QCD CONDENSATES

LECTURE II
TOY MODEL

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OUTLINE

- *Conditions in comapact stars*
- *Toy model*
- *Fluctuations*
- *Stability criteria*

CONDITIONS IN COMPACT STARS

The physics at high baryonic density is difficult to handle

1. QCD is nonperturbative
2. Lattice simulations do not work (Barbour et al. 1986 Nucl.Phys. B275 296)
3. No experimental facility (so far) can reproduce the correct conditions

Let me refrase it:

1. We do not know how to do computations
2. We do not have numerical methods for doing tests
3. We have no terrestrial lab for validating the theoretical results

The way out:

We can use symmetries and analogies for obtaining qualitative and semiquantitative results

We can use compact stars as the "lab"

Additional ingredients

We have to consider the typical environment of compact stars

- It is “cold”, with temperature of order tens of keV
- Matter is in weak equilibrium
- Matter is electrically neutral
- The strange quark mass might be comparable with the quark chemical potential

The first condition simplifies the treatment: we can take T=0.

The other conditions make the treament more complicated

Fermi mismatch *(unpaired quark matter)*

Large quark number chemical potential
Quark number densities of "free" quarks

$$300 \text{ MeV} < \mu < 1 \text{ GeV}$$

$$n_i = C_i \frac{k_{F,i}^3}{3\pi^2} \qquad k_{F,i} = \sqrt{\mu_i^2 - m_i^2}$$

Only u,d,s quarks are relevant. Light quarks can be treated as massless.

weak equilibrium

$$\left.\begin{array}{l} u \to d + \bar{e} + \nu_e \\ u \to s + \bar{e} + \nu_e \\ u + d \leftrightarrow u + s \end{array}\right. \longrightarrow \begin{array}{l} \mu_u = \mu_d - \mu_e \\ \mu_d = \mu_s \end{array}$$

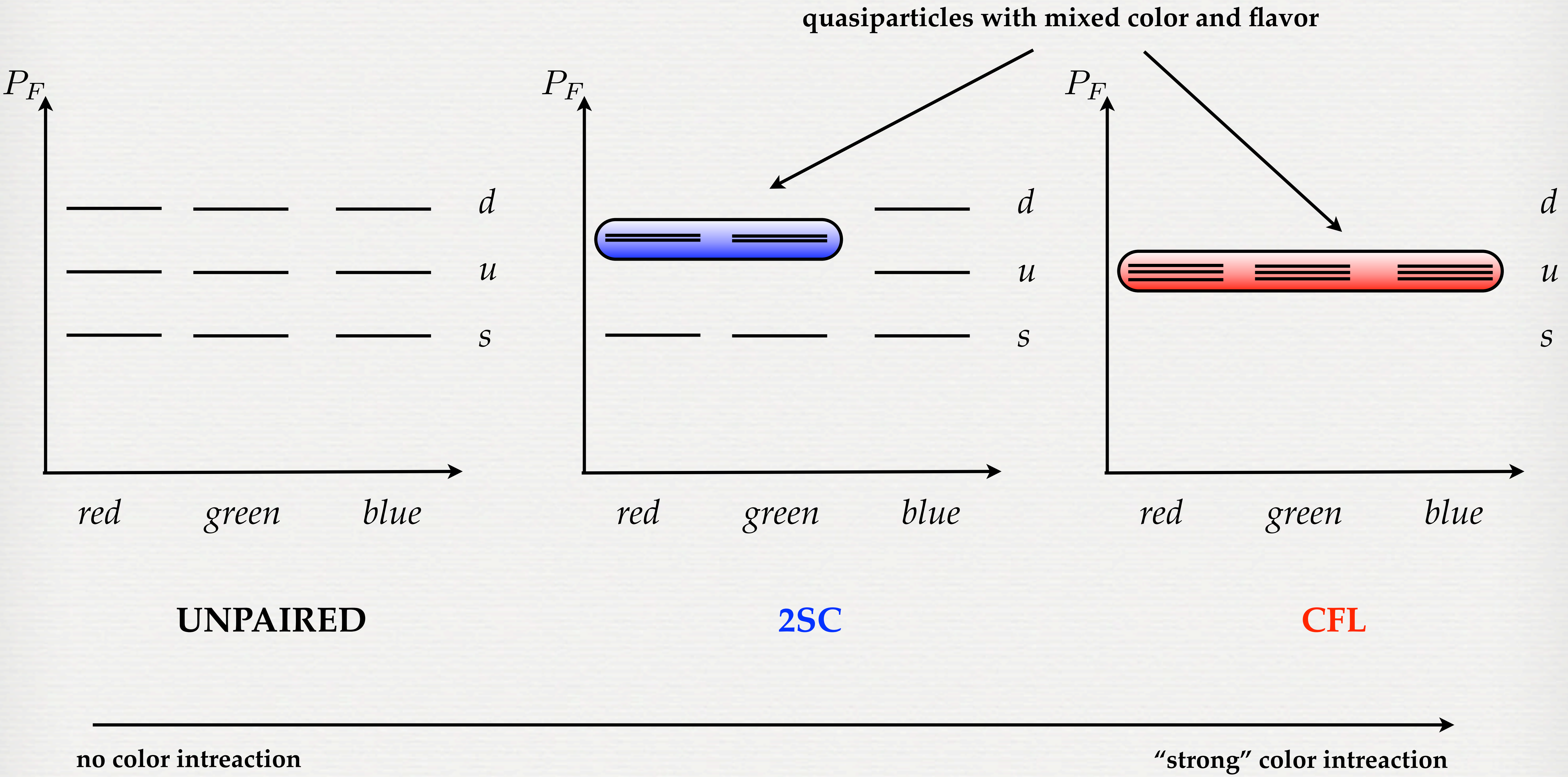
electric neutrality $\longrightarrow$ $$\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$$

