

# Nuclear Structure Information from Peripheral Nuclear Reactions

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## Lecture 3

### Reactions in the continuum

## Continuum discretization

Problem: continuum w.f. extends to infinity

Matrix elements for EM multipole operators

$\langle j||x||i \rangle$  explode

Solution: bunch states around discrete energy  $E_i$

CB, Canto, NPA 539, 163 (1992)

$$|\phi_0\rangle = |E_0, J_0 M_0\rangle e^{-iE_0 t / \hbar}$$

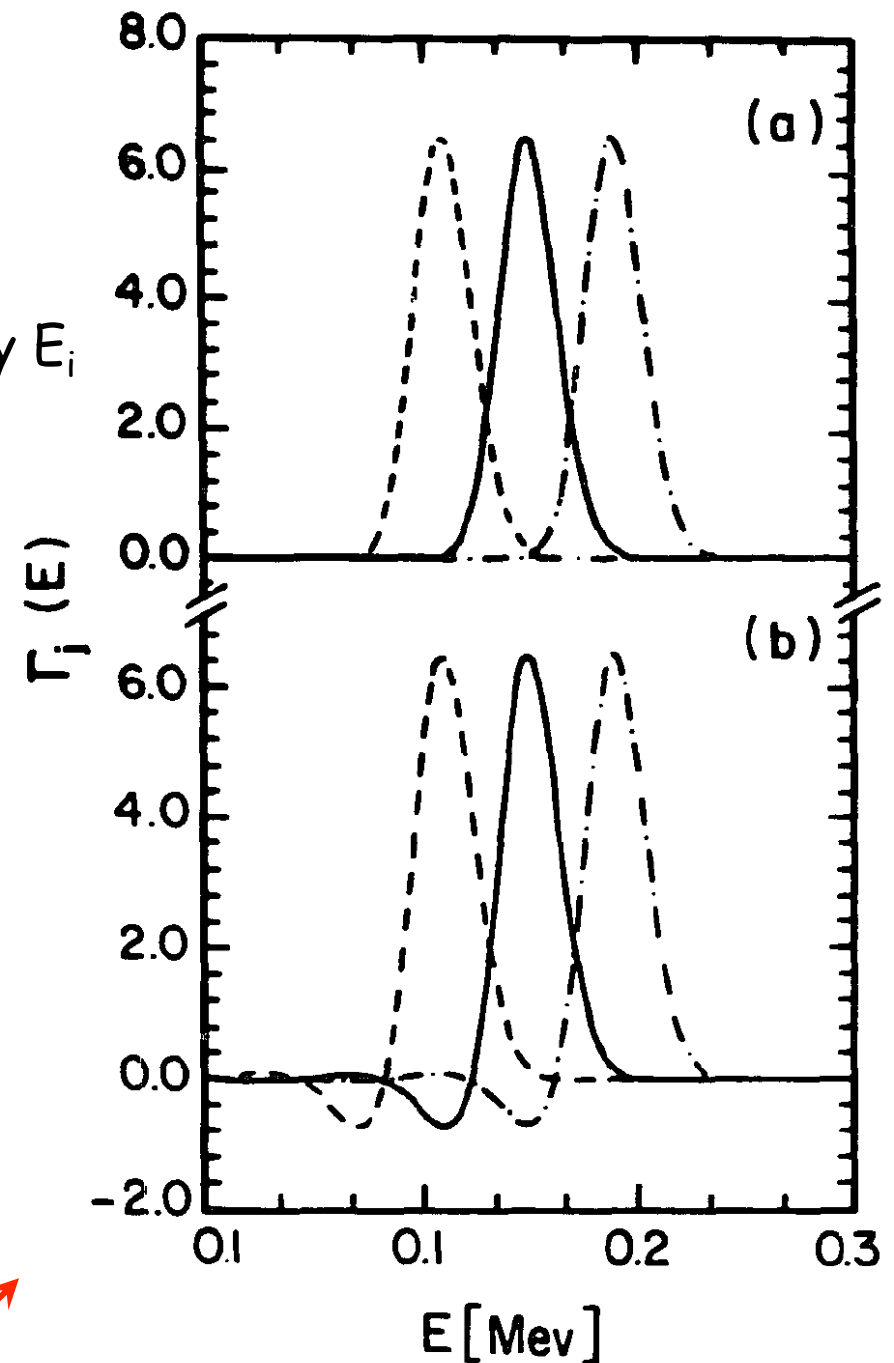
$$|\phi_{jJM}\rangle = e^{-iE_j t / \hbar} \int \Gamma_j(E) |E, JM\rangle dE$$

$$\int \Gamma_i(E) \Gamma_j(E) dE = \delta_{ij}$$

Ex: histograms

$$\Gamma_j(E) = \frac{1}{\sqrt{\sigma}}, \quad \text{for } (j-1)\sigma < E < j\sigma$$
$$= 0, \quad \text{otherwise}$$

Or, some other sort of smooth,  
orthogonal functions: **better**



# Continuum discretization

Application example: plane waves

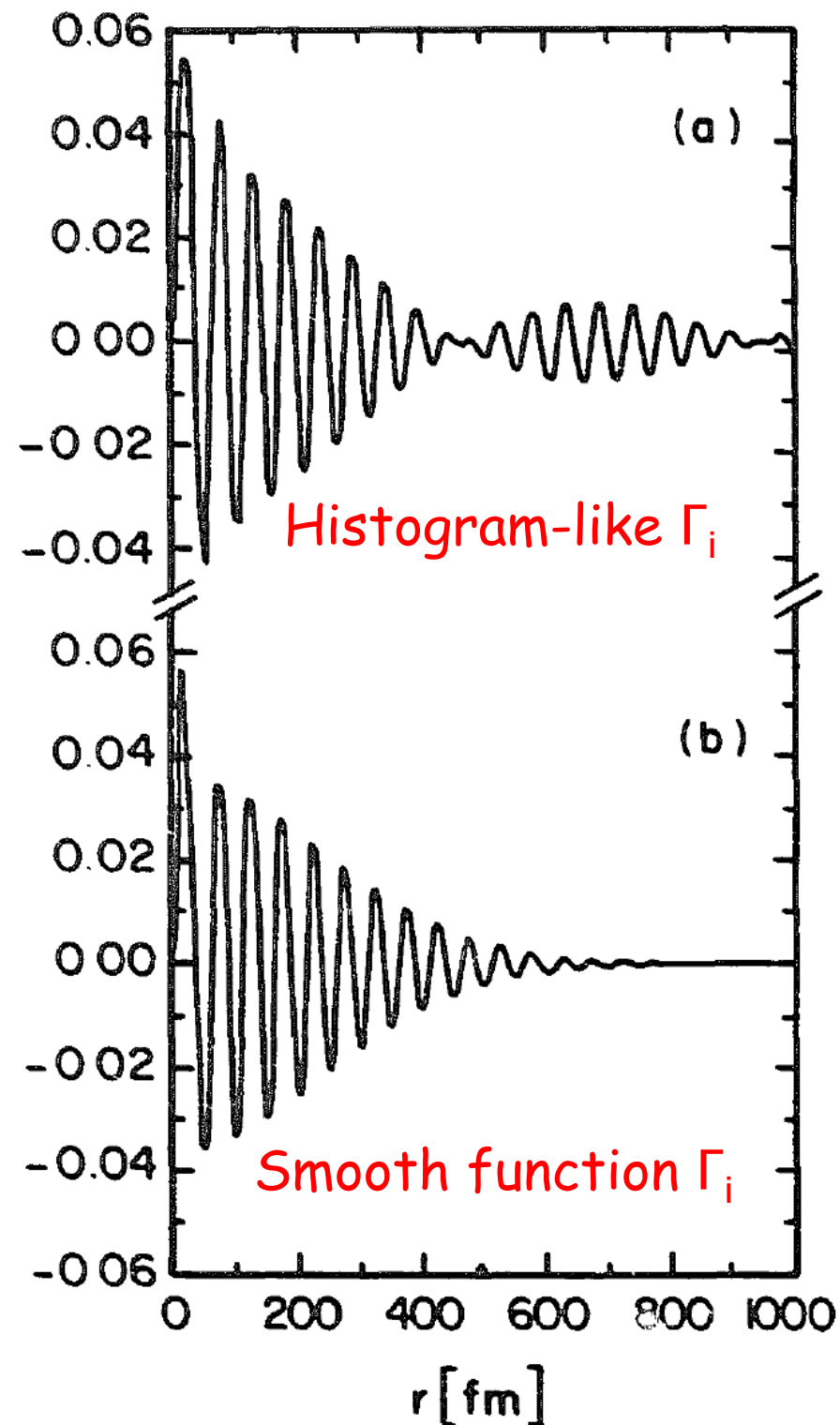
$$e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \sum_{lm} i^l j_l(qr) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{q}})$$



$$\begin{aligned} \langle \mathbf{r} | E l m \rangle &= u_{l,E}(r) Y_{lm}(\hat{\mathbf{r}}) \\ &= \left( \frac{2\mu}{\hbar^2} \right) \frac{E^{1/4}}{\sqrt{\pi}} j_l(qr) Y_{lm}(\hat{\mathbf{r}}) \end{aligned}$$

$$\langle E l m | E' l' m' \rangle = \delta_{ll'} \delta_{mm'} \delta(E - E')$$

Smooth, orthogonal functions: **better**  
(no beats, faster convergence)



# Continuum discretization

Application example: dipole transitions

$$I_{jl;j'l'} = \int r^2 dr \int dE \Gamma_j(E) \int dE' \Gamma_{j'}(E') u_{l,E}^*(r) r u_{l',E'}(r)$$



for histograms

$$I_{jl;j'l'} = \frac{\hbar^2}{\mu} \left\{ \frac{1+l'+2}{2} F_{jj'} + \delta_{l,l'+1} G_{j,j'} + \delta_{l+1,l'} G_{j'j} \right\}$$

$$|l-l'| = 1$$

$$F_{jj'} = \int dq \Gamma_j(E) \Gamma_{j'}(E)$$

$$F_{jj'} = \int dq \Gamma_j(E) \frac{d}{dq} \Gamma_{j'}(E)$$

$$E = \frac{\hbar^2 q^2}{2\mu}$$

$$E_0 = -\frac{\hbar^2 \eta^2}{2\mu}$$

Continuum

Bound state

$$I_{00;jl} = \frac{\sqrt{2\eta\sigma}}{\pi} \frac{E_j^{3/4}}{(E_0 + E_j)^2} \left( \frac{\hbar^2}{2\mu} \right)^{3/4}$$

