \langle 0|a_i^+ a_m|\nu\rangle \approx \langle HF|\left[a_i^+ a_m, Q_\nu^+\right]|HF\rangle = X^{\nu}_{mi} \\
\rho^{\nu}_{im} &= \langle 0|a_m^+ a_i|\nu\rangle \approx \langle HF|\left[a_m^+ a_i, Q_\nu^+\right]|HF\rangle = Y^{\nu}_{mi}
\end{aligned}
$$

→ **the quasi-boson approximation** is valid for very collective states: many transition amplitudes X of the same order of magnitude. Each single *ph*-component has only a small probability of being excited and the violation the Pauli principle can be neglected.

→ the amplitudes Y should be small compared to the coefficients X because they describe ground-state correlations. If the Y's are too large, the replacement of the correlated state |RPA> by |HF> is not justified.

…matrix elements for a Hermitian one-body operator in the RPA approx.

$$
\langle 0|F|\nu\rangle = \sum_{kk'} F_{kk'}\rho^{\nu}_{k'k} = \sum_{mi}\left(F_{im}X^{\nu}_{mi} + F_{mi}Y^{\nu}_{mi}\right)
$$

B. Normalization and closure relations

The RPA matrix is not Hermitian, and so its eigenvectors cannot be orthogonal in the usual sense.

$$|\nu\rangle = Q_\nu^+ |RPA\rangle$$

Orthogonality condition:

$$\langle \nu | \nu' \rangle = \delta_{\nu\nu'} = \langle RPA | [Q_\nu, Q_{\nu'}^+] | RPA \rangle \approx \langle HF | [Q_\nu, Q_{\nu'}^+] | HF \rangle$$

orthogonality:

$$\delta_{\nu\nu'} = \sum_{mi} \left(X_{mi}^{\nu *} X_{mi}^{\nu'} - Y_{mi}^{\nu *} Y_{mi}^{\nu'} \right)$$

closure:

$$\delta_{mm'} \delta_{ii'} = \sum_{\nu} (X_{mi}^{\nu} X_{m'i'}^{\nu *} - Y_{mi}^{\nu *} Y_{m'i'}^{\nu})$$

C. Representation by boson operators

→ quasiboson approximation: $a_m^+ a_i \longrightarrow B_{mi}^+ \qquad a_i^+ a_m \longrightarrow B_{mi}$

$$[B_{mi}^+, B_{m'i'}^+] = [B_{mi}, B_{m'i'}] = 0 \qquad [B_{mi}, B_{m'i'}^+] = \delta_{mm'}\delta_{ii'}$$

→ this is the lowest order approximation in which we retain the first term in the expansion of the fermion pair operator a^+a in a series of boson operators.

Boson representation of the hamiltonian:

$$H_B = E_{HF} + \sum_{minj} A_{minj} B_{mi}^+ B_{nj} + \frac{1}{2}\sum_{minj} \left(B_{minj} B_{mi}^+ B_{nj}^+ + h.c.\right)$$

def. BOSON OPERATOR

$$\boxed{O_\nu^+ = \sum_{mi} \left(X_{mi}^\nu B_{mi}^+ - Y_{mi}^\nu B_{mi}\right)}$$

⇒ when expressed in terms of the boson operators, the Hamiltonian is diagonal:

$$H_B = E_{RPA} + \sum_\nu \hbar\Omega_\nu \; O_\nu^+ O_\nu$$

with:

$$\begin{aligned} E_{RPA} &= E_{HF} - \frac{1}{2} Tr\ A + \frac{\hbar}{2} \sum_\nu \Omega_\nu \\ &= E_{HF} - \sum_\nu \hbar\Omega_\nu \sum_{mi} |Y_{mi}^\nu|^2 \end{aligned}$$

H_B → Hamiltonian of harmonic oscillators.

RPA → harmonic approximation, determines the uncoupled eigenmodes of the nuclear system.

D. Invariance and Spurious Solutions

→ assume that the exact two-body Hamiltonian H is invariant under a continuous symmetry operation generated by a one-body hermitian operator:

$$[H, \hat{P}] = 0$$

example: space translations

→ assume that the HF ground-state violates this symmetry (obvious for space translations)