$$\Longrightarrow \mu_e \simeq \frac{m_s^2}{4\mu}$$

$$\mu = \frac{\mu_u + \mu_d + \mu_s}{3}$$

$$p_d^F = \mu + \frac{1}{3}\mu_e \qquad p_u^F = \mu - \frac{2}{3}\mu_e \qquad p_s^F \simeq \mu - \frac{5}{3}\mu_e$$

Fixed mismatch, increasing coupling



Whenever there is BCS pairing, the Fermi surfaces have to match.

If the mismatch is too large, pairing cannot occur. The largest chemical potential mismatch which allows paring is named the **Chandrasekhar-Clogston limit** (derived it for weakly coupled two level systems)

Chemical potential stress on the CFL phase

At a given interaction strength a large chemical potential difference tends to disrupt the CFL paring

Free energy gain $\propto \Delta_{\rm CFL}$

Free energy cost $\propto \delta\mu \sim \dfrac{M_s^2}{4\mu}$

CFL favored for

$$\Delta_{\rm CFL} > c\frac{M_s^2}{\mu}$$

Various possibile transitions:

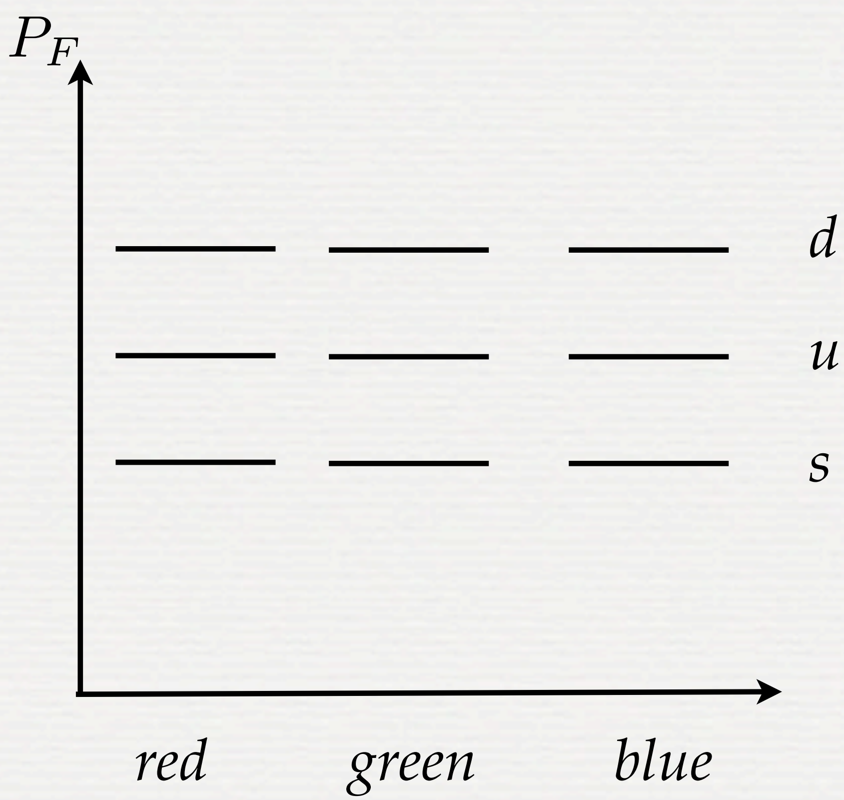
- **2SC phase**
- **gapless CFL phase (unstable)**
- **CFL-K0**
- **Crystalline color superconductors**