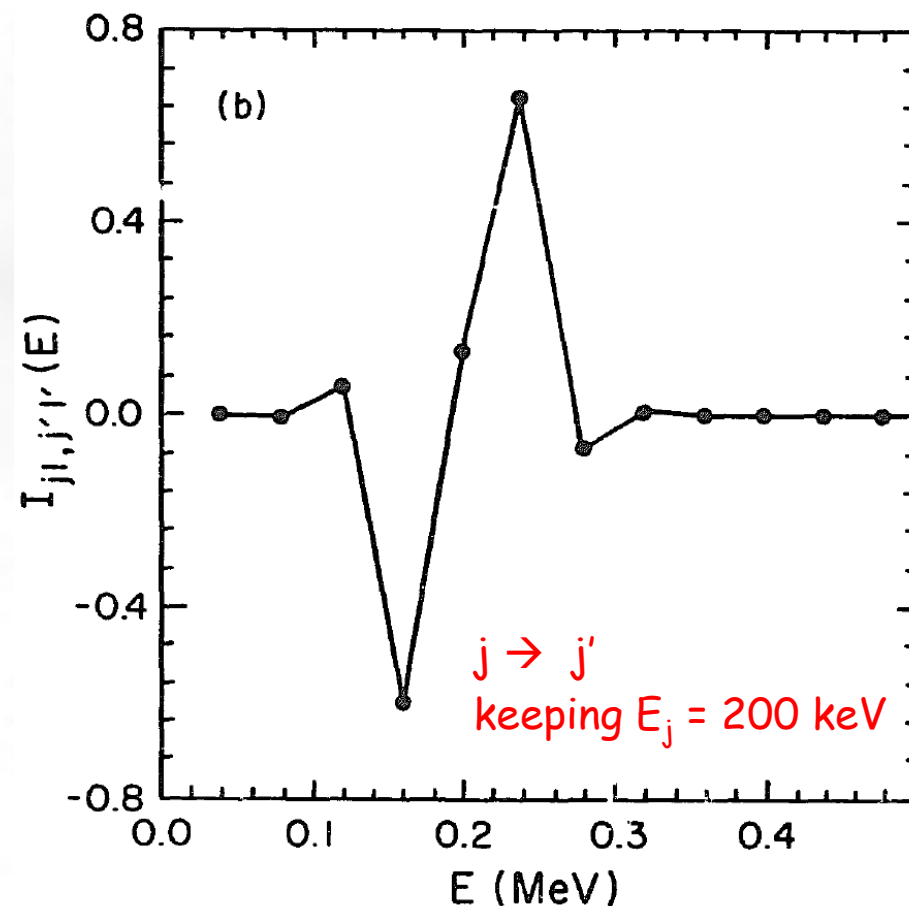
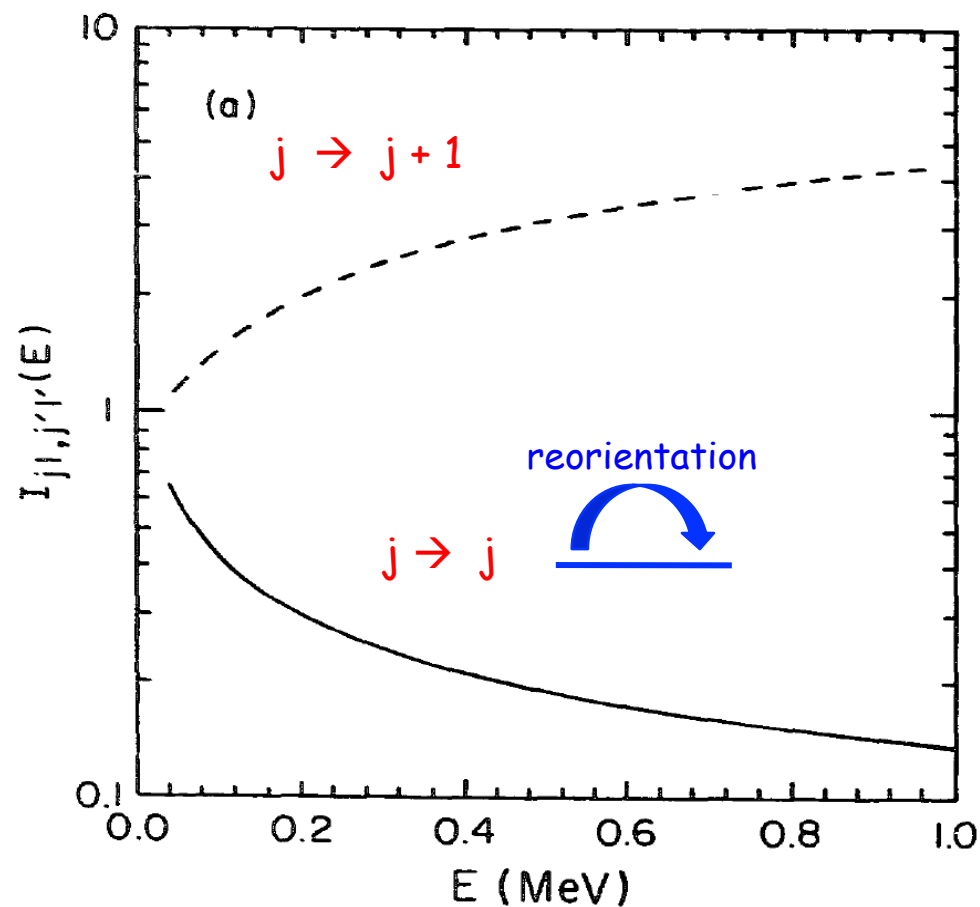


$$I_{jl;j'l'} = \hbar \sqrt{\frac{2}{\mu\sigma}} \begin{cases} \frac{1}{2}(1+l'+1) [\sqrt{j} - \sqrt{j-1}] & \text{if } j = j' \\ -(-1)^{(j+l-j'-l')} \sqrt{\frac{1}{2}(j+j'-1)} & \text{if } |j-j'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E_j = (j - 1/2)\sigma$$

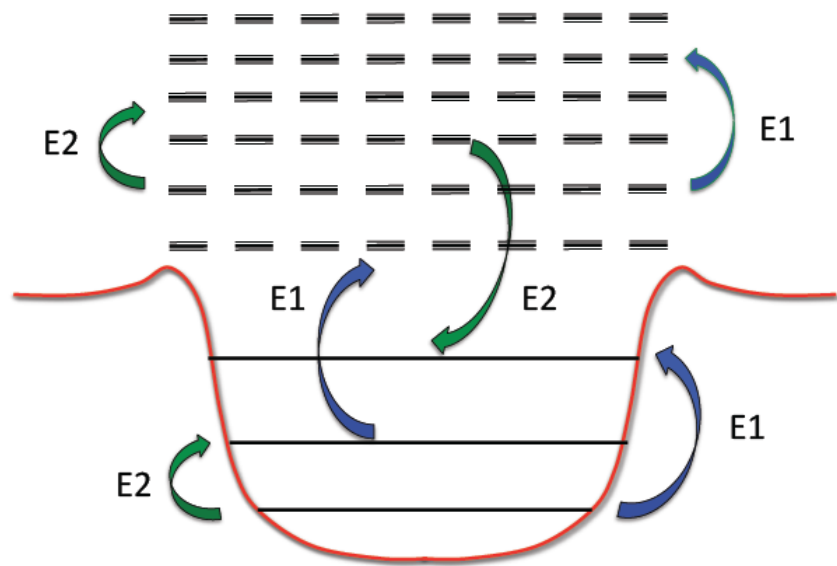
# Continuum discretization

Application example: dipole transitions  $\ell = 0 \rightarrow \ell' = 1$



CB, Canto, NPA 539, 163 (1992)

# Continuum Discretized Coupled Channels (CDCC) Time-dependent



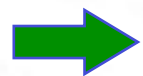
From lecture 1:

$$\frac{da_k}{dt} = -\frac{i}{\hbar} \sum_j a_j(t) U_{kj}(t) e^{i(E_k - E_j)t/\hbar}$$

$$U_{kj}(t) = \int \psi_k^* U(t) \psi_j d^3r$$

$$\psi(\mathbf{r}) = \sum_n a_n(t) \psi_n(\mathbf{r}) e^{-iE_n t/\hbar} \quad n = j(E)lm$$

(for bound and continuum states  $n$ )



Amplitudes  $a_n \rightarrow$  occupation probabilities  $|a_n|^2$

$\rightarrow$  wavefunctions  $\psi(\mathbf{r})$  (normalized if enough  $\#$   $n$ )

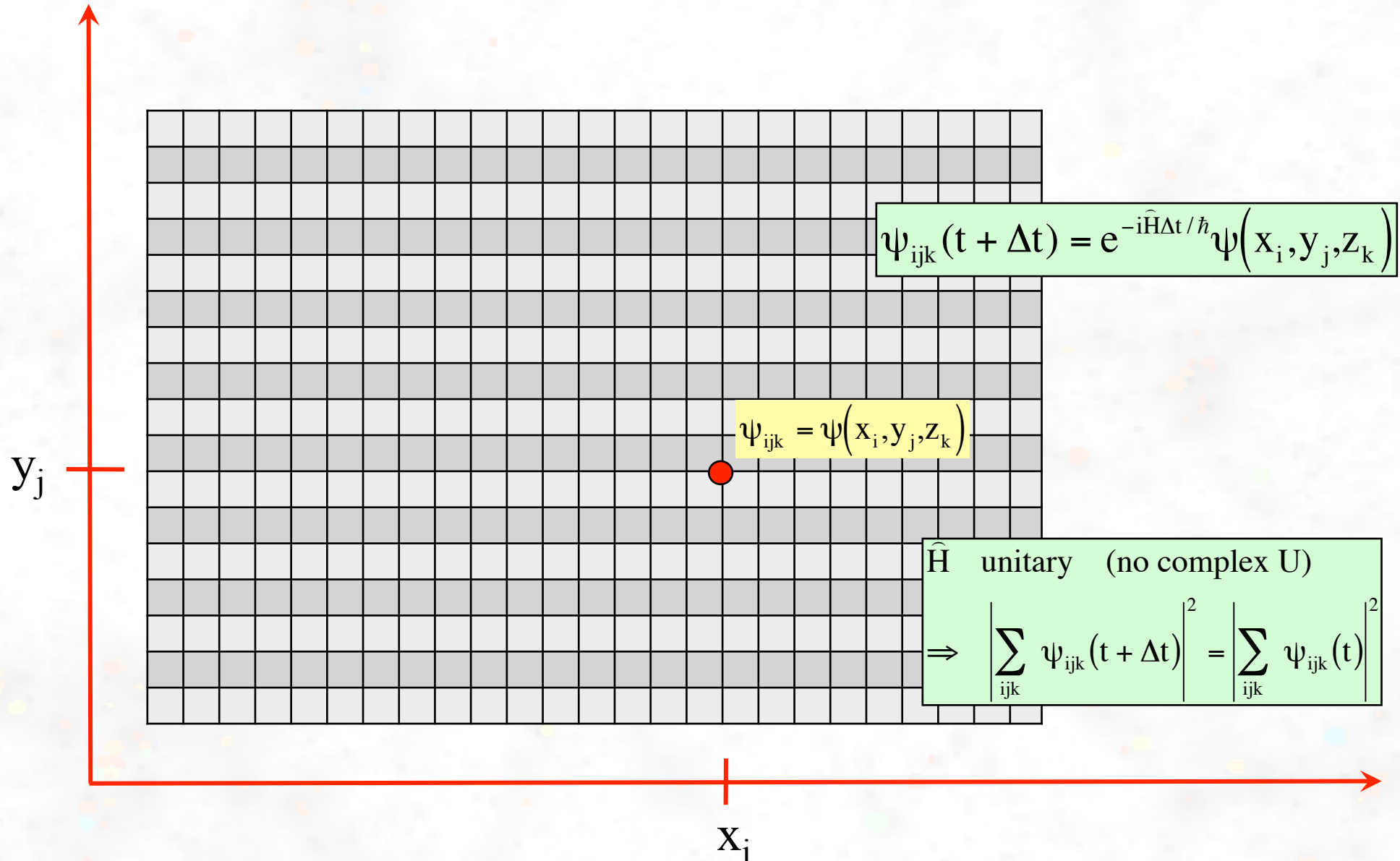
$\rightarrow$  calculate observables

# Schroedinger equation on a space-time lattice

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$H = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V(\mathbf{r}) + U(\mathbf{r}, t)$$