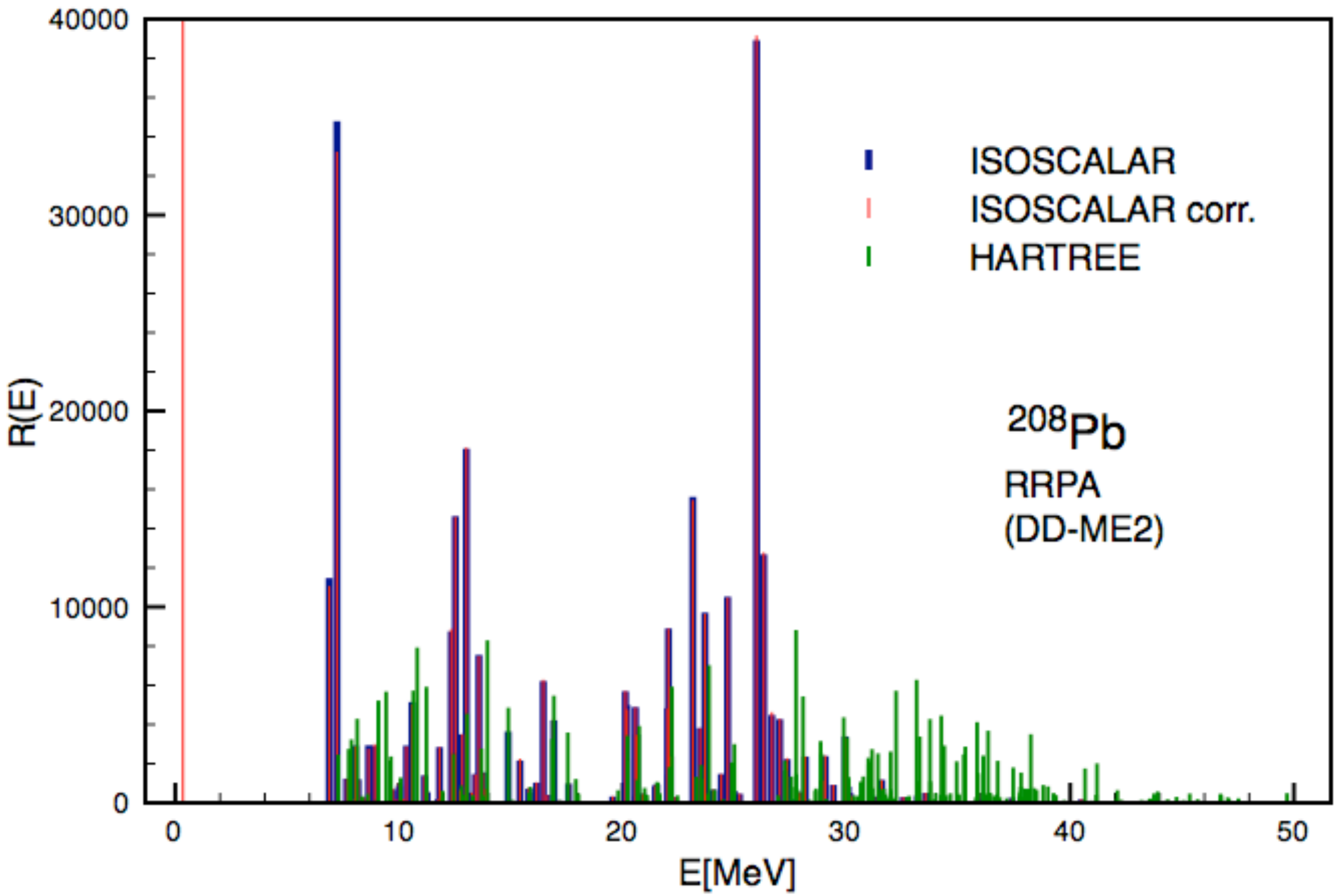
$$[\rho^{(0)}, \hat{P}] \neq 0$$

⇨ P is an exact but spurious solution of the RPA equation

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} P \\ -P^* \end{pmatrix} = 0$$

The corresponding state: $$|P\rangle = \sum_{mi} \left(P_{mi} a_m^+ a_i + P_{mi}^* a_i^+ a_m\right) |RPA\rangle$$

If the RPA is calculated using self-consistent single-particle energies and wave functions, the spurious excitations that correspond to a broken symmetry in the HF state - as, for example, translations of the nucleus - separate out. They are orthogonal to the other excitations and lie at zero excitation energy.



2. Linear Response Theory

… consider the response of a nuclear system to an external time-dependent field:

$$F(t) = Fe^{-i\omega t} + F^{+}e^{i\omega t}$$

with F a one-body operator: $$F(t) = \sum_{kl} f_{kl}(t) a_k^{+} a_l$$

→ assume that the field is weak, i.e. it causes only small changes of the nuclear density, which we can treat in linear order. The density oscillates with the external field and resonances arise whenever the frequency ω is close to an excitation energy of the system.

A. Derivation of the linear response equations

$|\Phi(t)\rangle$ → wave function of a nucleus in an external, time-dependent field

→ the one-body density: $$\rho_{kl}(t) = \langle \Phi(t) | a_l^{+} a_k | \Phi(t) \rangle$$

Assumptions:

(i) at any time ρ(t) corresponds to a Slater determinant: $\boxed{\rho^2 = \rho}$

equation of motion: $$\boxed{i\hbar\dot{\rho} = [\ h[\rho] + f(t), \rho\ ]}$$

Time-dependent Hartree-Fock equation

(ii) the external field f(t) is weak, i.e. it introduces only oscillations with small amplitudes around the stationary density $\rho^{(0)}$

$$\rho(t) = \rho^{(0)} + \delta\rho(t)$$

$$\delta\rho(t) = \rho^{(1)} e^{-i\omega t} + \rho^{(1)+} e^{i\omega t}$$

In the HF basis $\rho^{(0)}$ and $h[\rho^{(0)}]$ are diagonal:

$$\rho^{(0)}_{kl} = \delta_{kl}\rho^{(0)}_k = 0(\text{particle}), 1(\text{hole})$$

$$(h_0)_{kl} \equiv (h[\rho^{(0)}]_{kl} = \delta_{kl}\epsilon_k$$

$$\rho^2 = \rho \quad \Longrightarrow \quad \rho^{(0)}\delta\rho + \delta\rho\rho^{(0)} = \delta\rho$$