CFL-K0 phase

Kaons could be introduced for "stabilyzing" the system

P.F.. Bedaque and T. Schäfer, Nucl.Phys. A697 (2002) 802-822

When there is a chemical potential mismatch there is a lack of strange quarks

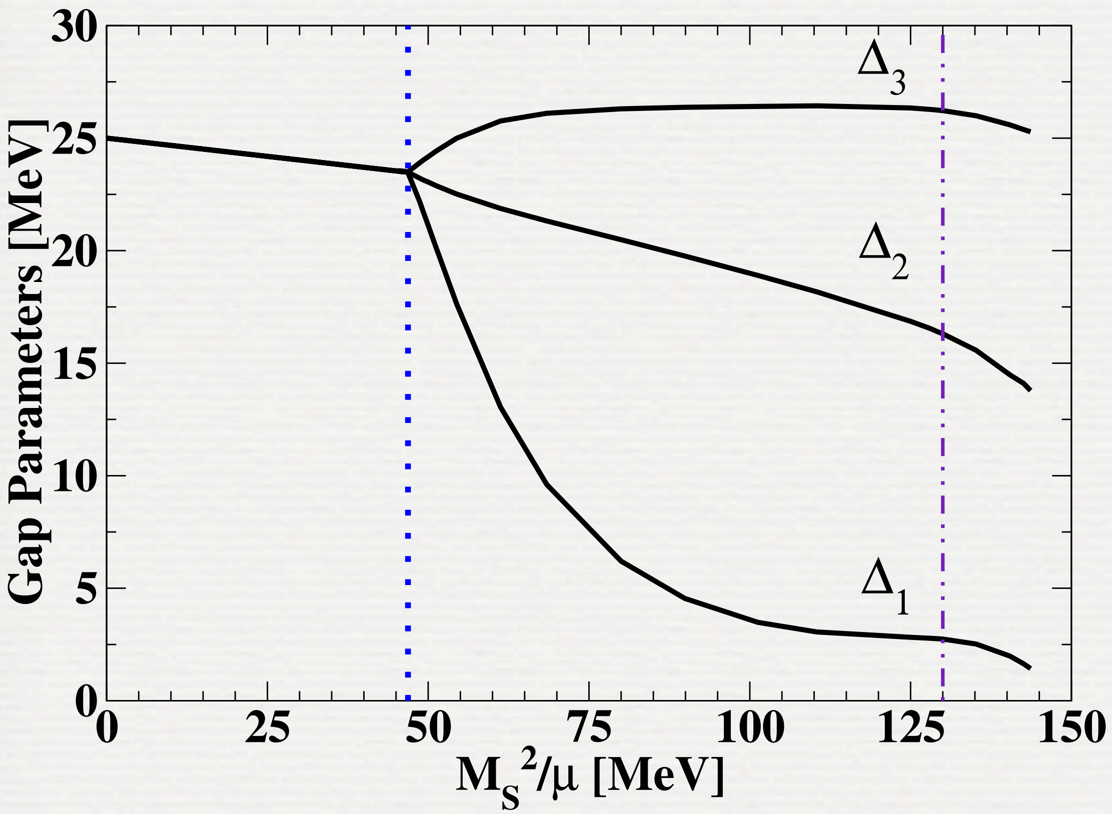


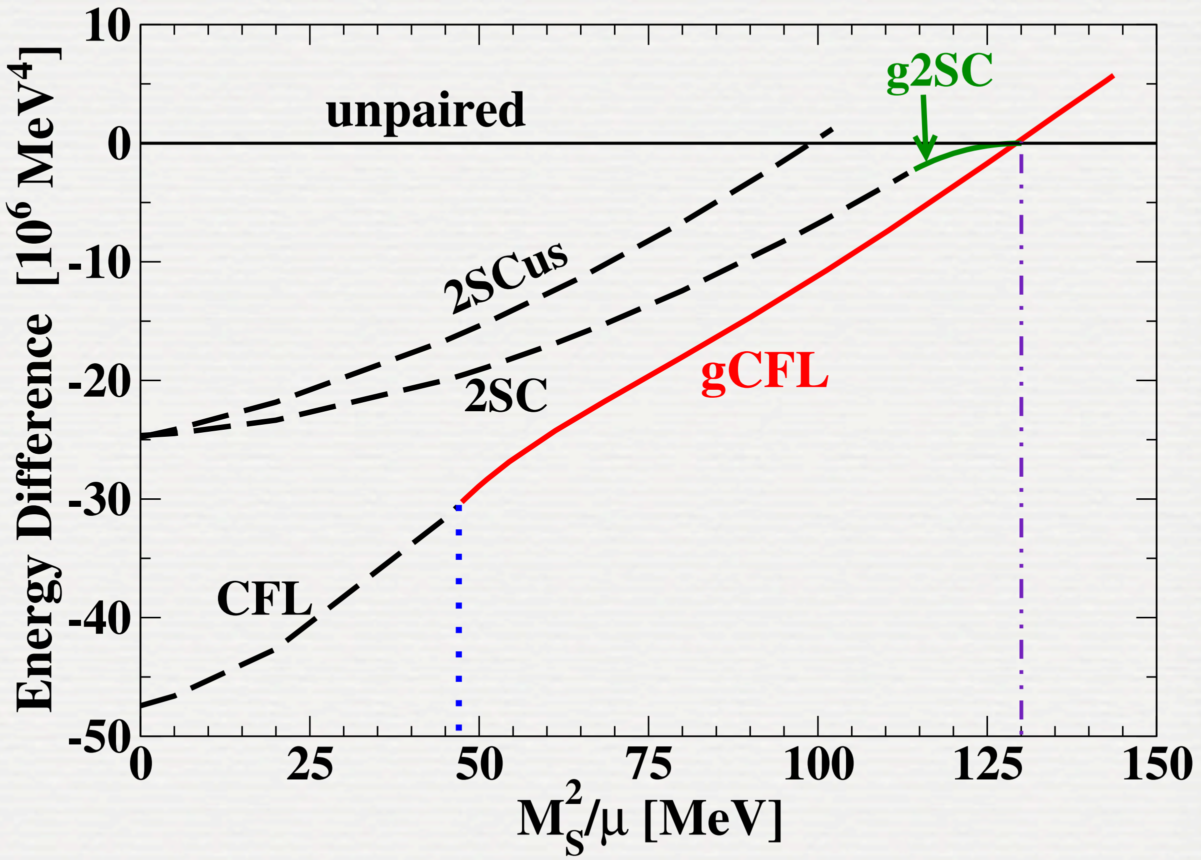
Adding $K_0 \sim$ "$d\bar{s}$"