Bertsch, CB  
NPA 556, 136 (1993)  
PRC 49, 2839 (1994)



# Schroedinger equation on a space-time lattice (one-dimension)

$$e^{-i\hat{H}\Delta t/\hbar} = 1 + \left(\frac{-i\hat{H}\Delta t}{\hbar}\right) + \left(\frac{-i\hat{H}\Delta t}{\hbar}\right)^2 + \dots$$

Often requires too many terms to preserve unitarity and is unstable

Use Crank-Nicolson operator instead:

Unitary and accurate to  $(\Delta t)^2$

Include  $U$ : define  $\hat{S}\psi_j(t) = \sum_k U_k \psi_k(t)$

$$\psi_j(t + \Delta t) = \frac{\left[1 - i\frac{\hat{H}\Delta t}{\hbar} + \Delta t \hat{S}\right]}{\left[1 + i\frac{\hat{H}\Delta t}{\hbar}\right]} \psi_j(t)$$

Good to order  $(\Delta t)^2$  and preserves unitarity if  $\Delta t$  small enough

Problem: needs to invert matrix

$$\frac{1}{\left[1 + i\frac{\hat{H}\Delta t}{\hbar}\right]} \psi_j(t)$$

$$\Delta^{(2)} = \left(\psi_{j-1} - 2\psi_j + \psi_{j+1}\right) / 2(\Delta x)^2$$



# Schroedinger equation on a space-time lattice (one-dimension)

Solution: use three-point second derivative

$$\Delta^{(2)} = (\psi_{j-1} - 2\psi_j + \psi_{j+1}) / 2(\Delta x)^2$$

Then

$$\psi_j(t + \Delta t) = \frac{\left[ \frac{1}{i\tau} + \Delta^{(2)} - \frac{\Delta t}{2\hbar\tau} V_j + \frac{\Delta t}{\hbar\tau} \hat{S} \right]}{\left[ \frac{1}{i\tau} - \Delta^{(2)} + \frac{\Delta t}{2\hbar\tau} V_j \right]} \psi_j(t)$$

involves only  $j, j-1$  and  $j+1$  for neighboring points

Instead of

$$u_j(t) = \frac{1}{\left[ 1 + i \frac{\hat{H}\Delta t}{\hbar} \right]} \psi_j(t)$$

solve for

$$\psi_j(t) = \left[ 1 + i \frac{\hat{H}\Delta t}{\hbar} \right] u_j(t)$$

But  $u_j$  not known !

**Solution:** using 3-point formula  $\rightarrow$  3-diagonal matrix

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} = \begin{bmatrix} O_{11} & O_{12} & 0 & 0 & 0 & 0 & & & \\ O_{21} & O_{22} & O_{23} & 0 & 0 & 0 & & & \\ 0 & O_{32} & O_{33} & O_{34} & 0 & 0 & & & \\ 0 & 0 & O_{43} & O_{44} & O_{44} & 0 & & & \\ & & & & & \dots & \dots & & \\ & & & & & \dots & \dots & & \\ & & & & & \dots & \dots & & \\ & & & & & & & & \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

←

# Schroedinger equation on a space-time lattice (one-dimension)

**First operation:**  $\psi_1 = O_{11} u_1 + O_{12} u_2 \rightarrow$  assume  $u_1 = f_1$  hold fixed by a boundary condition  $\rightarrow f_1$  known,  $u_1$  known  $\rightarrow u_2$  determined

**Second operation:** operation involves  $\psi_2$ ,  $u_1$ ,  $u_2$  and  $u_3$   
 $\rightarrow u_1$  and  $u_2$  (from previous)  $\psi_2$  known  $\rightarrow u_3$  determined

And so on:  $\rightarrow$  given  $\psi_j$  on lattice at  $t \rightarrow u_j$  obtained.

Finally,  $\psi_j(t + \Delta t) = \left[ 1 + i \frac{\hat{H} \Delta t}{\hbar} \right] u_j(t)$  is just a matrix multiplication

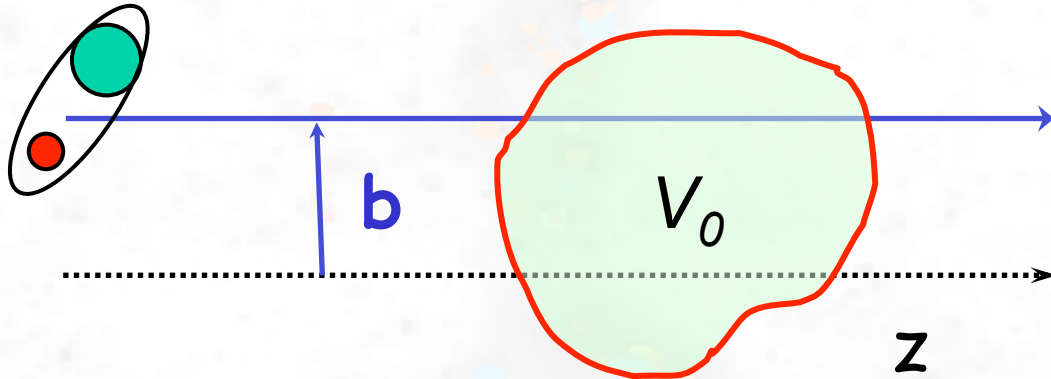
$\rightarrow$  One-particle in three-dimensions straightforward

$\rightarrow$  Extension to many-body: e.g., Time Dependent Hartree-Fock (TDHF)

# Time independent - CDCC

CB, PRL 94, 072701 (2005)

(at high energies)



$$\Psi(\mathbf{R}, \mathbf{r}) = \sum_{\alpha} S_{\alpha}(\mathbf{b}, z) e^{ik_{\alpha}z} \phi_{\alpha}(\mathbf{r}),$$

$$\mathbf{R} = (\mathbf{b}, z)$$

$$\frac{da_k}{dt} = -\frac{i}{\hbar} \sum_j a_j(t) U_{kj}(t) e^{i(E_k - E_j)t/\hbar}$$

with

$$\begin{array}{ll} a_{\alpha} & \rightarrow S_{\alpha} \\ vt & \rightarrow z \end{array}$$

$$iv\partial_z S_{\alpha}(\mathbf{b}, z) = \sum_{\beta} U_{\alpha\beta}(\mathbf{b}, z) S_{\beta}(\mathbf{b}, z) e^{i(k_{\beta} - k_{\alpha})z}$$

$$U_{\alpha\beta}(\mathbf{R}) = \langle \psi_{\alpha}(\mathbf{r}) | U(\mathbf{R}, \mathbf{r}) | \psi_{\beta}(\mathbf{r}) \rangle$$



$$f_{\alpha}(\mathbf{Q}) = -\frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{Q} \cdot \mathbf{b}} [S_{\alpha}(\mathbf{b}, z = \infty) - \delta_{\alpha,0}]$$

$$\mathbf{Q} = \mathbf{K}'_{\perp} - \mathbf{K}_{\perp} \quad \alpha = j l J M$$

Eikonal CDCC

**One needs potentials**

**High energies  $E > 50 \text{ MeV/nucleon}$**

**→ one needs relativistic corrections**

# Example: Rutherford scattering with relativity

Aguilar, Aleixo, CB, PRC 42, 2180 (1990)

$$L = L^{\text{LO}} + L^{\text{NLO}} + L^{\text{N}^2\text{LO}} + \dots$$

$$L^{\text{LO}} = \frac{1}{2} \mu c^2 \left( \frac{v}{c} \right)^2 - \frac{Z_1 Z_2 e^2}{r}$$

$$L^{\text{NLO}} = \frac{\mu^4 c^2}{8} \left[ \frac{1}{m_1^3} - \frac{1}{m_2^3} \right] \left( \frac{v}{c} \right)^4 - \frac{\mu^2 Z_1 Z_2 e^2}{2 m_1 m_2 r} \left[ \left( \frac{v}{c} \right)^2 + \left( \frac{\mathbf{v} \cdot \mathbf{r}}{c r} \right)^2 \right]$$

$$L^{\text{N}^2\text{LO}} = \frac{\mu c^2}{512} \left( \frac{v}{c} \right)^6 + \frac{Z_1 Z_2 e^2}{16 r}$$

$$\times \left[ \frac{1}{8} \left\{ \left( \frac{v}{c} \right)^4 - 3 \left( \frac{v_r}{c} \right)^4 + 2 \left( \frac{v_r v}{c} \right)^2 \right\} + \frac{Z_1 Z_2 e^2}{\mu c^2 r} \left\{ 3 \left( \frac{v_r}{c} \right)^2 - \left( \frac{v}{c} \right)^2 \right\} + \frac{4 Z_1^2 Z_2^2 e^4}{\mu^2 c^4 r^2} \right]$$