⇒implies that the only non-vanishing matrix elements of $\rho^{(1)}$ are the ph and hp elements, determined by the solution of the TDHF equation. Expansion in linear order in the external field:

$$i\hbar\delta\dot{\rho} = [\ h_0, \delta\rho\ ] + \left[\ \frac{\delta h}{\delta\rho}\delta\rho, \rho^{(0)}\ \right] + [\ f, \rho^{(0)}\ ]$$

with:

$$\frac{\delta h}{\delta\rho}\delta\rho \equiv \sum_{im}\left(\frac{\partial h}{\partial\rho_{mi}}\delta\rho_{mi} + \frac{\partial h}{\partial\rho_{im}}\delta\rho_{im}\right)_{\rho=\rho^{(0)}}$$

linear response equation for the ph and hp matrix elements:

$$\left\{\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \hbar\omega\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\begin{pmatrix} \rho^{(1)ph} \\ \rho^{(1)hp} \end{pmatrix} = -\begin{pmatrix} f^{ph} \\ f^{hp} \end{pmatrix}$$

with: $$A_{minj} = (\epsilon_m - \epsilon_i)\delta_{ij}\delta_{mn} + \frac{\partial h_{mi}}{\partial \rho_{nj}} \qquad B_{minj} = \frac{\partial h_{mi}}{\partial \rho_{jn}}$$

These matrices correspond exactly to the RPA matrices A and B if we use the residual interaction:

$$\overline{v}_{psqr} = \frac{\partial h_{pq}}{\partial \rho_{rs}} = \frac{\partial^2 E}{\partial \rho_{qp} \partial \rho_{rs}}$$

Self-consistent HF+RPA

linear relation between the external field f and the change in the density (the response of the system):

$$\boxed{\rho^{(1)}_{kl} = \sum_{pq} R_{klpq}(\omega) f_{pq}}$$

R_{klpq} → ***Response function*** - depends on the frequency of the external field; poles at the eigenfrequencies of the system.

To find the resonances ($\omega = \Omega_\nu$) one must consider the solutions of the homogeneous equation with vanishing external field ⇒ RPA equation. Its solution gives the transition densities:

$$\rho^{(1)}_{pq}(\Omega_\nu) = \langle 0 | a^+_q a_p | \nu \rangle$$

The RPA approximation is just the small amplitude limit of the time-dependent mean-field approach!

B. Strength function

Vibrational states can be excited by acting on the nucleus with an external field operator F. The imaginary part of the linear response function is related to the total transition probability. We define:

$$R_F(\omega) := Tr\left(f^+\rho^{(1)}(\omega)\right) = \sum_{pqp'q'} f^*_{pq} R_{pqp'q'}(\omega) f_{p'q'}$$

... using the spectral representation of the response function

$$Im\ R_F(\omega) = -\pi \sum_{\nu>0} |\langle\nu|F|0\rangle|^2 \delta(\hbar\omega - \hbar\Omega_\nu) \qquad \omega > 0$$

The ***strength function*** associated with the operator F:

$$\boxed{S(\omega) = -\frac{1}{\pi} Im\ R_F(\omega) = \sum_{\nu>0} |\langle\nu|F|0\rangle|^2 \delta(\hbar\omega - \hbar\Omega_\nu)}$$

C. Multipole transition operators

… electromagnetic excitation of a nucleus with real photons. The transition amplitude is the matrix element of the operator:

$$F_{JM}(\mathbf{r}) = \sum_{i=1}^{Z} e r_i^J Y_{JM}(\hat{r}_i)$$

Transitions induced by the strong interaction are typically studied by means of hadron inelastic scattering. The nuclear multipole transition operators:

$$F_{JM}(\mathbf{r}) = \sum_{i=1}^{A} r_i^J Y_{JM}(\hat{r}_i)$$

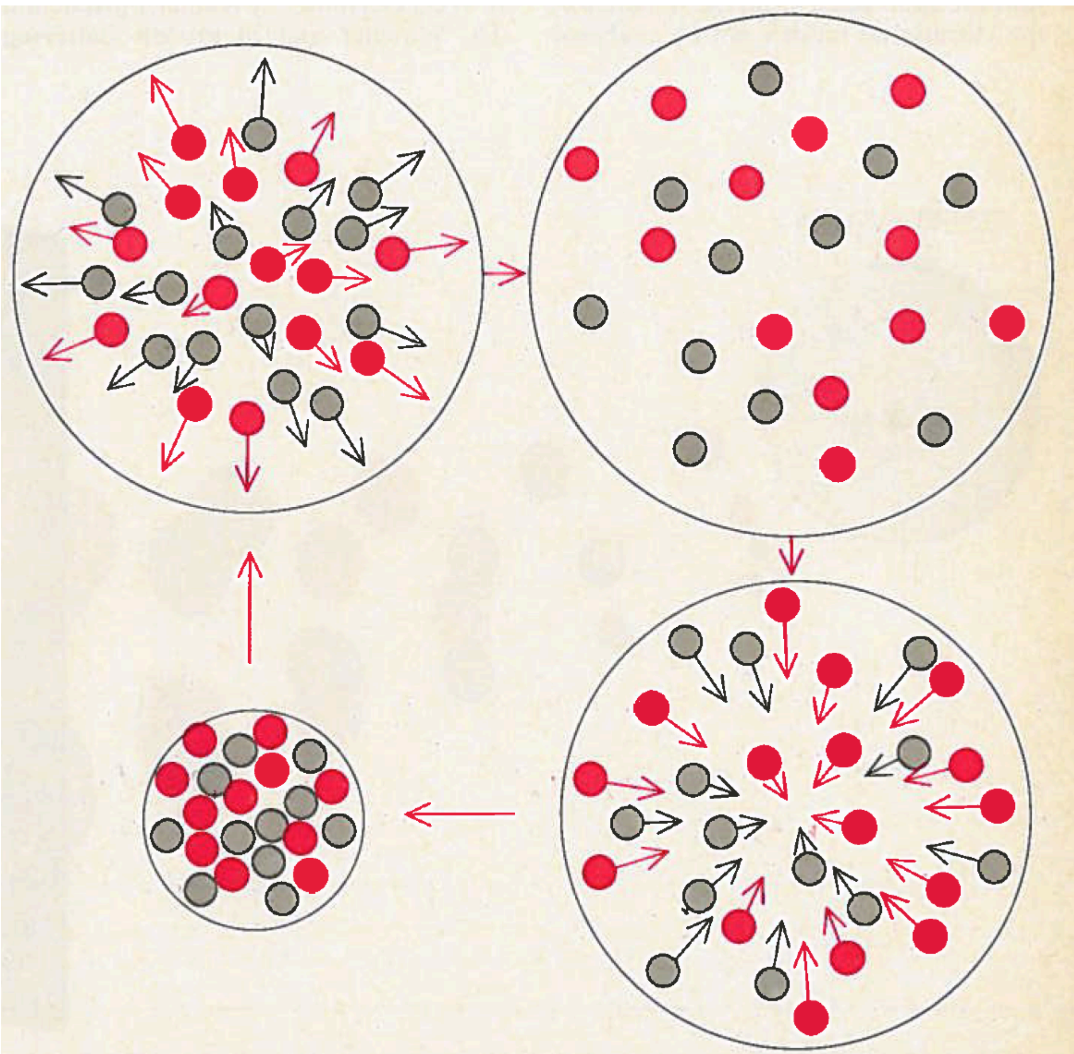
isoscalar excitations

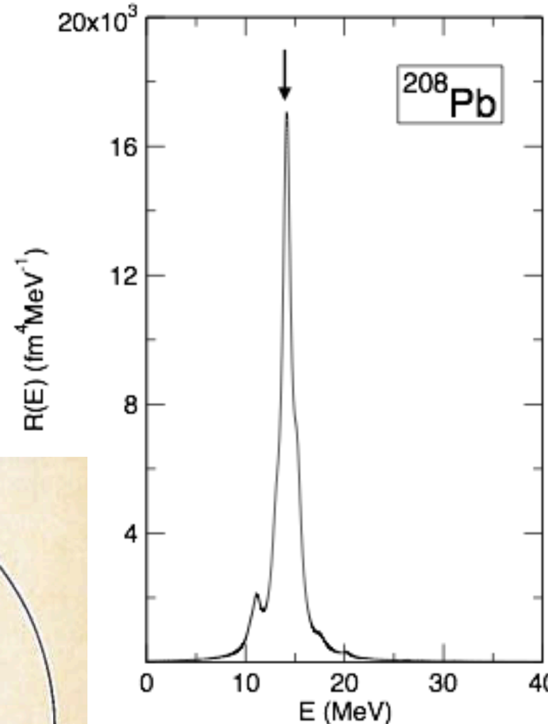
$$F_{JM}(\mathbf{r}) = \sum_{i=1}^{A} r_i^J Y_{JM}(\hat{r}_i)\tau_z(i)$$