we can effectively increase the number of strange quarks and reduce the number of *d* quarks

Kaon condensation occurs for $M_s \gtrsim m_u^{1/3} \Delta_{\rm CFL}^{2/3}$

gCFL phase





Phase with gapless modes may be favored.
One assumes that pairing is channel dependent and finds that some quasiparticles are massless

$$\langle \psi_{\alpha i} C \gamma_5 \psi_{\beta j} \rangle \propto \sum_{I=1}^{3} \Delta_{\mathrm{I}} \varepsilon^{\alpha\beta I} \epsilon_{ijI}$$

Alford et al. Phys.Rev.Lett. 92 (2004) 222001

It is chromomagnetically unstable

M.M et al. Phys.Lett. B605 (2005) 362-368,

Aiming at a simpler system

- **Quark condensate can break several symmetries.**

FIRST UNDERSTAND HOW THINGS WORK FOR GLOBAL AND GAUGE SIMMETRIES.

- **The analysis of complictae structures needs some preiminary work**

TOY MODEL WITH MISMATCHED FERMI SPHERES

- **Adding degrees of freedom will only make the game more complicated**

AND INTERSTING

TOY MODEL

Two-level system

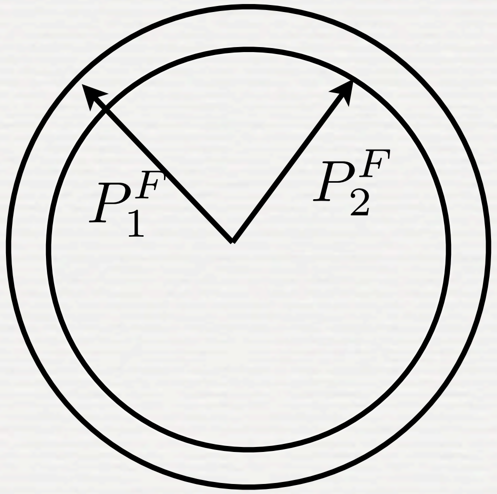
SYSTEM

Two non-relativistic species of fermions $\psi_1\ \psi_2$

different chemical potentials $\mu_1 = \mu + \delta\mu$
$\mu_2 = \mu - \delta\mu$

equal mass m

No interaction



$$\boxed{\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm I}}$$

Defining $\psi^t = (\psi_1, \psi_2)$

$$\mathcal{L}_{\rm free} = \psi^\dagger \left(i\partial_t - E(\boldsymbol{p}) + \mu + \delta\mu\sigma^3 \right) \psi$$

$$\xi(\boldsymbol{p}) = E(\boldsymbol{p}) - \mu = \frac{\boldsymbol{p}^2}{2m} - \frac{p_F^2}{2m} \simeq \frac{p_F}{m}(p - p_F) = v_F \ell$$

"residual energy" "residual momentum"

Residual momentum and energy can be above or below the Fermi surface, therefore they can be positive or negative

States with negative energy correspond to holes

$$\mathcal{L}_{\text{free}} = \psi^\dagger \underbrace{\left(i\partial_t - \xi(\boldsymbol{p}) + \delta\mu\sigma^3\right)}_{\text{inverse propagator}} \psi$$

Dispersion law $\epsilon = \pm\delta\mu + \xi(\boldsymbol{p})$

The effect of the chemical potential mismatch is a shift of the energy

Nmbu-Gorkov basis

1st step: Write

$$\mathcal{L}_{\text{free}} = \psi^\dagger \left(i\partial_t - \xi(\boldsymbol{p}) + \delta\mu\sigma^3\right)\psi = \frac{1}{2}\left[\psi^\dagger \left(i\partial_t - \xi(\boldsymbol{p}) + \delta\mu\sigma^3\right)\psi + \psi \left(i\partial_t + \xi(\boldsymbol{p}) - \delta\mu\sigma^3\right)\psi^*\right]$$