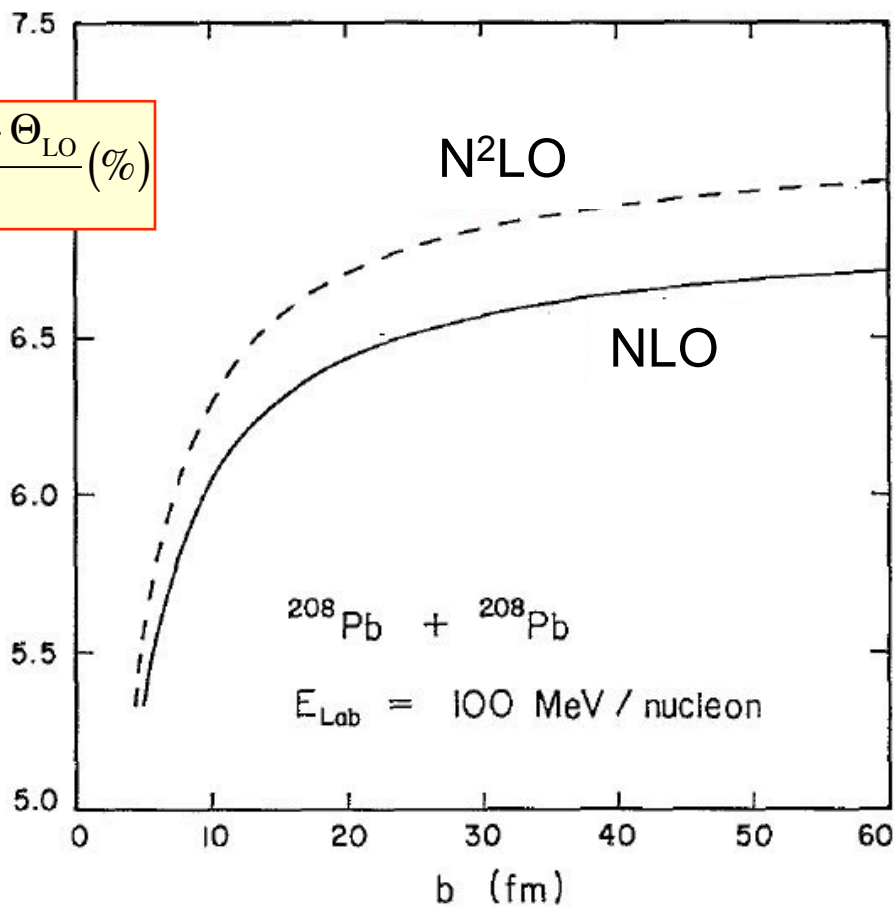
Equations of motion



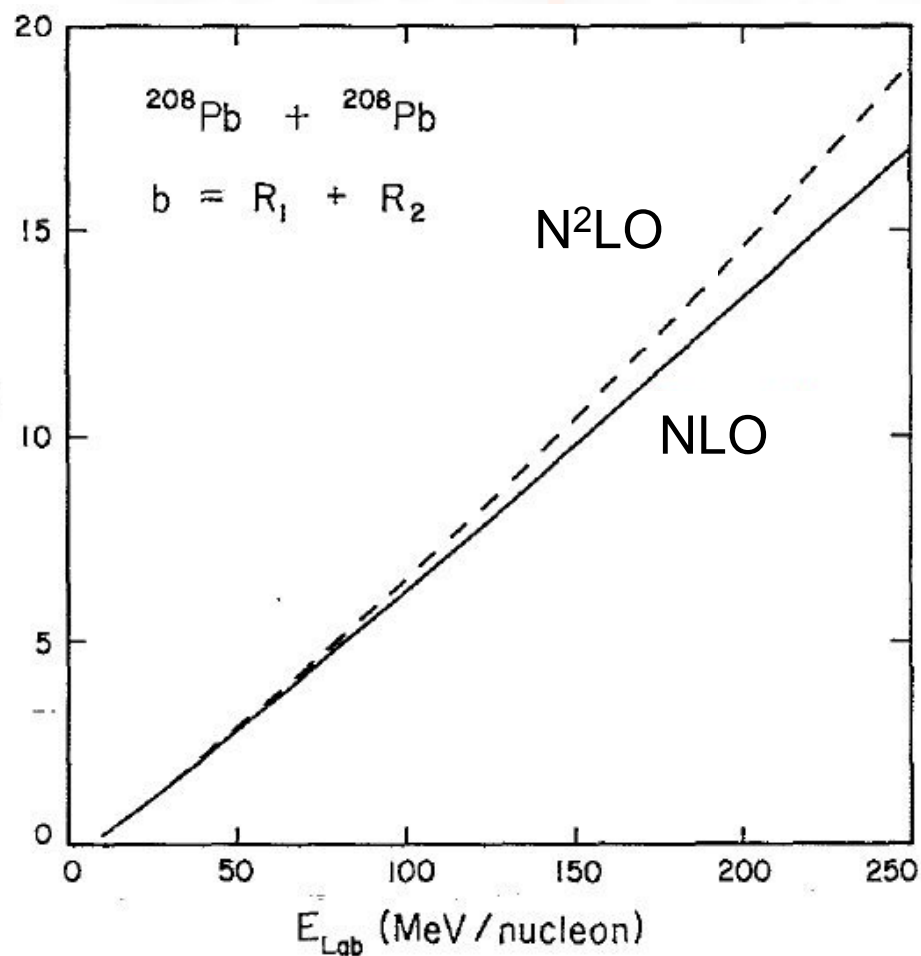
$\mathbf{r}(t)$ ,  $\mathbf{p}(t)$ ,  $\Theta(b)$ ,

$$\frac{d\sigma}{d\Omega}$$

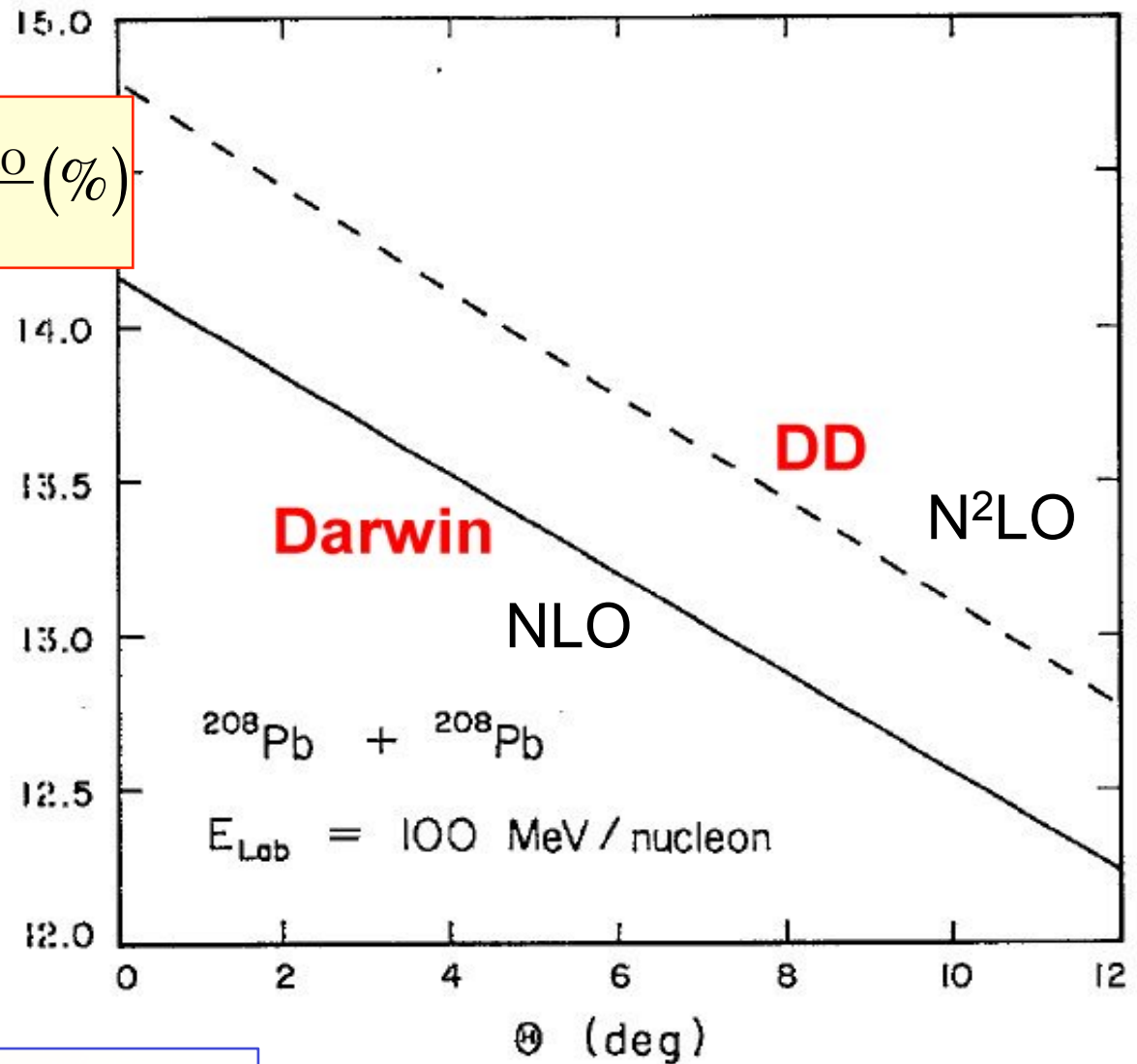
$$\frac{\Theta(E,b) - \Theta_{LO}}{\Theta_{LO}} (\%)$$



## Deviations from Rutherford



$$\frac{d\sigma(E,\theta) - d\sigma_{LO}}{d\sigma_{LO}} (\%)$$



important for elastic scattering:  
experimental data often reported

as

$$\frac{d\sigma_{elast}}{d\sigma_{Ruth}}$$



# Coulomb excitation with relativistic corrections

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{elast}} \times P_{\text{exc}} = \sum_{\pi\lambda\mu} |I(\pi\lambda\mu)|^2 |M_{\text{fi}}(\pi\lambda, -\mu)|^2$$

$$M_{\text{fi}}(\pi\lambda\mu) = \langle f | \text{EM Operator}(\lambda\mu) | i \rangle$$

$$I(\pi\lambda\mu) = \text{orbital integrals}$$

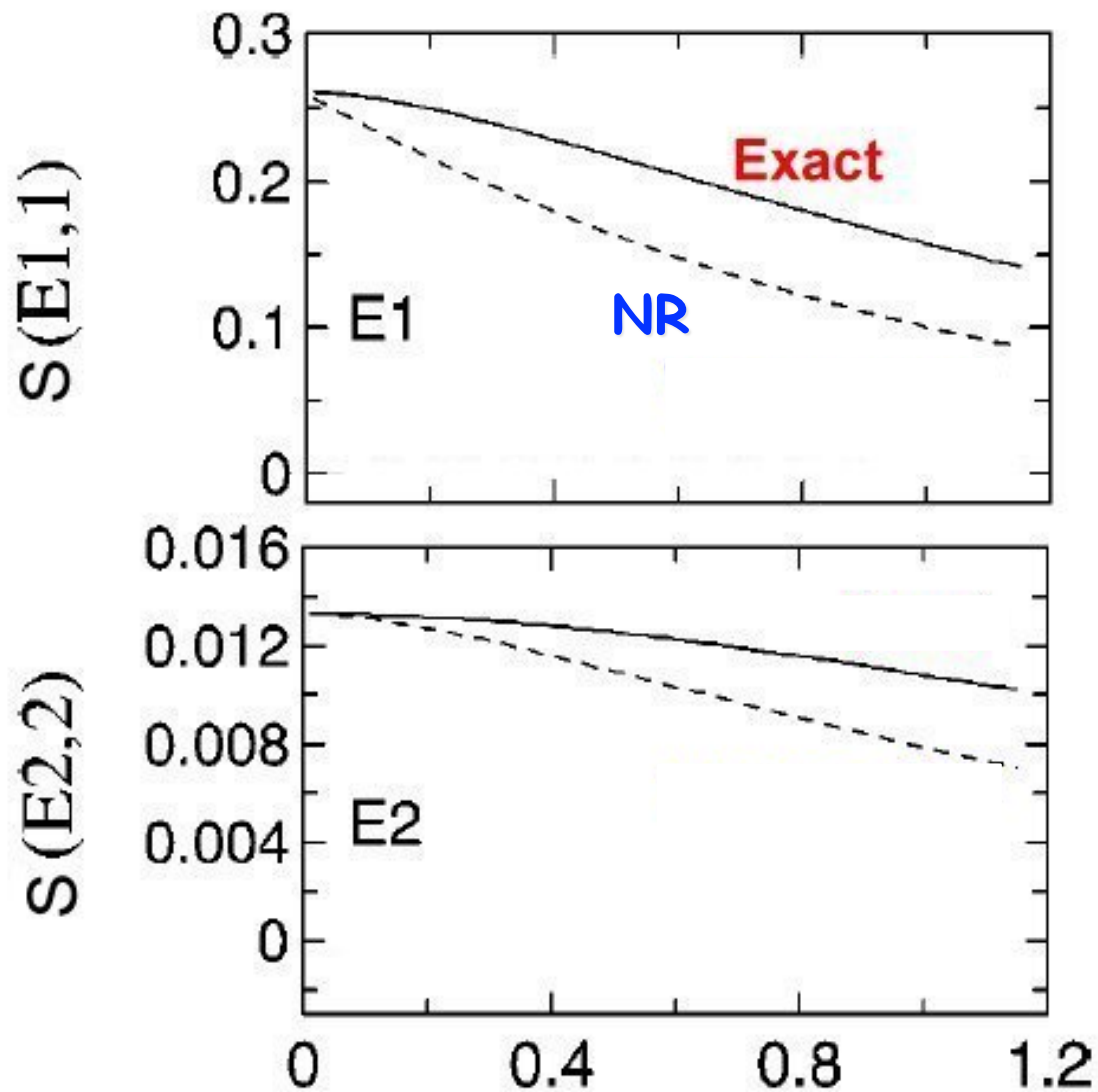
$I(\pi\lambda\mu)$  from Lecture 1:

$$I(\omega) = \int_{-\infty}^{\infty} dt \frac{1}{r^{\lambda+1}(t)} Y_{\lambda\mu}(\hat{\mathbf{r}}(t)) e^{i\omega t}$$

$$\frac{1}{r^{\lambda+1}(t)} Y_{\lambda\mu}(\hat{\mathbf{r}}(t)) \rightarrow \text{complicated functions of } t$$