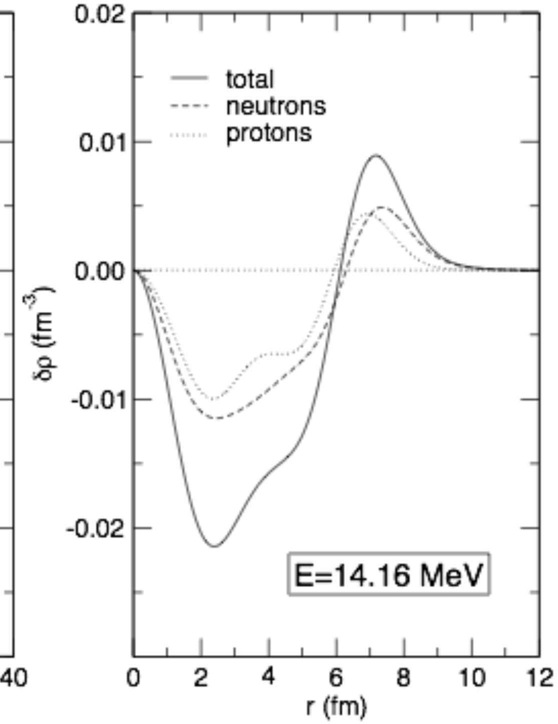
isovector excitations

D. Illustrative examples

Giant isoscalar monopole vibration

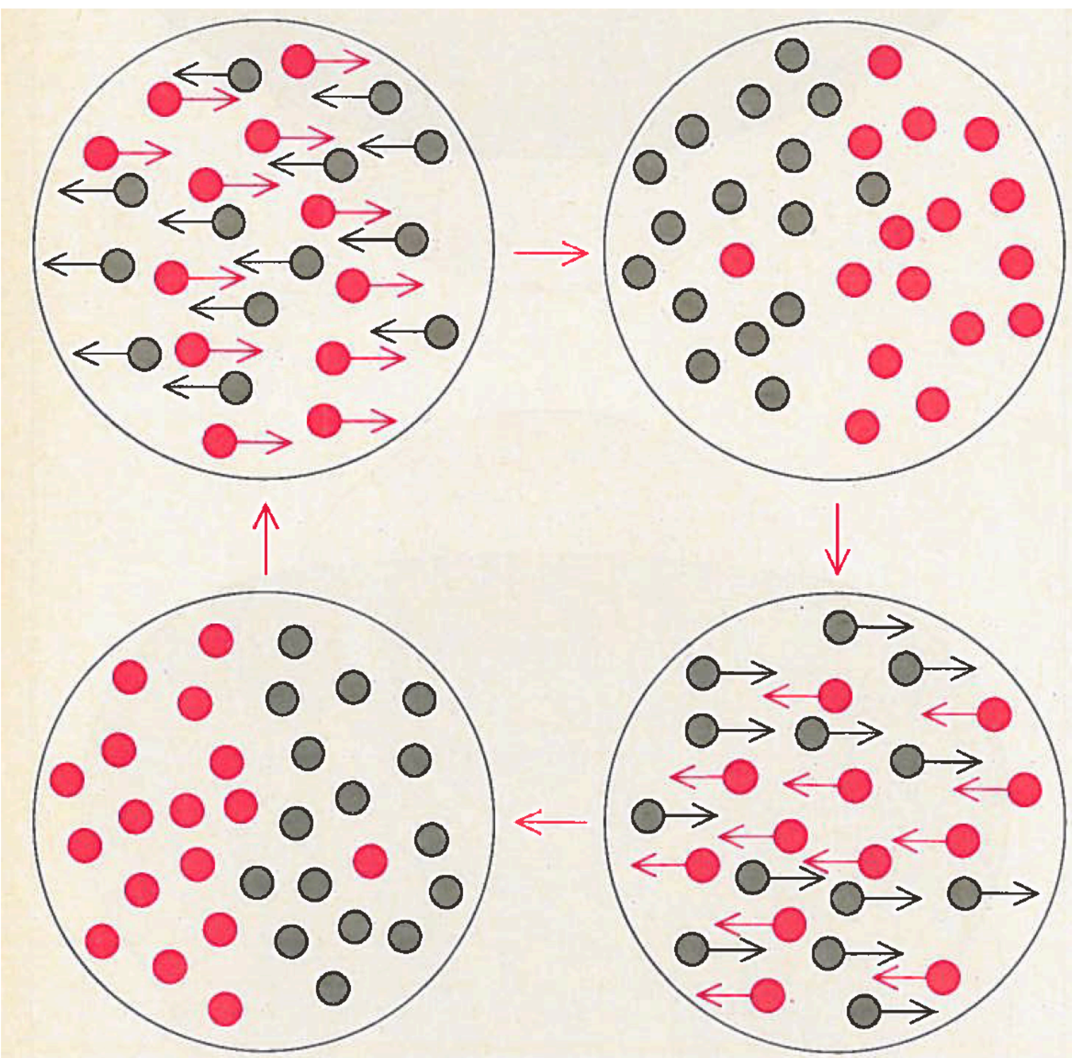


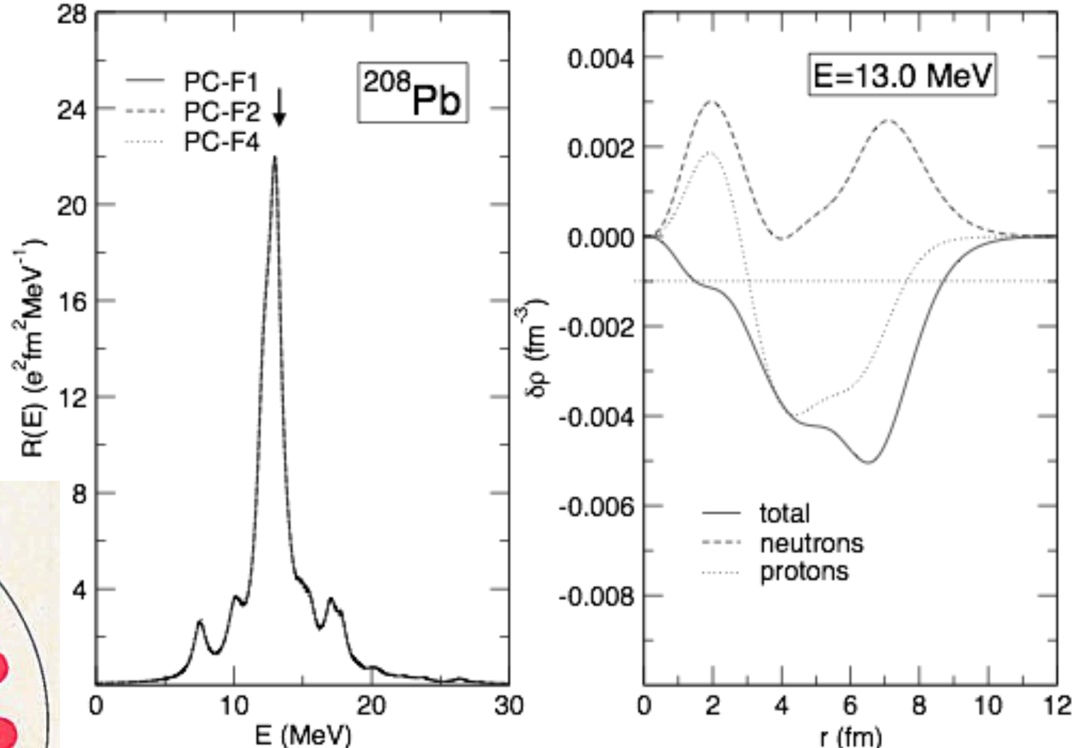




⇒ information about the nuclear matter incompressibility modulus.