2nd step: Introduce the Nambu-Gorkov spinor

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_1^* \\ \psi_2^* \end{pmatrix}$$

3rd step: Rewrite the Lagrangian in the matrix form

$$\boxed{\mathcal{L}_{\text{free}} = \chi^\dagger \begin{pmatrix} i\partial_t - \xi(\boldsymbol{p}) + \delta\mu\sigma^3 & 0 \\ 0 & i\partial_t + \xi(\boldsymbol{p}) - \delta\mu\sigma^3 \end{pmatrix}\chi}$$

Exercise

Using the Nambu-Gorkov spinor

$$\chi_{\text{new}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 \\ \psi_2^* \\ \psi_1^* \\ \psi_2 \end{pmatrix}$$

write the Lagrangian in a matrix form

Exercise. Compute the determinant of the inverse propagator and show that the dispersion laws are given by

particles	**holes**
$\epsilon_{\pm} = \pm\delta\mu + \lvert\xi(p)\rvert$	$\hat{\epsilon}_{\pm} = \pm\delta\mu - \lvert\xi(p)\rvert$

- **In the Nambu-Gorkov basis there is a DOUBLING of the degrees of freedom**

  Particle and hole states appear as zeros of the inverse propagator.

- **Of course the new theory is competeley equivalent to the standard one.**

  So, why is it useful?

Turning on the interaction

We use a local Fermi interaction

$$\boxed{\mathcal{L}_{\mathrm{I}} = \frac{\lambda}{2}\psi_s^\dagger(x)\psi_t^\dagger(x)\psi_t(x)\psi_s(x)}$$

The effect of the attractive interaction is to produce a difermion condensate

$$\langle\psi_s(x)\psi_t(x)\rangle = \frac{\Delta(x)}{\lambda}\varepsilon_{st}$$

Mean field: $$\psi_s^\dagger\psi_t^\dagger\psi_t\psi_s \to \langle\psi_s^\dagger\psi_t^\dagger\rangle\psi_t\psi_s + \psi_s^\dagger\psi_t^\dagger\langle\psi_t\psi_s\rangle - 2\langle\psi_s^\dagger\psi_t^\dagger\rangle\langle\psi_t\psi_s\rangle$$

Basically it is a smibosonization of the theory or a Hubbard-Stratanovich transformation.

Keeping the x dependence of the condensate means that we shall consider fluctuations of the condensate

Exercise. Which symmetry is broken by the condensate ?

Using the Nambu-Gorkov basis we obtain

$$\mathcal{L} = \chi^\dagger \begin{pmatrix} i\partial_t - \xi(\boldsymbol{p}) + \delta\mu\sigma^3 & -\Delta(x)\varepsilon \\ \Delta^*(x)\varepsilon & i\partial_t + \xi(\boldsymbol{p}) - \delta\mu\sigma^3 \end{pmatrix} \chi - \frac{|\Delta(x)|^2}{\lambda}$$

which is rather compact.

Exercise. Write the Lagrangian using $\chi_{\rm new}$

Note that the gap parameter appears as an off diagonal matrix. It is equivalent to a Majorana mass term for neutrinos

See for example F. Wilczek., arXiv:1401.4379

The dispersion laws of particles and holes are respectively

$$\epsilon_\pm = \pm\delta\mu + \sqrt{\xi^2 + \Delta^2} \qquad \hat{\epsilon}_\pm = \pm\delta\mu - \sqrt{\xi^2 + \Delta^2}$$

Free energy

$$\boxed{\Omega = -\frac{1}{2}\sum_{a=1}^{2}\int \frac{d^3p}{(2\pi)^3}\left\{\epsilon_a + 2T\log(1+e^{-\epsilon_a/T}) - \xi_a\right\} + \frac{\Delta^2}{G}}$$

At vanishing temperature

$$\boxed{\Omega = \frac{\Delta^2}{G} - \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Big[|\epsilon_1| + |\epsilon_2| - 2\xi\Big]}$$

The integral in this expression is ultraviolet divergent and can be regularized by considering the S-wave scattering length

$$\frac{m}{4\pi a} = -\frac{1}{G} + m\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}$$

Constraints

$$\frac{\partial\Omega}{\partial\Delta} = 0 \qquad \frac{\partial\Omega}{\partial\mu} = -n \qquad \frac{\partial\Omega}{\partial\delta\mu} = -\delta n$$

a) $$\Delta = G\Delta \int \frac{d^3p}{(2\pi)^3} \frac{1 - f(\epsilon_1) - f(\epsilon_2)}{2E_p}\,,$$

b) $$n = \int \frac{d^3p}{(2\pi)^3} \left\{ 1 - \frac{\xi}{E_p}[1 - f(\epsilon_1) - f(\epsilon_2)] \right\}\,,$$

c) $$\delta n = \int \frac{d^3p}{(2\pi)^3} \left\{ f(\epsilon_1) - f(\epsilon_2) \right\}.$$