Possible to include retardation, Lorentz contraction exactly

Aleixo, CB, NPA 505, 448 (1989)



Deviations from  
non-relativistic

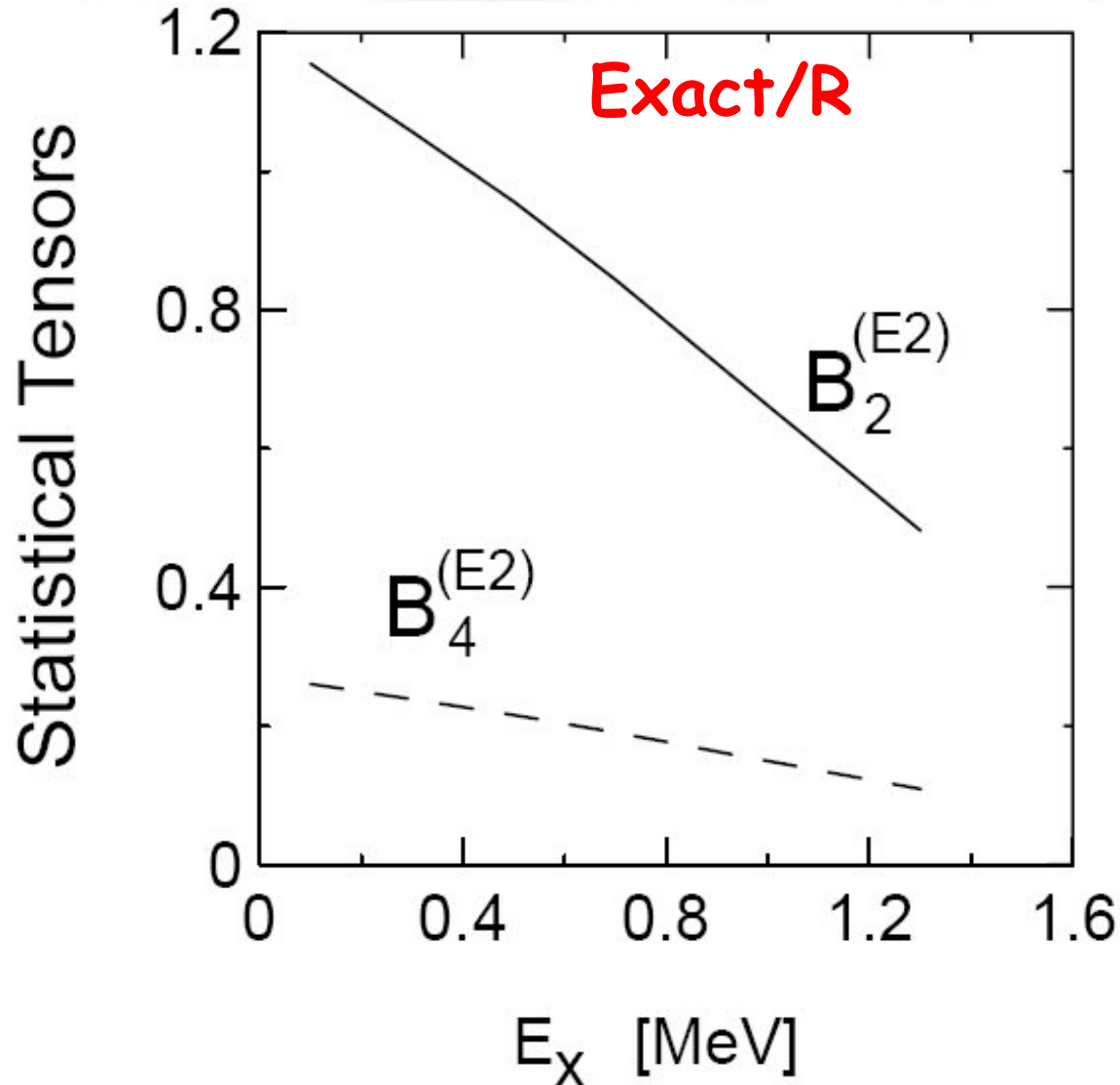
$^{40}\text{S}$  (100 MeV/nucleon) + Au

$$\xi = \frac{E_x b}{\gamma \hbar v}$$

Corrections important  
large  $b$ 's, large  $E_x$ 's

## De-excitation by $\gamma$ -ray emission

$$W_{\gamma}(\theta_{\gamma}) = 1 + \sum_{\kappa=2,4} B_{\kappa} Q_{\kappa}(E_{\gamma}) P_{\kappa}(\cos \theta_{\gamma})$$



Deviations from  
from relativistic  
theory

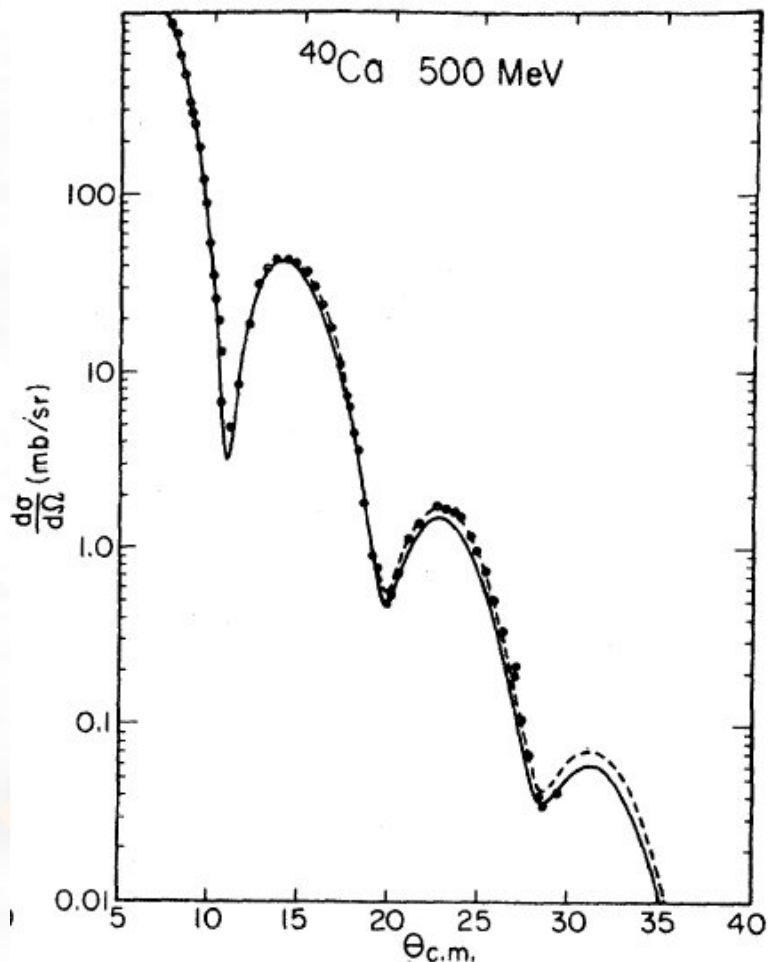
$^{38}\text{S}$  (100 MeV/nucleon) + Au

**Is it possible to do the same for  
nucleus-nucleus potential?**

# Proton-nucleus scattering at high energies

- meson exchange, two-nucleon interaction
- mean field approximation,  $U_0$  ( $\omega$  exchange),  $U_s$  ( $2\pi$  exchange)

$$\left[ E - V_C - U_0 - \beta(mc^2 + U_s) \right] \Psi = -i\hbar c \boldsymbol{\alpha} \cdot \nabla \Psi$$



non-relativistic reduction

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + U_{\text{cent}} + \left( \frac{\hbar}{2mc} \right)^2 \frac{1}{r} \frac{d}{dr} U_{\text{so}} \boldsymbol{\sigma} \cdot \mathbf{L} \right] \varphi = E \varphi$$

$$U_{\text{cent}} = m^* (U_0 + U_s) + \dots$$

$$m^* = 1 - \frac{U_0 - U_s}{2mc^2} + \dots$$

$$U_{\text{so}} = U_0 - U_s + \dots$$

Arnold, Clark, PLB 84, 46 (1979)

Dirac phenomenology

# Relativistic MF nucleus-nucleus potential

Long, CB, PRC 83, 024907 (2011).

$\sigma, \omega, \rho$  and  $\gamma$  exchange

$$E = \int d^3r \sum_a \bar{\psi}_a (-i\gamma \cdot \nabla + M) \psi_a + \frac{1}{2} \sum_{\phi=\sigma,\omega,\rho,\gamma} \int d^3r d^3r' \sum_{ab} \bar{\psi}_a(\mathbf{r}) \bar{\psi}_b(\mathbf{r}') \Gamma_\phi(\mathbf{r}, \mathbf{r}') D_\phi(\mathbf{r} - \mathbf{r}') \psi_a(\mathbf{r}) \psi_b(\mathbf{r}')$$

$$\Gamma_\phi(\mathbf{r}, \mathbf{r}') = -g_\sigma(\mathbf{r}) g_\sigma(\mathbf{r}')$$

$$\Gamma_\omega(\mathbf{r}, \mathbf{r}') = -\left(g_\omega \gamma^\mu\right)_r \cdot \left(g_\omega \gamma_\mu\right)_{r'}$$

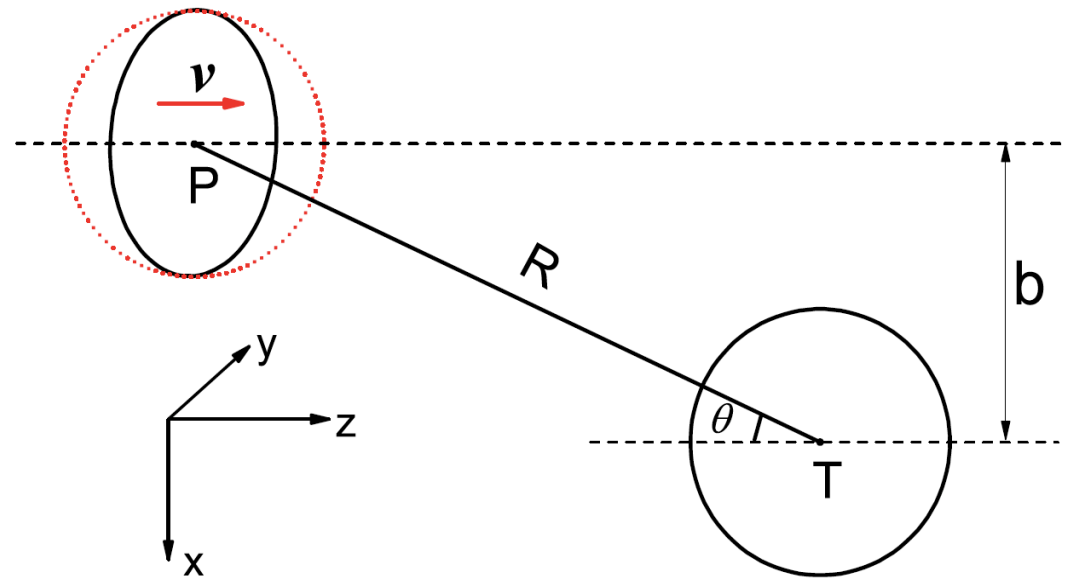
$$\Gamma_\rho(\mathbf{r}, \mathbf{r}') = -\left(g_\rho \gamma^\mu \vec{\tau}\right)_r \cdot \left(g_\rho \gamma_\mu \vec{\tau}\right)_{r'}$$

$$\Gamma_\gamma(\mathbf{r}, \mathbf{r}') = \frac{e^2}{4} \left[ \gamma^\mu (1 - \tau_z) \right]_r \cdot \left[ \gamma_\mu (1 - \tau_z) \right]_{r'}$$

$$D_\phi = \frac{1}{4\pi} \frac{e^{m_\phi |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}$$

$$D_\gamma = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

Lorentz transform



$$x_p = x_t + b, \quad y_p = y_t$$

$$z_p = \gamma(z_t + R \cos \theta)$$

$$E(A_t, A_p, v) = E(A_t) + E(A_p, v) + \mathcal{E}(A_t, A_p, v)$$

$$\mathcal{E}(A_t, A_p, v) = \sum_{\phi=\sigma,\omega,\rho,\gamma} \int d^3r \int d^3r' \sum_{ab} \bar{\psi}_{t,a}(\mathbf{r}) \bar{\psi}_{p,b}(\mathbf{r}') \Gamma_{\phi}(\mathbf{r}, \mathbf{r}') D_{\phi}(\mathbf{r} - \mathbf{r}') \psi_{t,a}(\mathbf{r}) \psi_{p,b}(\mathbf{r}')$$