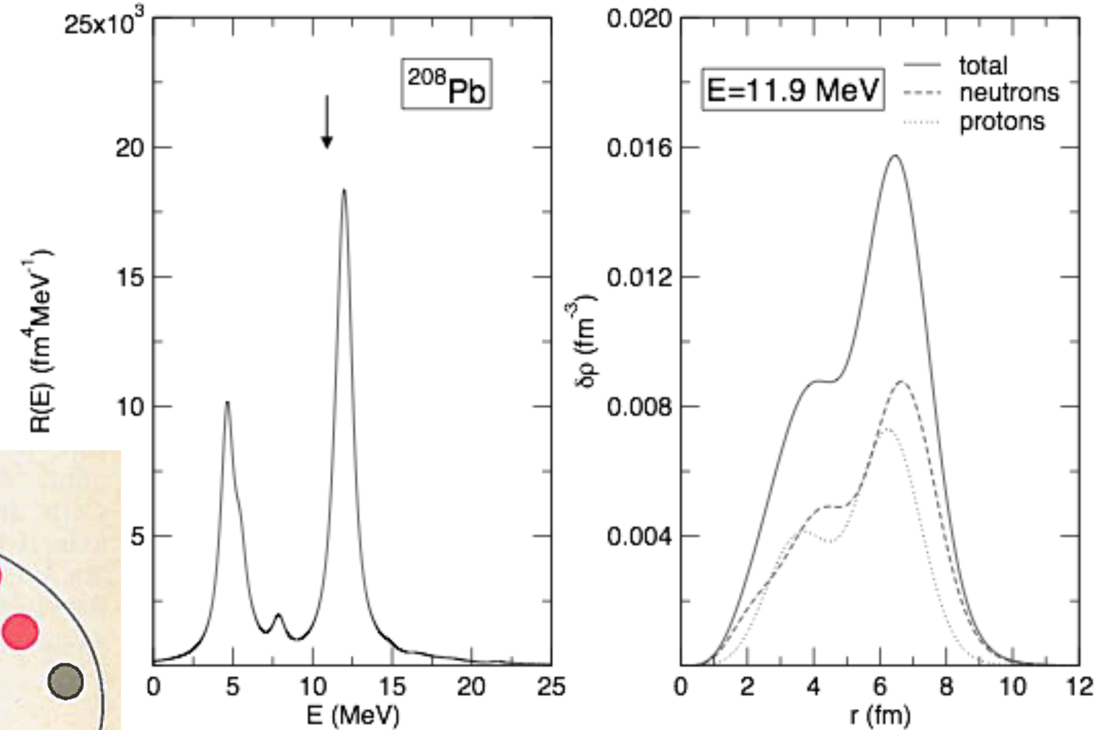
Giant isovector dipole vibration

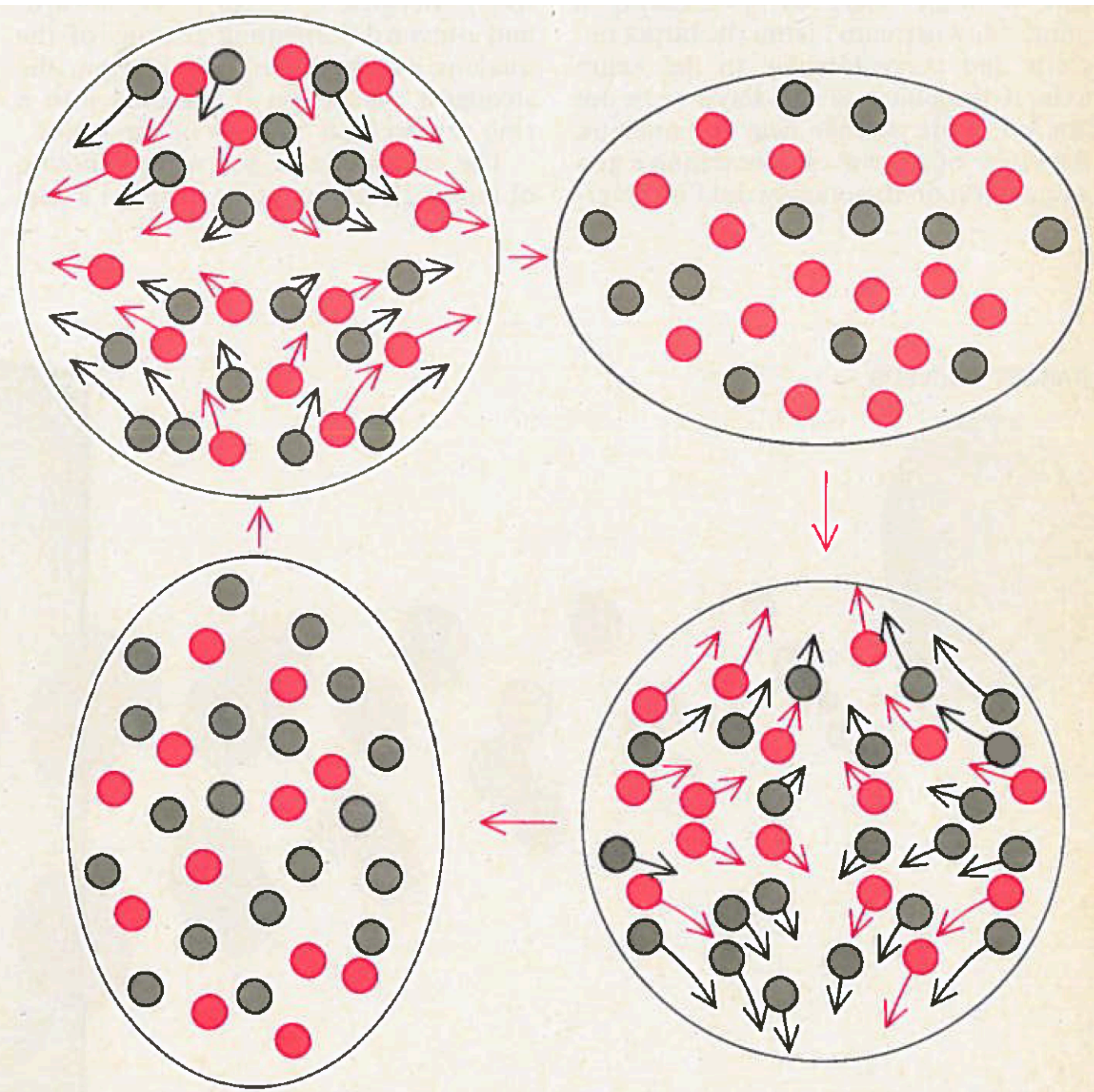




⇒ information about nuclear matter symmetry energy.

Giant isoscalar quadrupole vibration





⇒ information about the nucleon effective mass in nuclear matter.

4. Beyond the Mean-Field Approximation