Several possibilities:

1) Fixed chemical potentials; only a)
2) Fixed total number density; a) and b)
3) Fixed differential number density; a) and c)
4) Fixed number densities; a), b) and c)

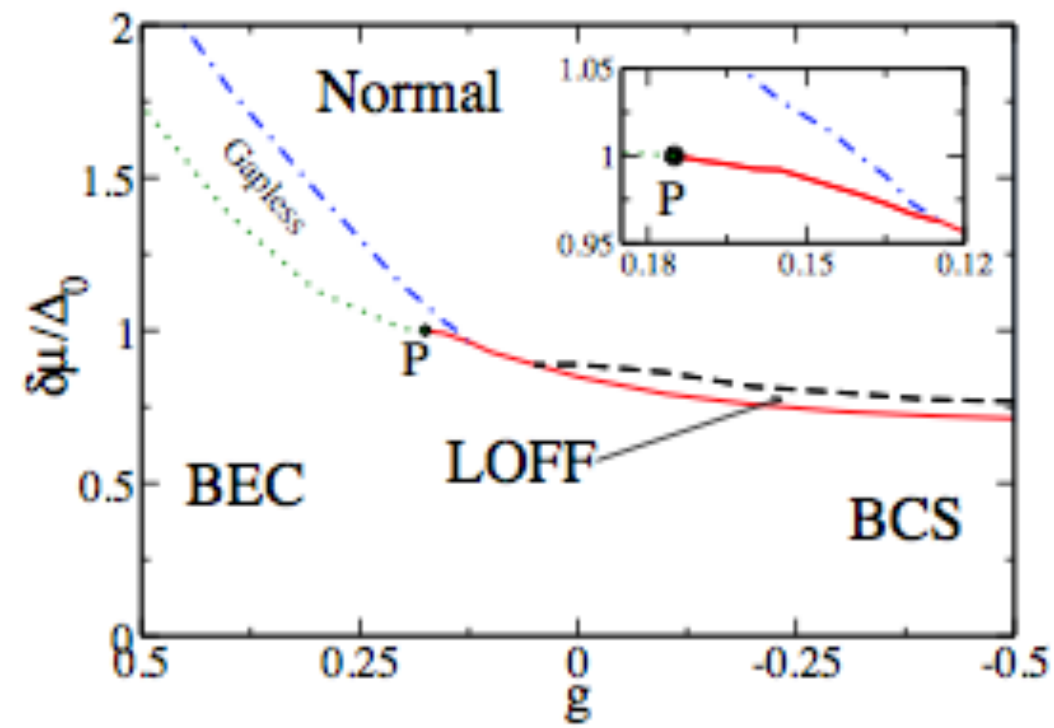
Compact stars

Ultracold atoms

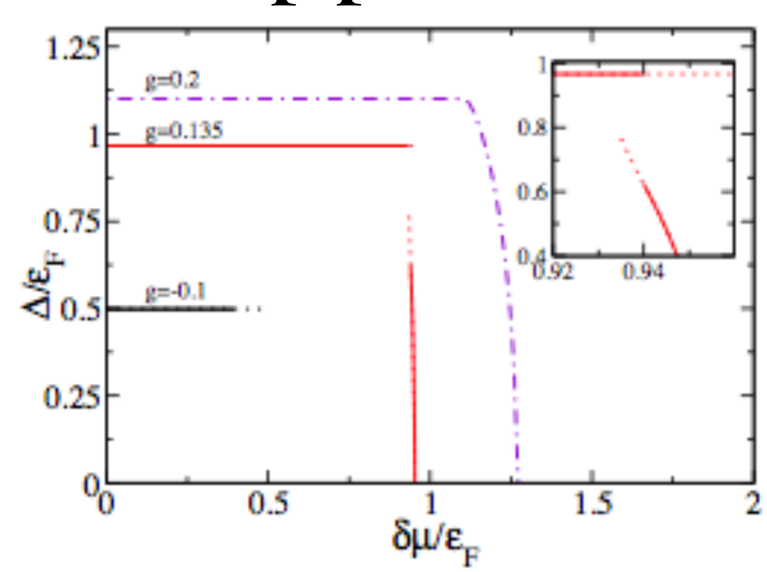
Results for cold atoms

Mean field analysis

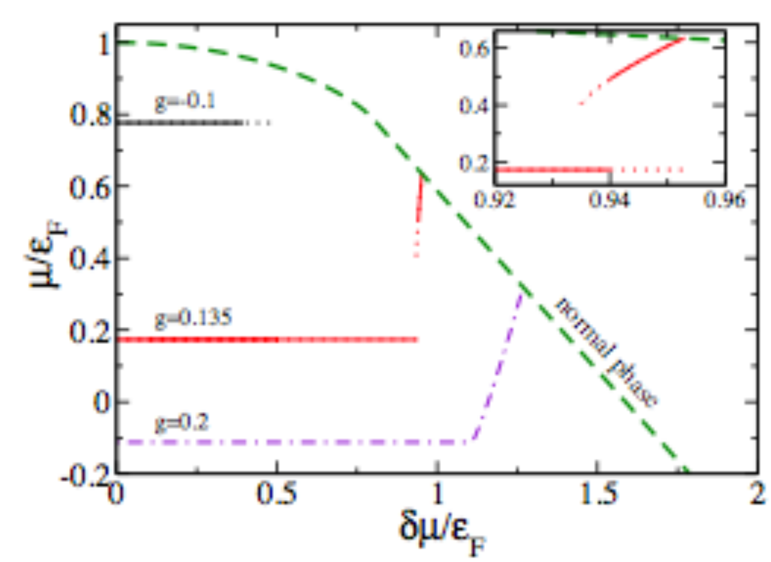
Phase diagram



Gap parameter



Chemical potential



M.M et al. Phys.Rev. A74 (2006) 033606

FLUCTUATIONS

Introducing the fluctuations

We now expand around the mean-field solution, writing

$$\boxed{\Delta(x) = \Delta_0 + \eta(x)}$$

The partition function can be written as $Z = \int \mathcal{D}\eta^* \mathcal{D}\eta \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^{-\mathcal{S}[\Psi^\dagger, \Psi, \eta, \eta^*]}$

Where the Wick rotated action is given by

$$\boxed{\mathcal{S}[\Psi^\dagger, \Psi, \eta, \eta^*] = \int d^4x \left\{ \frac{1}{\lambda} |\Delta + \eta(x)|^2 - \Psi^\dagger \begin{pmatrix} -\partial_{x^4} - \xi(\boldsymbol{p}) + \delta\mu\sigma^3 & -(\Delta + \eta(x))\epsilon \\ (\Delta + \eta^*(x))\epsilon & -\partial_{x^4} + \xi(\boldsymbol{p}) - \delta\mu\sigma^3 \end{pmatrix} \Psi \right\}}$$

To find the effective action for the fluctutions, we integrate out the fermionic fields, which can be done because the action is quadratic in these fields.