Ex:  $\sigma$  and  $\omega$  contributions

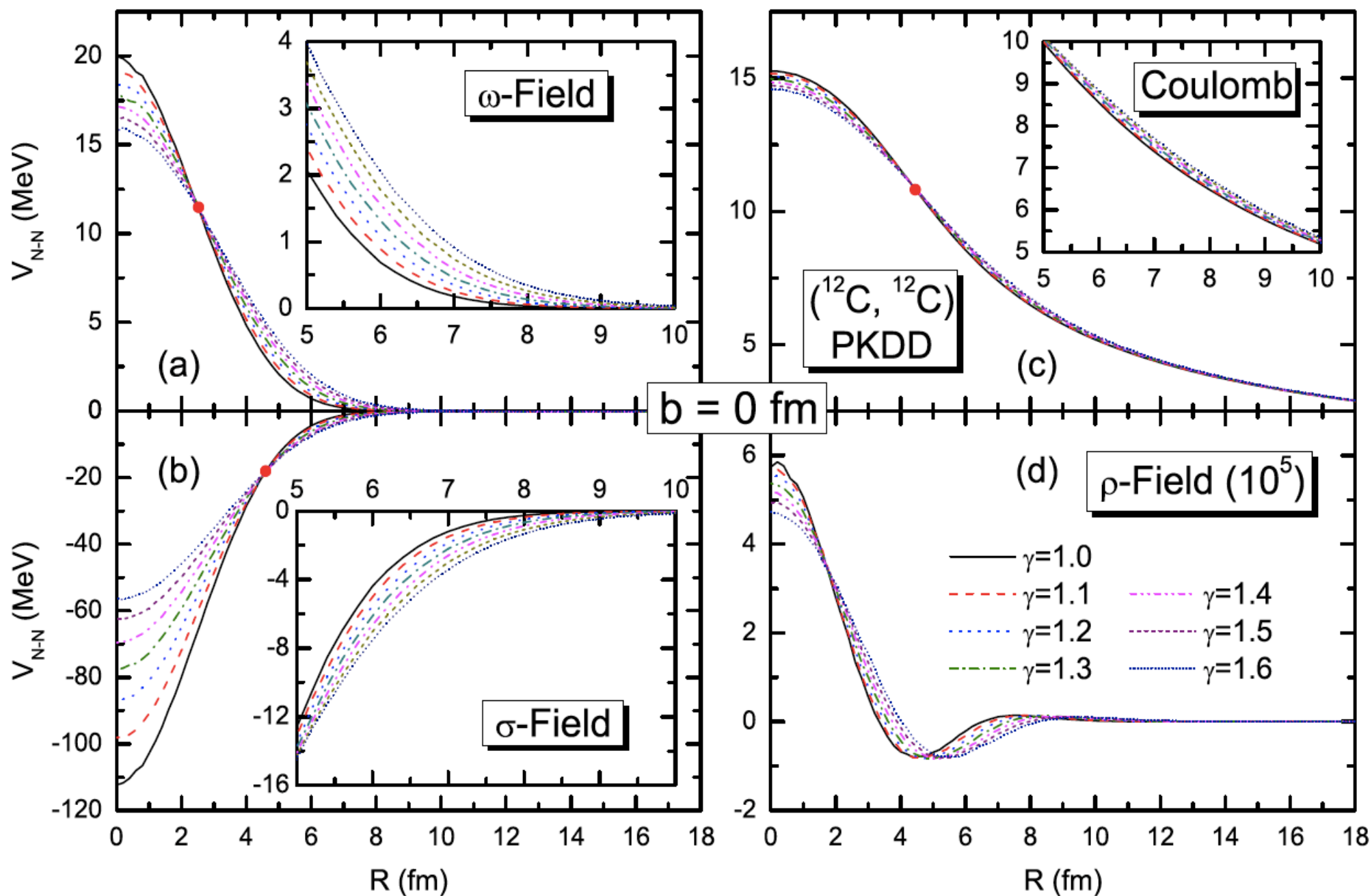
$$\mathcal{E}_{\sigma} = -\frac{1}{\gamma} \int d^3r_t \int d^3r'_p g_{\sigma}(\mathbf{r}_t) \rho_{s,t}(\mathbf{r}_t) D_{\sigma}(\mathbf{r} - \mathbf{r}') \rho_{s,p}(\mathbf{r}'_p) g_{\sigma}(\mathbf{r}'_p)$$

$$\mathcal{E}_{\omega} = \int d^3r_t \int d^3r'_p g_{\omega}(\mathbf{r}_t) \rho_{b,t}(\mathbf{r}_t) D_{\omega}(\mathbf{r} - \mathbf{r}') \rho_{b,p}(\mathbf{r}'_p) g_{\omega}(\mathbf{r}'_p)$$

$$\rho_s(\mathbf{r}) = \sum_a \bar{\psi}_a(\mathbf{r}) \psi_a(\mathbf{r}), \quad \rho_b(\mathbf{r}) = \sum_a \bar{\psi}_a(\mathbf{r}) \gamma^0 \psi_a(\mathbf{r})$$

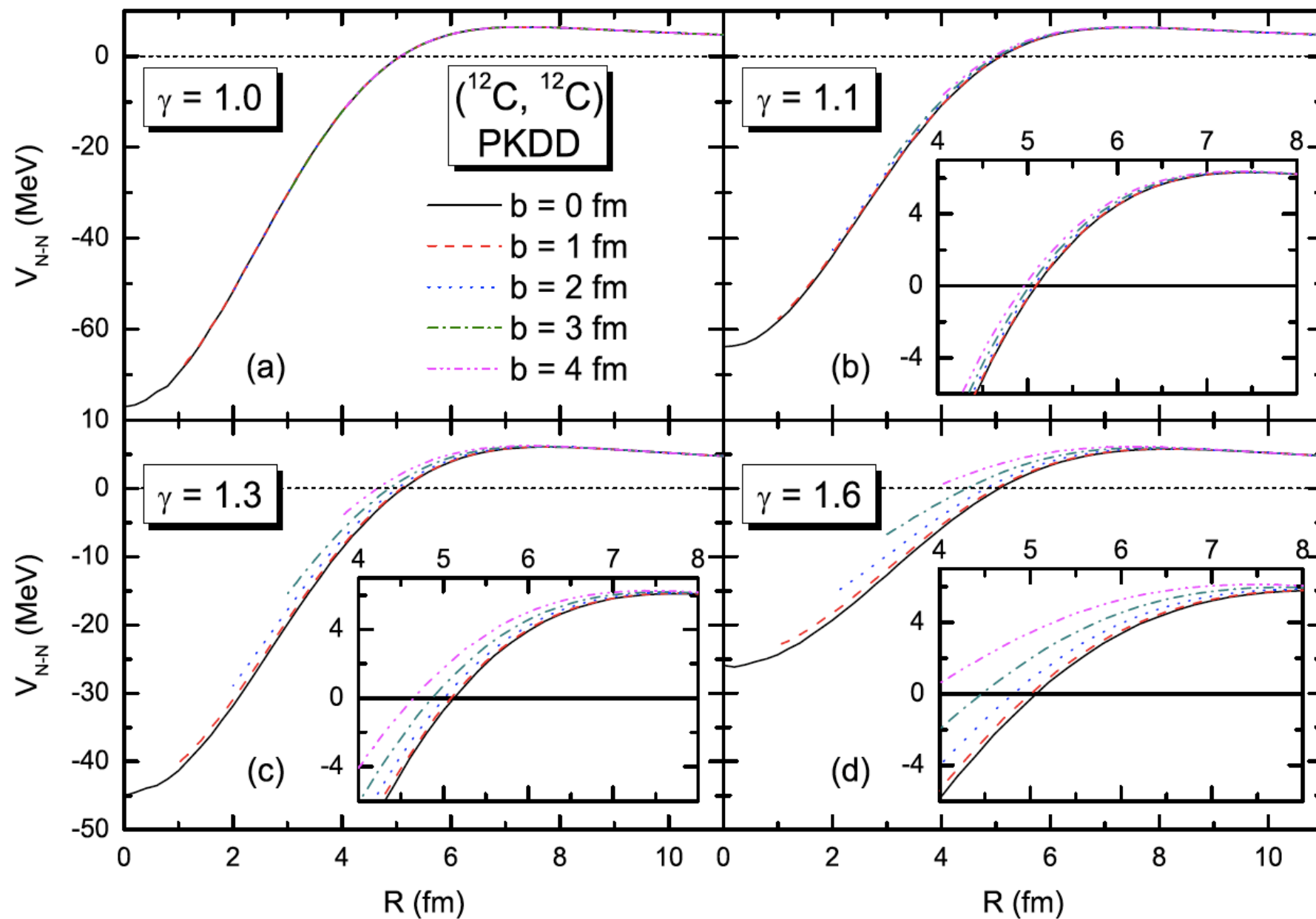
Projectile densities boosted to the target frame