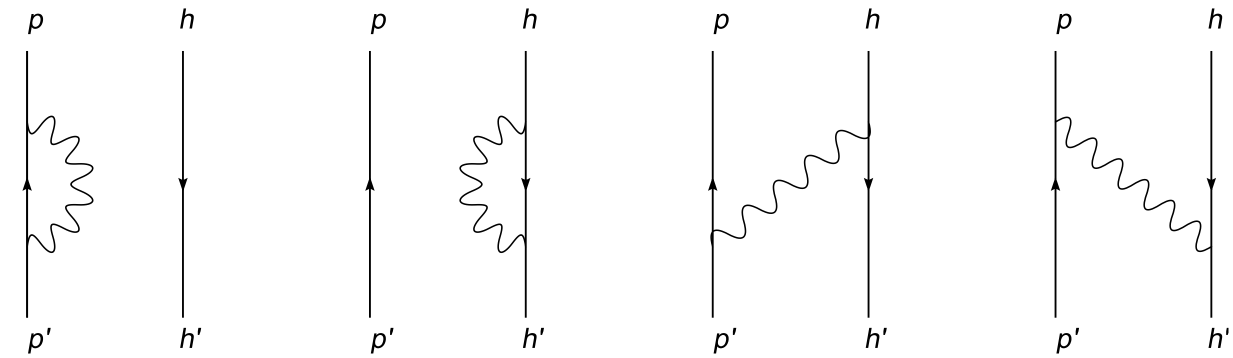
At the RPA level, nuclear collective motion is represented as a coherent superposition of *1p−1h* states. The energy and angular momentum of these vibrations can be released to other degrees of freedom, because vibrational states are embedded in a dense background of excited states.