$$Z = \int \mathcal{D}\eta^* \mathcal{D}\eta e^{-\mathcal{S}[\eta,\eta^*]}$$

$$S[\eta,\eta^*] = \int d^4x \left\{ \frac{1}{\lambda} |\Delta + \eta(x)|^2 \right\} - \left\{ \frac{1}{2} \mathrm{Tr} \log \begin{pmatrix} -\partial_{x^4} - \xi(\boldsymbol{p}) + \delta\mu & -(\Delta + \eta(x)) \\ -(\Delta + \eta^*(x)) & -\partial_{x^4} + \xi(\boldsymbol{p}) + \delta\mu \end{pmatrix} + (\delta\mu \to -\delta\mu) \right\}$$

Exercise. Which change of basis have we done here ?

We expand the logarithm in increasing powers of η and η^*

$$\mathrm{Tr} \log(\hat{O} + \hat{V}) = \mathrm{Tr} \log(\hat{O}) + \mathrm{Tr} \Big(\sum_{n=1}^{\infty} \frac{-1}{n} (-\hat{O}^{-1}\hat{V})^n \Big)$$

$$\mathcal{S}[\eta,\eta^*] = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \ldots$$

free energy of the system in the absence of fluctuations

vanishes why?

lowest order non-trivial term in the expansion of the action. Effective LO action for the fluctuations

$$\eta(x) = \frac{1}{\sqrt{2}}(\lambda(x) + i\theta(x))$$

Higgs mode: radial oscillation

Goldstone boson: breaking of U(1) associated with total number conservation

From the LO effective action we obtain

$$\boxed{\mathcal{L}_{\lambda,\phi} = A(\partial_t\phi(x))^2 - \frac{B}{3}(\vec{\nabla}\phi(x))^2 - C\lambda(x)^2 + D(\partial_t\lambda(x))^2 - \frac{E}{3}(\vec{\nabla}\lambda(x))^2}$$

