Problem 1: Geometrical units

A geometric unit system is a system of natural units in which the base physical units are chosen so that the speed of light in vacuum, c, and the gravitational constant, G, are set equal to unity, i.e., $c = 1$ and $G = 1$.

1.1 Show that for the geometric unit system the mass of the sun is $M_\odot = 1.47$ km.

1.2 Show that for $\hbar = 1$ and $c = 1$ one obtains the conversion factor $1 = 197.3$ MeV fm.

1.3 Using geometric units, estimate the Schwarzschild radius, $R_s \equiv 2M$, of the Earth ($M_{\text{Earth}} = 10^{-6} \, M_\odot$), Sun ($M = 1 \, M_\odot$), and a neutron star ($M_{\text{NS}} = 2 \, M_\odot$).

1.4 What happens to an object if it is shrunk below it's respective Schwarzschild radius?

Problem 2: Thickness of a neutron star atmosphere

The atmosphere of Earth has a thickness of over 100 kilometers, however the Chandra X-Ray Observatory recently detected an atmosphere on the neutron star Cassiopeia-A that measured only 10 centimeters thick. How can a small planet have a deeper atmosphere than an entire stars-worth of matter? The size of an atmosphere depends on a balance between the force of gravity pulling it towards the center of the planet, and the pressure of the atmosphere due to its temperature and density, pushing in the opposite direction. A pair of simple equations then defines how the density of the atmosphere has to rearrange itself with height above the surface so that gravity and pressure are always in balance. The equations look like this:

$$n(z) = n_0 \, \exp(z/H), \quad \text{where} \quad H = kT/mg, \tag{1}$$

where k is Boltzman's constant which equals 1.38×10^{-23} Joules/degree. The exponential equation says that as you get farther from the surface, the density of the gas, n, drops very fast. The quantity, H, in meters, is called the *scale height* and its value is defined by the atmospheres temperature, T, in Kelvins, and the acceleration of gravity at the surface, g, in multiples of Earth's acceleration (9.8 m/s^2). It also depends on the average mass, m, of the particles in the atmosphere. A light atmosphere made from hydrogen ($m = 1$) will produce a value for H that is much larger than one made from pure oxygen ($m = 16$). The surface acceleration of the neutron star is 10^{11} times that of Earth, the temperature of the gas is 3×10^6 Kelvins compared to Earths of 300 Kelvins, and the neutron star atmosphere is composed of carbon ($A = 12$) rather than

Earth's mixture of nitrogen and oxygen ($A = 28$).

2.1 From the formula for H, and the way in which it scales with m, T and g, what would you predict as the scale height for the neutron star atmosphere if for Earth, $H = 8$ kilometers?

2.2 How far from the surface would you have to travel in order for the density of the atmosphere to fall by 10^6 times for: A) Earth and B) a neutron star?

Problem 3: Stars in hydrostatic equilibrium

Consider a star that is in hydrostatic equilibrium. The forces acting on a mass shell inside of such a star are gravity, which acts radially inward, and the pressure force produced by the matter inside the mass shell, which acts in the radial outward direction (see Figure 1). The mathematical relation between pressure P and energy density ϵ of the matter is called the equation of state, $P(\epsilon)$. The mass of the mass shell
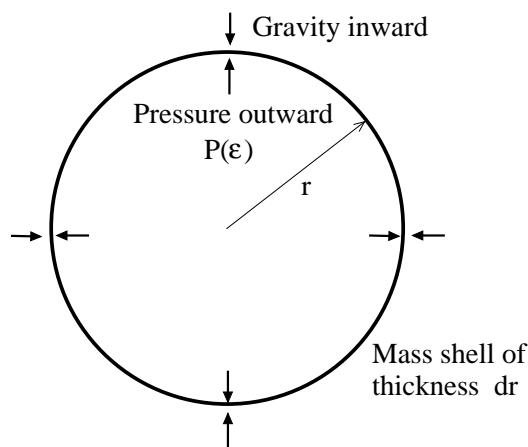

Figure 1: Hydrostatic equilibrium of a mass shell of thickness dr.

is $dm = 4\pi\epsilon r^2 dr$.

3.1 Show that in Newtonian physics the pressure gradient is given by ($G = c = 1$)

$$\frac{dP}{dr} = -G\frac{m\,\epsilon}{r^2}, \quad \text{where} \quad m = 4\pi \int_0^r dr\, r^2\, \epsilon\,. \tag{2}$$

The surface, R, of the star is defined by $P(r = R) = 0$ and the star's total gravitational mass, M, is given by $M \equiv m(r = R)$. Recall that pressure inside a hydrostatically

stable star is monotonically decreasing from the center to the surface.

3.1 Qualitatively, plot the pressure profile of hydrostatically stable stars, as described by Equation (2), and compare it with the pressure profile of hydrostatically stable stars that follows from Einstein's field equations of General Relativity theory as $(G = c = 1)$

$$\frac{dP}{dr} = -G\frac{(\epsilon + P)(4\pi r^3 P + m)}{r^2\left(1 - \frac{2m}{r}\right)}.$$ (3)

Equation (3), know as Tolman-Oppenheimer-Vokoff equation, will be derived during my lectures. Comment on the qualitative differences. Which stars (classical or General Relativistic ones) will have smaller radii?

