QCD CONDENSATES

LECTURE III
TOOLS

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OUTLINE

- *Nambu-Jona Lasinio (NJL) model*
- *High Density Effective Theory (HDET)*
- *Low energy effetive theory*

From the previous lectures

- We have seen that we can easily solve a toy model with a local four Fermi interaction

- In the weak coupling limit, the phases with gapless fermionic modes are unstable

- The instability suggests that some inhomogenous phase dhould be realized

The path of these two lectures:

1) Build an EFT which is as similar as possible to the TOY MODEL

2) Study crystalline color superconducting (CCSC) phase, which is one inhomogeneous candidate phase

The physics at high baryonic density is difficult to handle

1. QCD is nonperturbative
2. Lattice simulations do not work (Barbour et al. 1986 Nucl.Phys. B275 296)
3. No experimental facility (so far) can reproduce the correct conditions

Let me refrase it:

1. We do not know how to do computations
2. We do not have numerical methods for doing tests
3. We have no terrestrial lab for validating the theoretical results

The way out:

We can use symmetries and analogies for obtaining qualitative and semiquantitative results

We can use compact stars as the "lab"

Additional ingredients

We have to consider the typical environment of compact stars

- It is “cold”, with temperature of order tens of keV
- Matter is in weak equilibrium
- Matter is electrically neutral
- The strange quark mass might be comparable with the quark chemical potential

The first condition simplifies the treatment: we can take T=0.

The other conditions make the treament more complicated

QCD Lagrangian

$$\mathcal{L}_{QCD} = \sum_{i=1}^{3} \bar{\psi}_i \left(i\gamma_\mu D^\mu - m_i + \mu_i \gamma_0 \right) \psi_i - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}$$

Minimal coupling

Chemical potential

Gluon kinetic term
+ nonabelian interactions

Breaking of Lorentz symmetry

The fact that we can write the Lagrangian does not mean that we can solve it

However, we do not want to solve QCD.
We want to understand how things work at high density and low temperature

"Transform" QCD in a EFT that is as similar as possible to the TOY MODEL

We want to do two things

1) Simplify the interaction; we have somehow to get rid of gluons

2) Simplify the structure; we have somehow to get rid of spinorial indices

We cannot get rid of color and flavor, because these are the symmetries that play a role in the condensation

What we know

- **Perturbatively the color antitriplet channel determines the attractive interaction**

THIS TELLS WHAT KIND OF CHANNEL WE WANT IN THE EFT

- **Gluons are "screened"**

THIS SUGGESTS THAT A LOCAL INTERACTION SHOULD NOT BE BAD

- **"Active fermions" are close to the Fermi sphere**

THIS TELLS THAT ANTIPARTICLES SHOULD NOT BE IMPORTANT

- **The residual symmetries and the low energy degrees of freedom**

THIS SUGGEST THE FORM OF THE CONDENSATE AND OF THE LOW ENERGY EFT

Nambu-Jona Lasinio (NJL) model

"Getting rid of gluons"

Simplifying the interaction

The most important part of QCD (for our puroposes) is the interaction between fermions and gauge fields

NJL Lagrangian $\mathcal{L}_{\rm NJL} = \bar\psi\left(i\gamma_\mu\partial^\mu - m + \boldsymbol{\mu}\gamma_0\right)\psi + \mathcal{L}_{int}$

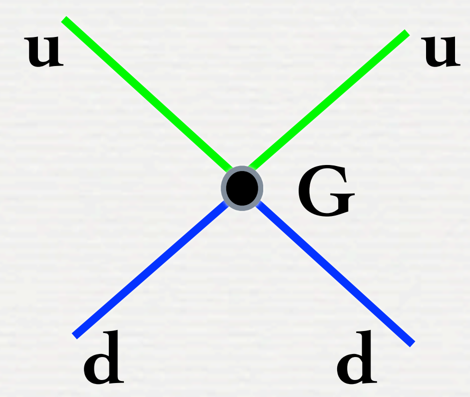
no minimal coupling

$$\boldsymbol{\mu} \equiv \mu_{ij,\alpha\beta} = (\mu\ \delta_{ij} - \mu_e\ Q_{ij})\,\delta_{\alpha\beta} + \frac{2}{\sqrt{3}}\mu_8(T_8)_{\alpha\beta}\delta_{ij}$$