The **width of a giant resonance** originates from:

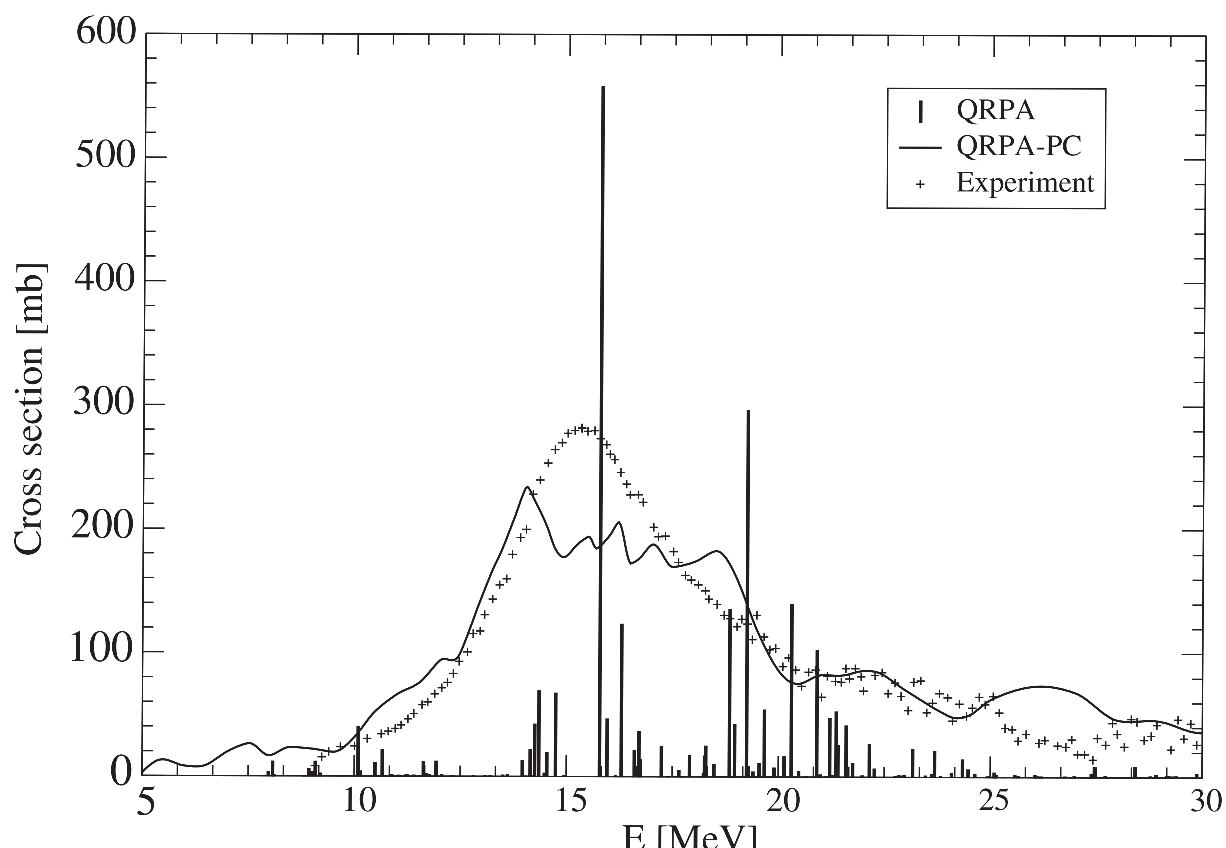
(i) When the energy of a vibrational state lies above the particle emission threshold, the state can decay by neutron or proton emission. This damping mechanism is associated with the escape width $\Gamma_{\uparrow}$, which can be taken into account within the framework of continuum-RPA → use scattering solutions of the s.p. HF equations in the calculation of R(ω).

(ii) The spreading width $\Gamma^{\downarrow}$ arises because the energy and angular momentum of coherent vibrations can be transferred to more complicated nuclear states, of *2p−2h* (and eventually *3p-3h*, . . . ,*np-nh*) character. In order to describe $\Gamma^{\downarrow}$ a theoretical framework must include the coupling to these complex configurations: Second RPA, particle-vibration coupling model, ETFFS, QPM, ...

Diagrams which correspond to the coupling of the p – h components of a giant resonance with phonon states.



Photoabsorbtion cross section for ^{120}Sn, calculated with the QRPA (bars) and QRPA-PC (solid curve).



Quasiparticle-Phonon Model:

Fragmentation of the low-lying electric dipole strength in ^{138}Ba. Calculations are performed in the one-phonon approximation (top panel), and taking into account the coupling to two-phonon configurations (middle panel), or to two- and three-phonon configurations (bottom panel).