# Contribution of mesons fields

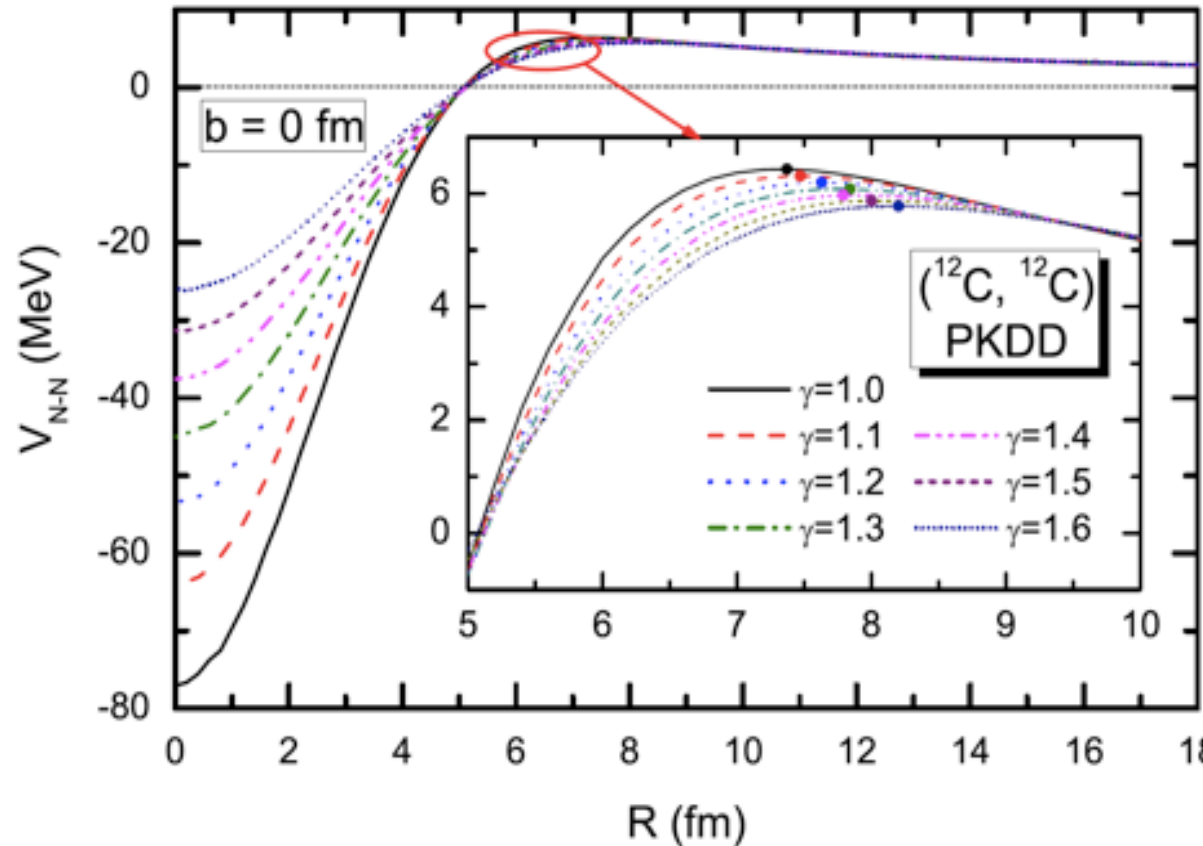




# Dependence on energy and impact parameter

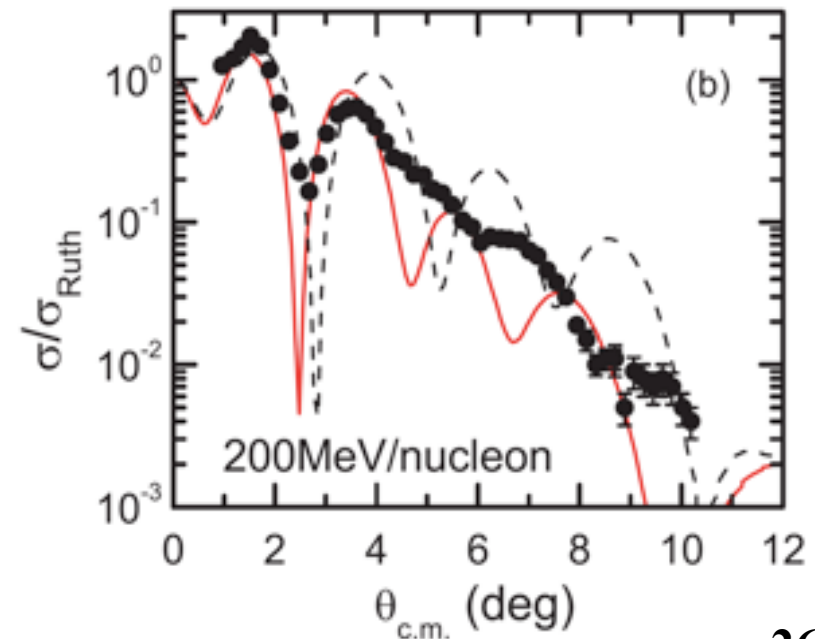
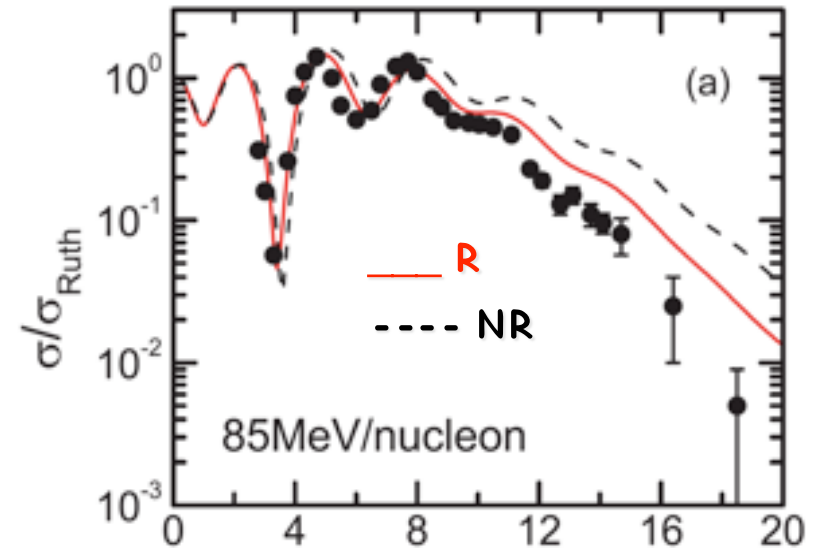


# Application to elastic scattering



And the imaginary part of  $U_{\text{opt}}$ ?

→ No easy solution. Here put by hand



# Practical calculations

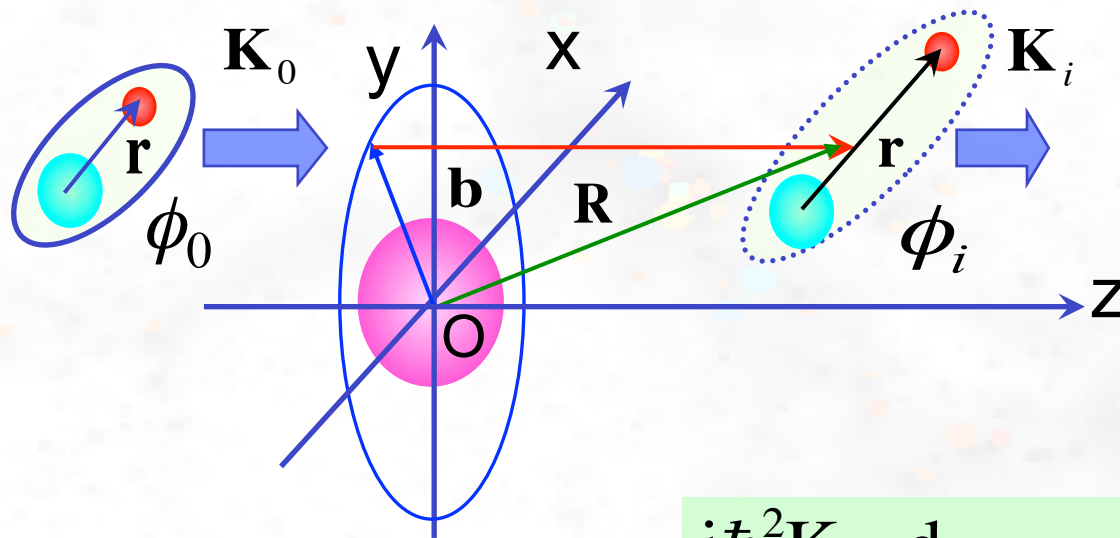
# From low to high energies

Eikonal scattering wave amplitudes  $S_i(\mathbf{K}_i, \mathbf{R})$

$$\psi^{\text{Eikonal}} = \sum_i \phi_i(\mathbf{r}) S_i(b, z) \exp(i\mathbf{K}_i \cdot \mathbf{R})$$

$$K_i = \sqrt{2\mu_R (E - \varepsilon_i)} / \hbar,$$

Energy conservation



● Boundary condition

$$S_i(b, z) \xrightarrow{z \rightarrow -\infty} \delta_{i,0}$$

$$\Delta S_i(b, z) \cong 0 \quad \rightarrow \quad \frac{i\hbar^2 K_i}{\mu_R} \frac{d}{dz} S_i^{(b)}(z) = \sum_{i'} F_{ii'}^{(b)}(z) S_{i'}^{(b)}(z) e^{i(K_{i'} - K_i)z}$$

$$f_{i,0}^E = \sum_L f_L^E \equiv \sum_L \frac{2\pi}{iK_i} \sqrt{\frac{2L+1}{4\pi}} i^m Y_{Lm}(\Omega) [S_{i,0}^{b(L;i)} - \delta_{i,0}]$$

# Relativistic CDCC

## Form factor of non-rel. E-CDCC

$$F_{c'c}^{(b)}(z) = \langle \Phi_{c'} | U_{\text{nucleon-T}} + U_{\text{core-T}} | \Phi_c \rangle e^{-i(m-m')\phi} = \sum_{\lambda} F_{c'c}^{(b)\lambda}(z)$$

## Lorentz tranform of form factor and coordinates

$$F_{c'c}^{(b)\lambda}(z) \rightarrow f_{\lambda, m'-m} \gamma F_{c'c}^{(b)\lambda}(\gamma z)$$

$$f_{\lambda, m'-m}^{\text{Coul}} = \begin{cases} 1/\gamma, & (\lambda = 1, m'-m = 0) \\ \gamma, & (\lambda = 2, m'-m = \pm 1) \\ 1, & (\text{otherwise}) \end{cases}$$