Color and electrical chemical potentials are introduced by hand. (in gauge theories the gauge fields that drive the system to a “neutral state”)

For the interaction term we try something that mimics QCD and that is simple to handle

Local Fermi interaction



Two-flavor quark matter

$$\mathcal{L}_{int} = G_S \left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \boldsymbol{\tau} \psi)^2 \right] + G_D \left[(\bar{\psi}^C \epsilon \varepsilon i\gamma_5 \psi)_{k\gamma} (i\bar{\psi} \epsilon \varepsilon i\gamma_5 \psi^C)_{k\gamma} \right]$$

$$(\bar{\psi}^C \epsilon \varepsilon i\gamma_5 \psi)_{k\gamma} \equiv (\bar{\psi}_i^{\alpha C} \epsilon_{ijk} \varepsilon^{\alpha\beta\gamma} i\gamma_5 \psi_j^\beta)$$

- Analogous of the four-fermi interaction in the electroweak sector.
- Analogous of the interaction we have seen in the TOY MODEL
- The interaction diquark channel is in the relevant color antitriplet

Shortcomings

1) No confinement

2) High energy cut-off

3) Non-renormalizable

4) Couplings have to be fixed

5) Missing Yang-Mills contributions (PNJL does a bit better)

NJL helps to have a qualitative and semiquantitative description

Example: what happens changing the coupling

As we have seen in the previous lecture that at a fixed mismatch changing the coupling we have different phases

Two-flavor homogeneous phases

BEC-like
- For $G_D/G_S \gtrsim 0.8$, *strong coupling*, the 2SC phase is the only **homogeneous** stable phase

BCS-gapless
- For $0.7 \lesssim G_D/G_S \lesssim 0.8$, *intermediate coupling*, the g2SC phase is allowed for $\delta\mu > \Delta_{2SC}$

Unpaired phase
- For $G_D/G_S \lesssim 0.7$, *weak coupling*, only unpaired quark matter is favored.

The BCS-like gapless phase named g2SC is chromomagnetically unstable

This instability is the analologous of the instability we found in the two-level system and in three-flavor quark matter in the gCFL phase

The qualitative aspect: "the gapless intermediate coupling phase is unstable" is trustable
What we do not know is in which of the three regimes above we are

High Density Effective Theory

“Getting rid of the spinorial structure”

Start from the QCD (or NJL) Lagrangian for obtaining a simpler Lagrangian with no Dirac structure