Problem 4: Numerical modeling of compact stars

4.1 Write a numerical program (Fortran, C, or C++) that solves the stellar structure equation (3) numerically using the bag model equation of state

$$P = (\epsilon - 4B)/3,$$ (4)

which describes the stellar interior in terms of a relativistic gas of deconfined quarks. Here P and ϵ denote pressure and energy density in units of MeV/fm^3, and $B = 57$ MeV/fm^3 is a constant. Compute 50 stellar models whose central densities ϵ_c range from $5B$ to around $20B$. (A sample Fortran 90 code is provided below.)

4.2 Design your program such that it writes the stellar radius R (in km) and mass M (in units of the mass of the sun) of an entire stellar sequence to an output file.

4.3 Plot M as a function of R. Repeat the computation but this time for Newtonian stars. Show the mass-radius relationship of this stellar sequence in the plot that you just created. Comment on any differences between Newtonian stars and General Relativistic stars..

4.4 Modify your code such that the gravitational redshift

$$z = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1$$

of light emitted from the surface of a compact star is computed for a given mass and radius, both for Newtonian stars and General Relativistic stars. Illustrate the results graphically.

3

```fortran
program relativistic_stars

! Fortran 90 sample code which solves both the Newtonian & General Relativistic
! equations of stars which are in hydrostatic equilibrium.

      implicit none
      real :: e_c, p_c, e, p, pdr, msun_km=1.475, deltaec
      integer :: choice

      double precision, parameter :: msun_mev = 1.115829d60
      double precision, parameter :: c18 =1.0d18
      double precision :: cf, mass, r, dr

      real, parameter :: pi=acos(-1.), bag=57.0    !* Bag constant in MeV/fm^3

      open (unit=10, file='MvsR.dat', status='unknown')

      write(*,*) ' Newtonian (1) or General Relativistic (2) calculation?'
      read(*,*) choice

         if (choice == 1) then
            write(*,*) ' A Newtonian solution will be computed.'
            else if (choice == 2) then
            write(*,*) ' A General Relativistic solution will be computed.'
            else
            write(*,*) ' Input error -> computation terminated!'
            stop
         end if

      cf     = msun_km*c18/msun_mev  !* Compute conversion factor
      e_c    = 4.2*bag               !* Initialize central density of 1st stellar model
      deltaec = bag/10.


      do while (e_c <= (20.*bag))    !* Compute sequence of stellar models
        p_c = (e_c - 4.*bag) / 3.    !* Central pressure
        mass= 0.                     !* Initialize star's mass
        r   = 1.0e16                 !* Initialize radial distance in fermi
        dr  = 1.0e16                 !* Initialize step size in fermi
        e   = e_c ; p = p_c          !* Initialize e, p

              pdr = 1.
        call TOV(pdr, choice, mass, cf, r, e, p, pi, dr, bag)
```

```fortran
      write(*,*)   ' R=', r/c18, 'km,', '  M=', mass/msun_mev,'M_sun'
      write(10,20) r/c18, mass/msun_mev   !* Output r in km, mass in M_sun
 20   format(2x, f8.4, 4x, f8.4)

      e_c = e_c + deltaec        !* Central density of next stellar model
    end do

  close(unit=20)

  stop
end program relativistic_stars4

!*
    subroutine TOV(pdr, choice, mass, cf, r, e, p, pi, dr, bag)
!*
!* This subroutine integrates the TOV equation

    implicit none
    integer, intent(in) :: choice
    real, intent(in) :: pi, bag
    double precision, intent(in) :: cf
    real, intent(inout) :: pdr, e, p
    double precision, intent(inout) :: mass, r, dr
    real :: f

        do while (pdr > 0.)
            if (choice == 1) then
            f = e * mass * cf / (r*r)
            else if (choice == 2) then
            f = (e+p) * (4.*pi*r**3*p+mass) * cf / (r*r*(1.-2.*mass*cf/r))
            end if
          mass = mass + 4.*pi*r*r*e*dr
          pdr  = p - f * dr
          p = pdr
          e = 3.*pdr + 4.*bag
          r = r + dr
        end do

    return
end subroutine TOV
```

Problem 5: Metric tensor

Consider the fictitious two-dimensional line element given by

$$ds^2 = g_{ij}dx^i dx^j = x^2 dx^2 + 2dxdy - dy^2 \, . \tag{5}$$

5.1 From Equation (5), read off the components of the covariant metric tensor, (g_{ij}).

5.2 Use $g^{ik}g_{kj} = \delta^i{}_j$ to determine all contravariant components (g^{ij}).

5.3 Raise and lower indices on the vectors $(A_i) = (1,1)$, $(B^j) = (0,1)$.