Stability analysis

Role of the fluctuations

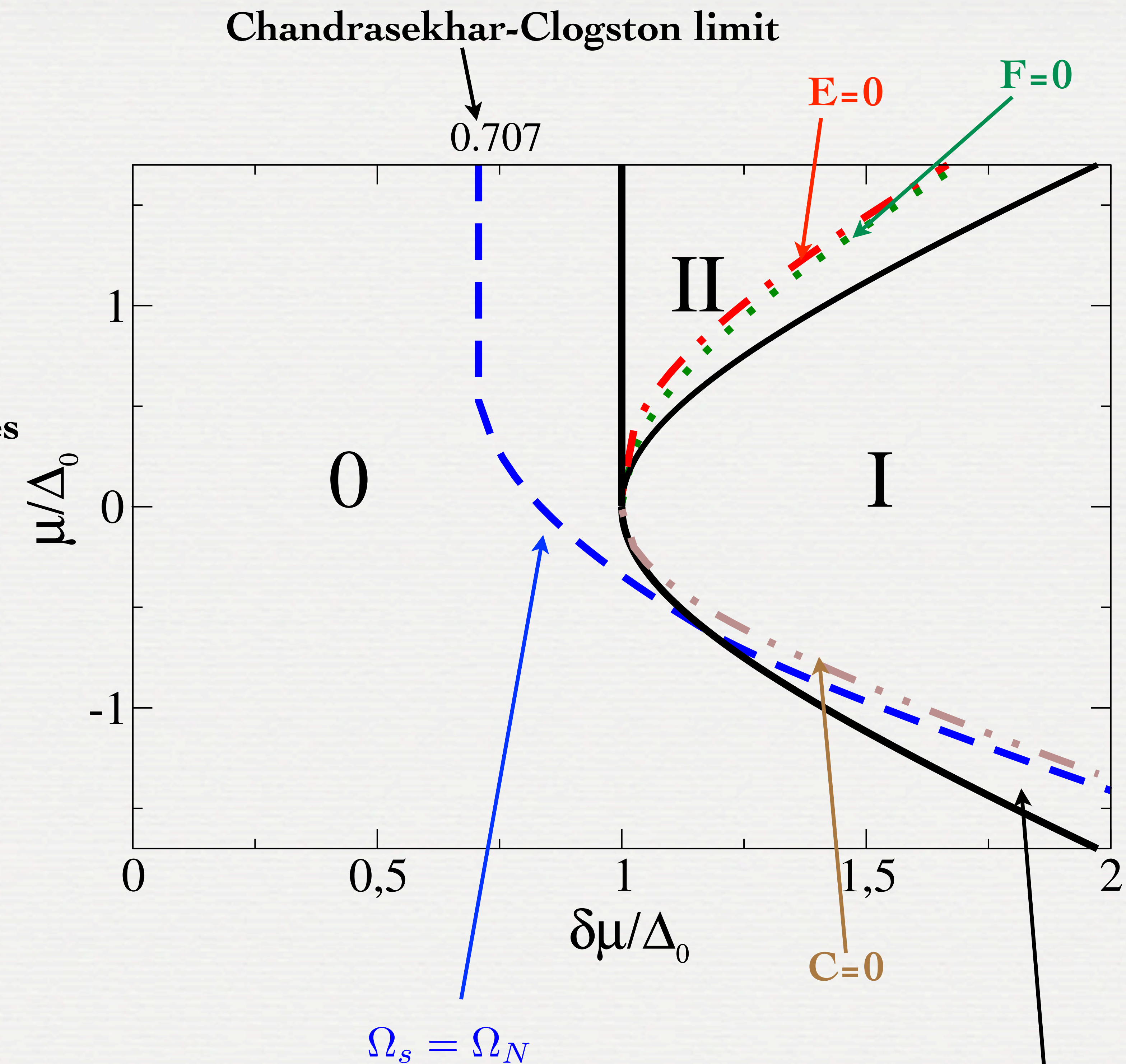
The fluctuations allow to probe the stability of the mean field solution. We are essentially probing the functional space arounf the stationary point.

$$\boxed{\mathcal{L}_{\lambda,\phi} = A(\partial_t \phi(x))^2 - \frac{B}{3}(\vec{\nabla}\phi(x))^2 - C\lambda(x)^2 + D(\partial_t \lambda(x))^2 - \frac{E}{3}(\vec{\nabla}\lambda(x))^2}$$

The stability of the system is guaranteed when all the coefficients are positive and if the free-energy of the system is a global minimum.

A and D turn to be always positive. These are the kinetic coefficients, no surprises here.

0: no gapless fermionic mode
I: one gapless fermionic mode
II: two gapless fermionic modes



At the right of the dashed line, metastable or unstable regions

Stable gapless phase!!

M.M et al. Annals Phys. 325 (2010) 1987-2017

Gauging the U(1)

We have so far assumed that the U(1) is a global symmetry, what happens if we gauge it?

This happens if we switch on a charge. In this case the system becomes a superconductor.

The goldstone boson is "eaten" by the longitudinal component of the gauge field.

$$\boxed{\partial^\mu \phi \to A^\mu}$$

$$\mathcal{L}_{\lambda,A} = A A_0^2 - \frac{B}{3}\boldsymbol{A}^2 \underbrace{- C\lambda(x)^2 + D(\partial_t \lambda(x))^2 - \frac{E}{3}(\vec{\nabla}\lambda(x))^2}_{}$$

Debye mass squared

Meissner mass squared of the gluon

This remains unchanged: probes the radial direction

One of the stability condition is related with having a non-tachionic photon