First we focus on noninteracting degenerate massless quarks

Then we shall add the condensate and the fluctuations

Various steps

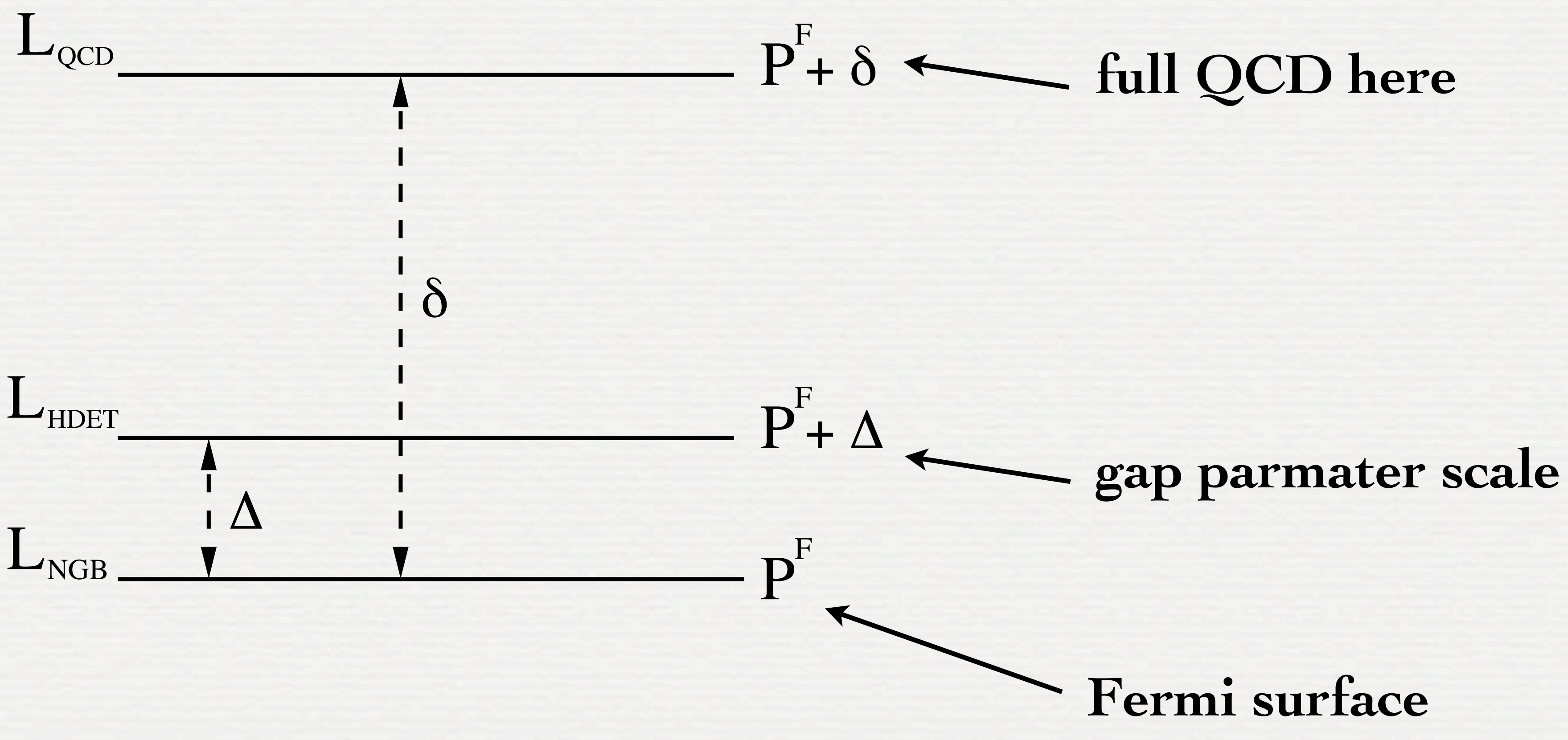
- Identify relevant scales
- Momentum decomposition
- Projecting out negative energy fields
- Rewriting the Lagrangian with positive energy fields
- Get rid of the Dirac structure
- Expanding the Lagrangian

Relevant scales

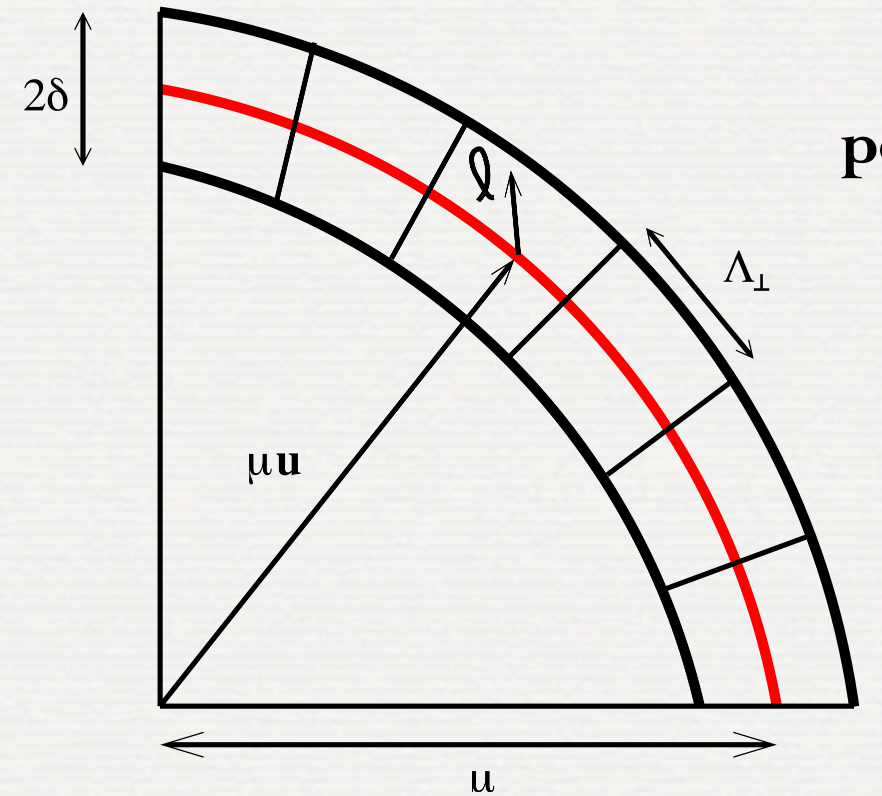
Given the large scale μ the idea is to expand