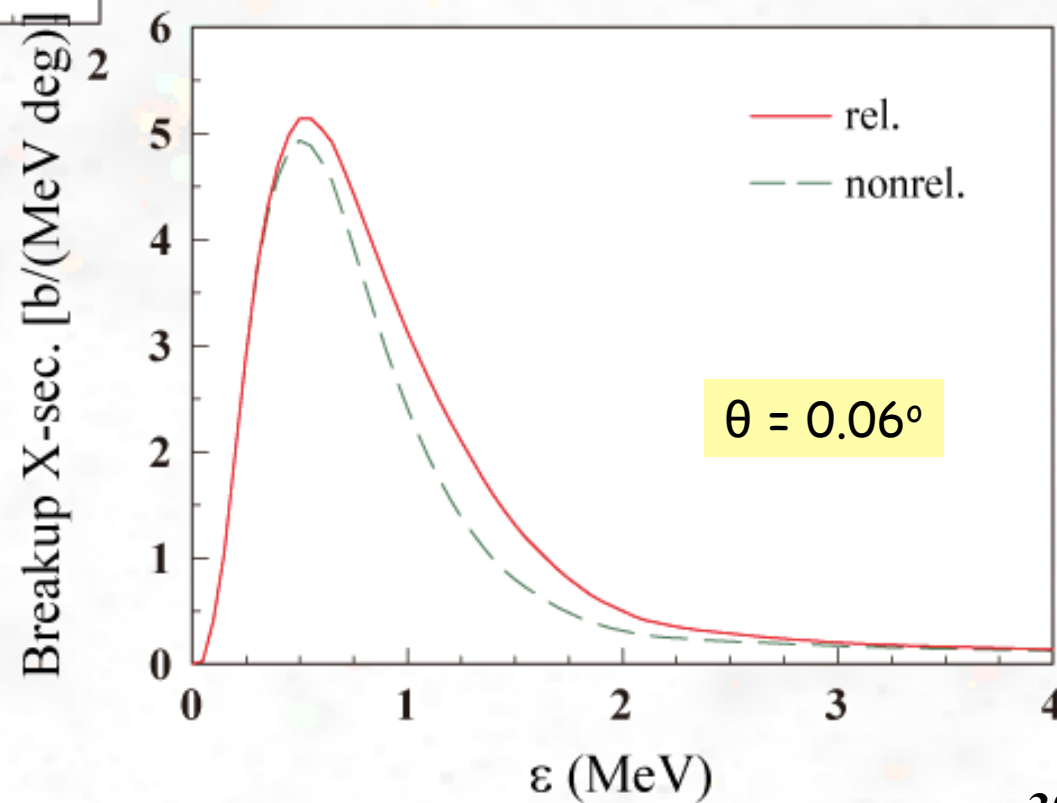
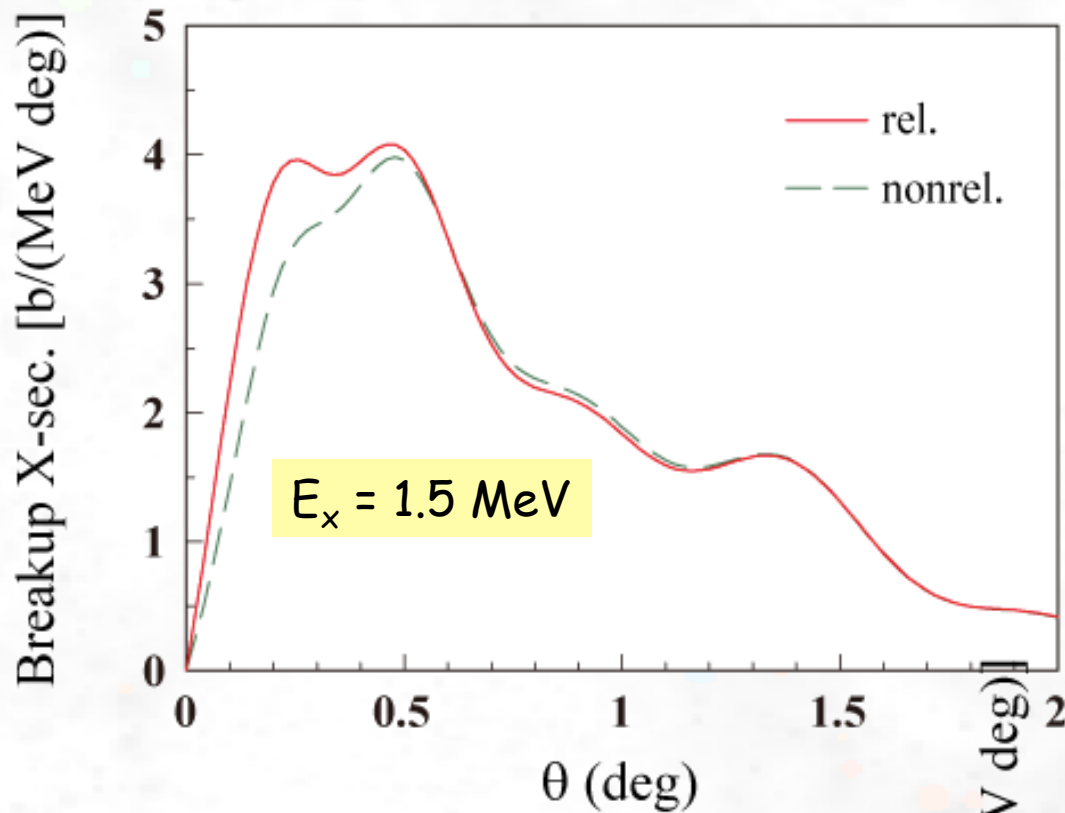
$$f_{\lambda, m'-m}^{\text{nuc}} = 1$$

## Assumptions

- ✓ Point charges for 1, 2 and A
- ✓ Neglecting far-field ( $r_i > R$ ) contribution
- ✓ Correction to nuclear form factor

Ogata, CB, PTP 121 (2009), 1399  
PTP, 123 (2010) 701

# $\text{Pb}(^8\text{B}, p^7\text{Be})$ at 250 MeV/nucleon

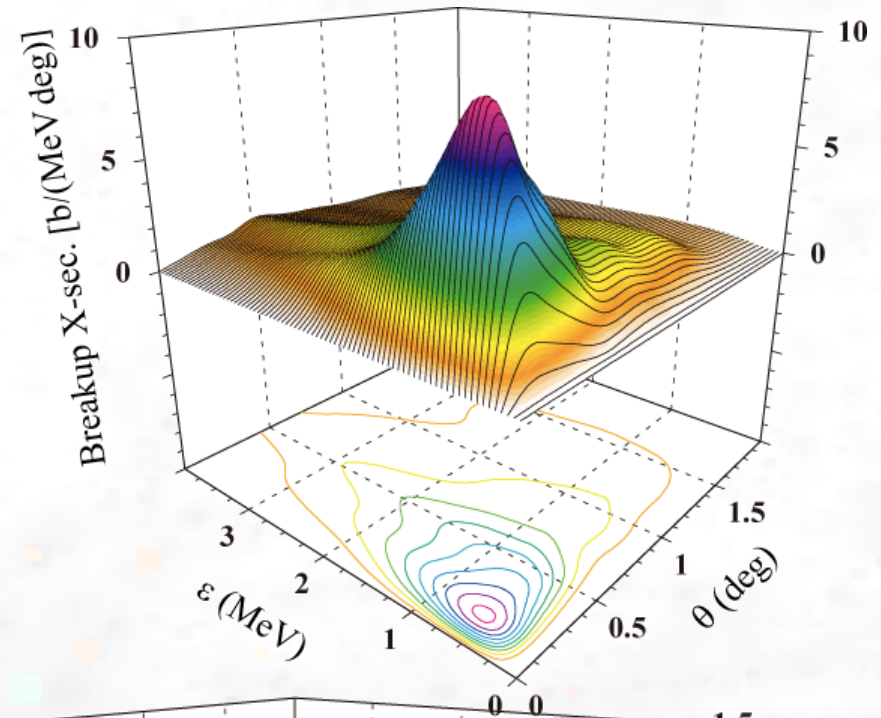
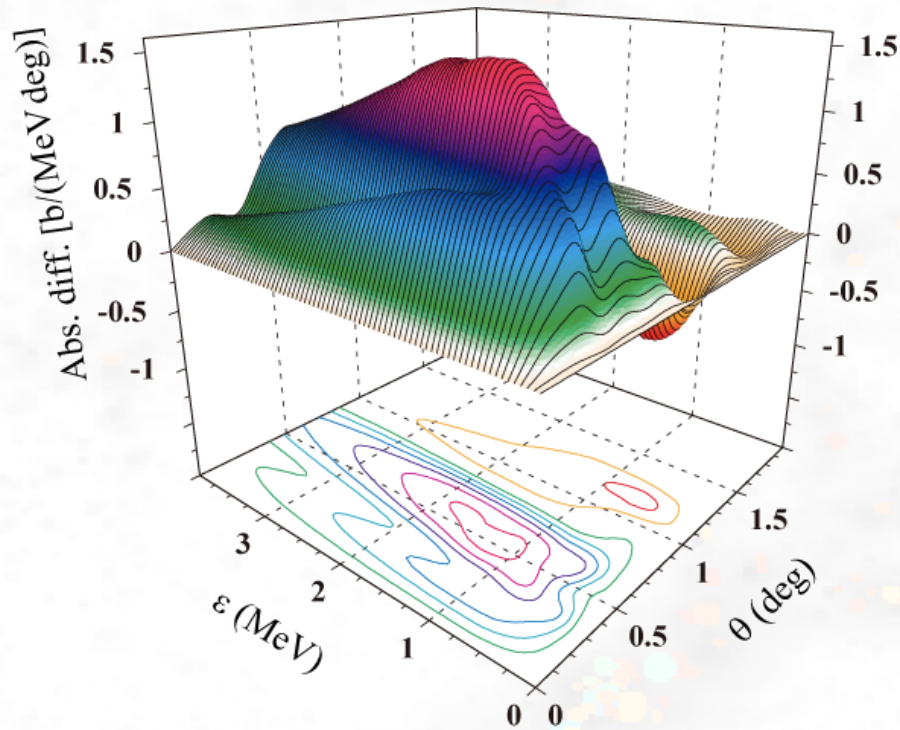




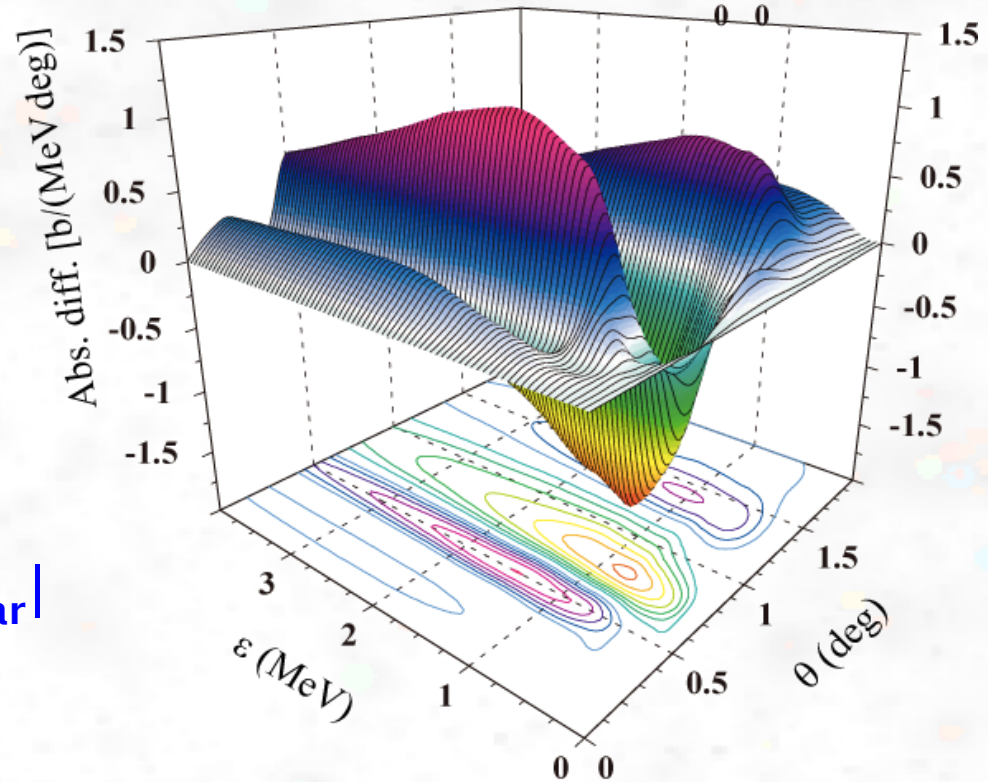
# $\text{Pb}(^8\text{B}, p^7\text{Be})$ at 250 MeV/nucleon

all orders

$|\sigma_{\text{all}} - \sigma_{\text{NR}}|$



$|\sigma_{\text{all}} - \sigma_{\text{no-nuclear}}|$



**End Lecture 3**