$$\boxed{\mathcal{L} = \mathcal{L}_0 + \frac{1}{\mu}\mathcal{L}_I + \frac{1}{\mu^2}\mathcal{L}_{II} + \ldots}$$

In general, the presence of a hierearcy implies that one can develop an EFT



Momentum decomposition



portion of the Fermi sphere

Momentum separation

$$\boxed{p^\nu = \mu v^\nu + \ell^\nu}$$

where $\ell^\nu = (\ell_0, \ell_{||}\boldsymbol{v})$ $\quad v^\nu = (0, \boldsymbol{v})$ $\quad |\boldsymbol{v}| = 1$

therefore $p^0 = \ell^0$ $\quad \boldsymbol{p} = (\mu + \ell_{||})\boldsymbol{v}$

velocity dependent fields

$$\boxed{\psi(x) = \int \frac{d\boldsymbol{v}}{4\pi} e^{-i\mu v\cdot x}\psi_{\boldsymbol{v}}(x)}$$

Positive and negative energy fields

Positive and negative energies $\epsilon_{\pm} = -\mu \pm |\boldsymbol{p}|$

considering that $\boldsymbol{p} = (\mu + \ell_{\parallel})\boldsymbol{v}$

$\epsilon_{+} = \ell_{\parallel}$ ← **particle on the top of the Fermi sphere**

$\epsilon_{-} = -2\mu + \ell_{\parallel}$ ← **antiparticle deep in the Dirac sea**

Antiparticles, at the leading order, decouple

Energy projectors $P_{\pm} = \dfrac{1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{v}}{2}$

Exercise: show that this operator project positive and negative energy states

Positive and negative energy velocity dependent fields $\psi_{\pm,\mathbf{v}}(x) = P_{\pm}\psi_{\mathbf{v}}(x)$

Rewriting the Lagrangian

$$\mathcal{L} = \sum_{\mathbf{v}} \underbrace{\psi_+^\dagger iV\cdot D\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_-}_{\text{no spinorial structure}} + \underbrace{\psi_+^\dagger\gamma^0 i\not{D}_\perp\psi_- + \psi_-^\dagger\gamma^0 i\not{D}_\perp\psi_+}_{\text{“residual” spinorial structure}}$$

light-like vecotors $V^\mu = (1, \boldsymbol{v})$ $\tilde{V}^\mu = (1, -\boldsymbol{v})$

velocity projectors $L^{\mu\nu} = \dfrac{V^\mu\tilde{V}^\nu + V^\nu\tilde{V}^\mu}{2}$ $P^{\mu\nu} = g^{\mu\nu} - L^{\mu\nu}$

Exercise. Show that for any vector field

$$\begin{aligned} A_\perp^\mu &= P^{\mu\nu}A_\nu = (0, \boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{v})\boldsymbol{v}) \\ A_\parallel^\mu &= L^{\mu\nu}A_\nu = (A_0, (\boldsymbol{A}\cdot\boldsymbol{v})\boldsymbol{v}) \end{aligned}$$

Disappearence of spinorial structure? How does that happen?

Consider the current

$$J^\nu(x) = \bar{\psi}(x)\gamma^\nu\psi(x) = \int \frac{d\mathbf{v}}{4\pi}\frac{d\mathbf{v}'}{4\pi} e^{-i\mu(v'-v)\cdot x}\bar{\psi}_{\mathbf{v}'}(x)\gamma^\nu\psi_{\mathbf{v}}(x)$$

for large chemical potentials $\quad v' = v \quad$ this is a “selection rule”

Gordon decomposition (for massive fields)

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{p^\mu + p'^\mu}{2m} + i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right)u(p)$$

since $\quad \mathbf{v}' = \mathbf{v} \quad$ then

$$\begin{aligned} p^\nu &= \mu v^\nu + \ell^\nu \\ p'^\nu &= \mu v^\nu + \ell'^\nu \end{aligned}$$

Show that $\quad \bar{u}(p')\gamma^\mu u(p) \simeq \dfrac{\mu\, v^\mu}{m}\bar{u}(\ell')u(\ell)$

Integrating out the negative energy fields

Equations of motion:

$$\begin{aligned} iV\cdot D\psi_+ + i\gamma^0 \not{D}_\perp \psi_- &= 0 \\ (2\mu + i\tilde{V}\cdot D)\psi_- + i\gamma^0 \not{D}_\perp \psi_+ &= 0 \end{aligned}$$

then we get $$\psi_- = -i\frac{1}{2\mu + i\tilde{V}\cdot D}\gamma^0 \not{D}_\perp \psi_+$$

The negative energy states represent the "small" component
The positive energy states are the "large" component

Integrating out the "small" component we obtain the HDET LAGRANGIAN

$$\mathcal{L}_I = \int \frac{d\mathbf{v}}{4\pi}\left[\psi_+^\dagger (iV\cdot D)\psi_+ - P^{\mu\nu}\psi_+^\dagger D_\mu \frac{1}{2\mu + i\tilde{V}\cdot D} D_\nu \psi_+\right]$$

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{\mu}\mathcal{L}_I + \frac{1}{\mu^2}\mathcal{L}_{II} + \dots$$

Turning on pairing and fluctations

Nambu-Gorkov field
(a bit more complicated than in the toy model)

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{+,+} \\ C\psi^*_{+,-} \end{pmatrix} \qquad \mathcal{L}_a = \mathcal{L}_0 \qquad \mathcal{L}_b = \frac{1}{\mu}\mathcal{L}_I + \frac{1}{\mu^2}\mathcal{L}_{II} + \dots$$

with $\psi_+(\pm\mathbf{v}) \equiv \psi_{+,\pm}$

$$\mathcal{L}_a = \int \frac{d\mathbf{v}}{4\pi} \Psi^\dagger \begin{pmatrix} iV^\mu\partial_\mu - V^\mu A_\mu & \Delta + \rho \\ \Delta + \rho & i\tilde{V}^\mu\partial_\mu + \tilde{V}^\mu A_\mu \end{pmatrix} \Psi$$

$$\mathcal{L}_b = \int \frac{d\mathbf{v}}{4\pi} \Psi^\dagger P^{\mu\nu} A_\mu A_\nu \begin{pmatrix} -2\mu + V\cdot\ell + V\cdot A & -\Delta \\ -\Delta & 2\mu + \tilde{V}\cdot\ell - \tilde{V}\cdot A \end{pmatrix} \frac{1}{L}\Psi$$

with $A^\mu = \partial^\mu \phi \qquad L = (2\mu + \tilde{V}\cdot\ell - \tilde{V}\cdot A)(-2\mu + V\cdot\ell + V\cdot A) - \Delta^2 - i\epsilon$

$\mathcal{L}_a$ **is basically the TOY MODEL Lagrangian**

$$\mathcal{L}_{\text{toy model}} = \Psi^\dagger \begin{pmatrix} i\partial_0 - \xi(\boldsymbol{p}) + \delta\mu\sigma^3 & -\Delta \\ \Delta^* & i\partial_0 + \xi(\boldsymbol{p}) - \delta\mu\sigma^3 \end{pmatrix} \Psi$$

Integrating over the positive energy fields **(as in the toy model, but here we keep Δ as a field)**

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \frac{\int [d\Delta, d\Delta^*] \exp\left[\frac{1}{g}\int d^4x\, \Delta^2\right] \det[S^{-1}]^{1/2}}{\det[S_0^{-1}]^{1/2}}$$

the full inverse propagator is $S^{-1} \equiv S_{MF}^{-1} + \Gamma$

Effective action

$$\mathcal{S}_{\text{eff}} \sim -\frac{i}{g}\int d^4x\, \left[\rho^2 + 2\rho\Delta\right] - \frac{i}{2}\mathrm{Tr}\ln\left(1 + S_{MF}\Gamma\right)$$

As in the toy model we now expand

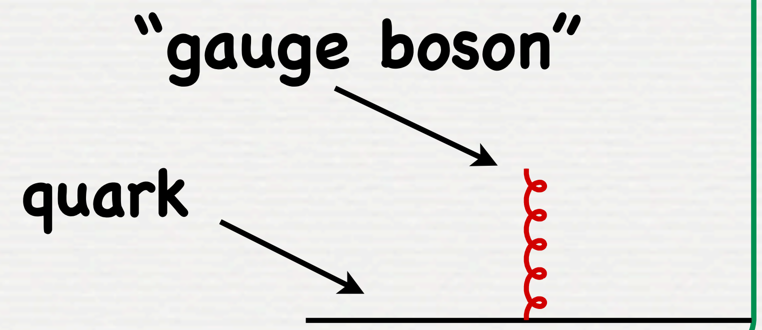
$$\mathrm{Tr}\ln\left(1 + S_{MF}\Gamma\right) = \mathrm{Tr}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(S_{MF}\Gamma\right)^n\right]$$

where $\Gamma = \Gamma_\rho + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \ldots$ **with** $\Gamma_\rho = \begin{pmatrix} 0 & -\rho \\ -\rho & 0 \end{pmatrix}$

Interaction with fictitious gauge bosons

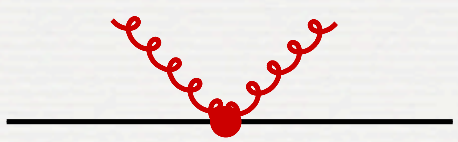
Minimal coupling

$$\Gamma_1 = \begin{pmatrix} -V\cdot A & 0 \\ 0 & \tilde{V}\cdot A \end{pmatrix}$$

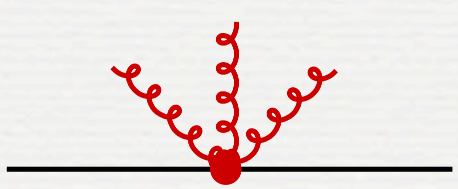


Nonminimal couplings

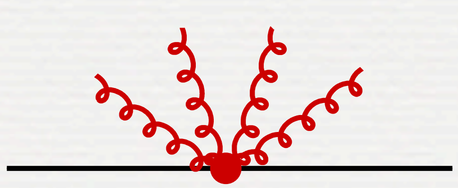
$$\Gamma_2 = \frac{P^{\mu\nu}A_\mu A_\nu}{L_0}\begin{pmatrix} -2\mu + V\cdot\ell & -\Delta \\ -\Delta & 2\mu + \tilde{V}\cdot\ell \end{pmatrix}$$



$$\Gamma_3 = -\frac{\Gamma_2}{L_0}[V\cdot A(2\mu + \tilde{V}\cdot\ell) + \tilde{V}\cdot A(2\mu - V\cdot\ell)] - \frac{\Gamma_1}{L_0}P^{\mu\nu}A_\mu A_\nu$$

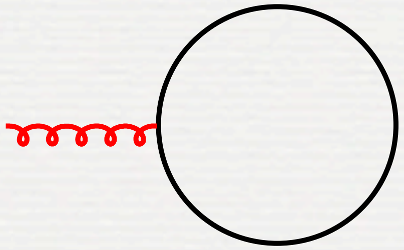


$$\Gamma_4 = \frac{\Gamma_3}{L_0}[V\cdot A(2\mu + \tilde{V}\cdot\ell) + \tilde{V}\cdot A(2\mu - V\cdot\ell)]$$

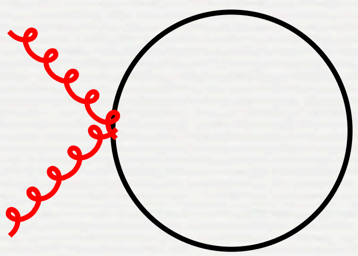


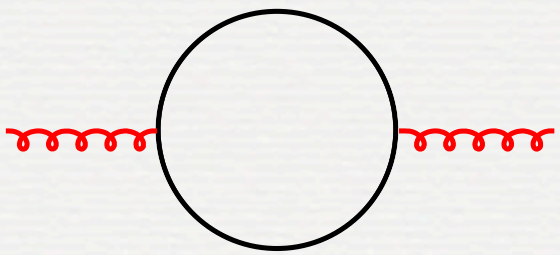
One-point function

$$\mathcal{L}_1 = -i\mathrm{Tr}[S\Gamma]\Big|_A = \left(-\frac{3}{\pi^2}\mu^3 + \frac{6}{\pi^2}\mu^2\Delta\right)\partial_0\varphi$$



Self-energy





$$\begin{aligned}\mathcal{L}_2 &= -i\left(\mathrm{Tr}[S\Gamma] - \frac{1}{2}\mathrm{Tr}[S\Gamma S\Gamma]\right)\Big|_{A^2} \simeq \frac{9\mu^2}{2\pi^2}\left(1 - 2\frac{\Delta^2}{\mu^2}\right)A_0^2 - \frac{3\mu^2}{2\pi^2}\left(1 - 2.1\frac{\Delta^2}{\mu^2}\right)\mathbf{A}^2 \\ &= \frac{1}{2}m_D^2 A_0^2 - \frac{1}{2}m_M^2\mathbf{A}^2\end{aligned}$$

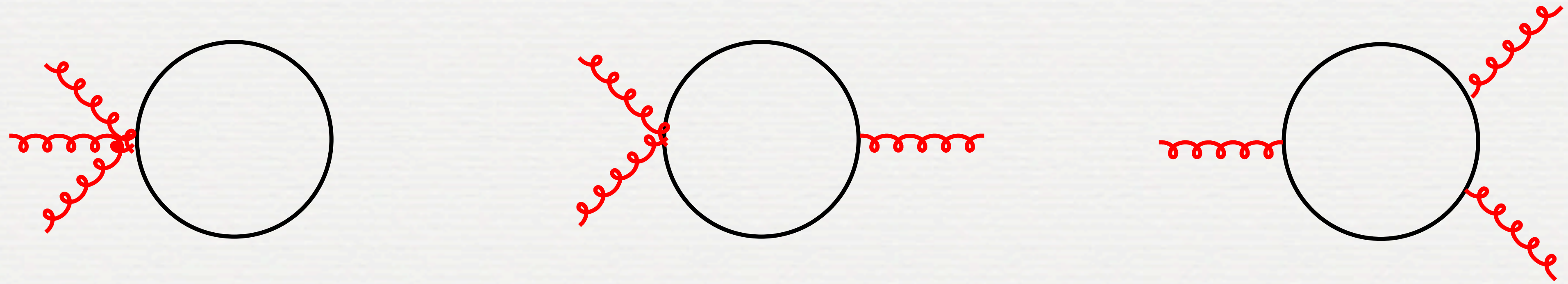
after renormalization $\mathcal{L}_2 = \frac{1}{2}g_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi + \mathcal{O}\left(\frac{\Delta^2}{\mu^2}\right)$

$$c_s^2 = 1/3$$

$$g_{\mu\nu} = \mathrm{diag}(1, -1/3, -1/3, -1/3)$$

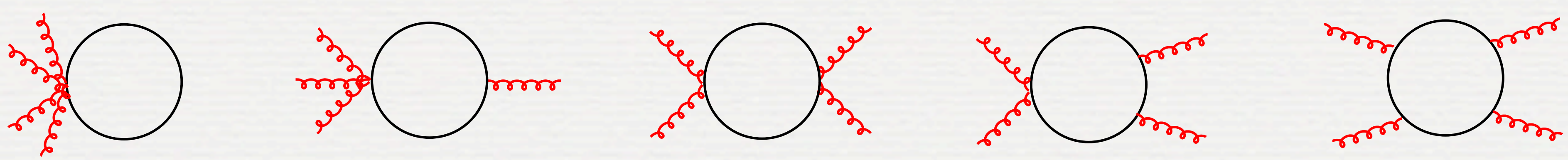
induced metric

Three-point function



$$\mathcal{L}_3 = -i \left(\mathrm{Tr}[S\Gamma] - \frac{1}{2}\mathrm{Tr}[S\Gamma S\Gamma] + \frac{1}{3}\mathrm{Tr}[S\Gamma S\Gamma S\Gamma] \right)\Big|_{A^3} \simeq \frac{3\mu}{\pi^2}\left(1 - \frac{\Delta^2}{\mu^2}\right) A_0 \mathbf{A}^2 - \frac{3\mu}{\pi^2}\left(1 - \frac{3\Delta^2}{2\mu^2}\right) A_0^3$$

Four-point function



$$\mathcal{L}_4 = -i \sum_{k=1}^{4} \mathrm{Tr}[(S\Gamma)^4]\Big|_{A^4} \simeq \frac{3}{4\pi^2} A_0^4 + \frac{3}{4\pi^2}\mathbf{A}^4 - \frac{3}{2\pi^2} A_0^2 \mathbf{A}^2$$

Low energy effective theory

Nambu-Goldstone bosons of CFL

Since the color superconductors are realized in compact stars (ultracold systems) we can integrate out the fermionic degrees of freedom, focusing on the low energy NGBs

Octet

$$\mathcal{L}_{\rm eff} = \frac{f_\pi^2}{4}{\rm Tr}[\partial_0\Sigma\partial_0\Sigma^\dagger - v_\pi^2\partial_i\Sigma\partial_i\Sigma^\dagger]$$

$$f_\pi^2 = \frac{21-8\log 2}{18}\frac{\mu^2}{2\pi^2}$$

$$v_\pi^2 = \frac{1}{3}$$

$\Sigma = e^{i\phi^a\lambda_a/f_\pi}$ ϕ^a **describes the octet** $(\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$

Casalbuoni and Gatto, Phys. Lett. B 464, (1999) 111

Since chiral symmetry is explicitly broken by the quark masses, these are pseudo NGBs

Masses

In standard chiral perturbation theory, one considers a mass term like

$$\Delta\mathcal{L}_{\rm eff} = \cancel{{\rm Tr}(M^\dagger\Sigma + \Sigma^\dagger M)}$$

not allowed by $U(1)_A$ symmetry

Thus, we are forced to terms like

$$\Delta\mathcal{L}_{\rm eff} = -c\,{\rm Det}M\,{\rm Tr}(M^{-1}\Sigma) - c'\,{\rm Det}\Sigma\,[{\rm Tr}(M\Sigma^\dagger)^2 + ({\rm Tr}M\Sigma^\dagger)^2] + {\rm h.c.}$$

$$\boxed{\begin{aligned} m^2_{\pi^\pm} &= A\,(m_u + m_d)m_s \\ m^2_{K^\pm} &= A\,(m_u + m_s)m_d \\ m^2_{K^0,\bar{K}^0} &= A\,(m_d + m_s)m_u \end{aligned}}$$

kaons are lighter than mesons!

$$A = \frac{3\Delta^2}{\pi^2 f_\pi^2}$$

Son and Sthephanov, Phys. Rev. D 61, (2000) 74012

Qualitative reason:

$\pi^+ \sim (\bar{d}\bar{s})(us)$ has strangeness 0

$K^+ \sim (\bar{d}\bar{s})(ud)$ has strangeness 1

"H-Phonon"

There is an additional massless NGB, φ, associated to $U(1)_B \longrightarrow Z_2$

Quantum numbers $\varphi \sim \langle \Lambda \Lambda \rangle$ like the H-dibaryon of Jaffe, Phys. Rev. Lett. 38, 195 (1977)

Effective Lagrangian up to quartic terms

$$\mathcal{L}_{\text{eff}}(\varphi) = \frac{3}{4\pi^2}\left[(\mu - \partial_0\varphi)^2 - (\partial_i\varphi)^2\right]^2$$

Son, hep-ph/0204199

bulk

"sound" or phonon

$$\varphi(x) = \bar{\varphi}(x) + \phi(x)$$

classical field

long-wavelength fluctuations

Phenomenology

Dissipative processes due to vortex-phonon interaction damp r-mode oscillations of CFL stars rotating at frequencies < 1 Hz

Phys. Rev. Lett. 101, 241101 (2008)