Superfluidity and Superconductivity in nucleonic Neutron Stars matter

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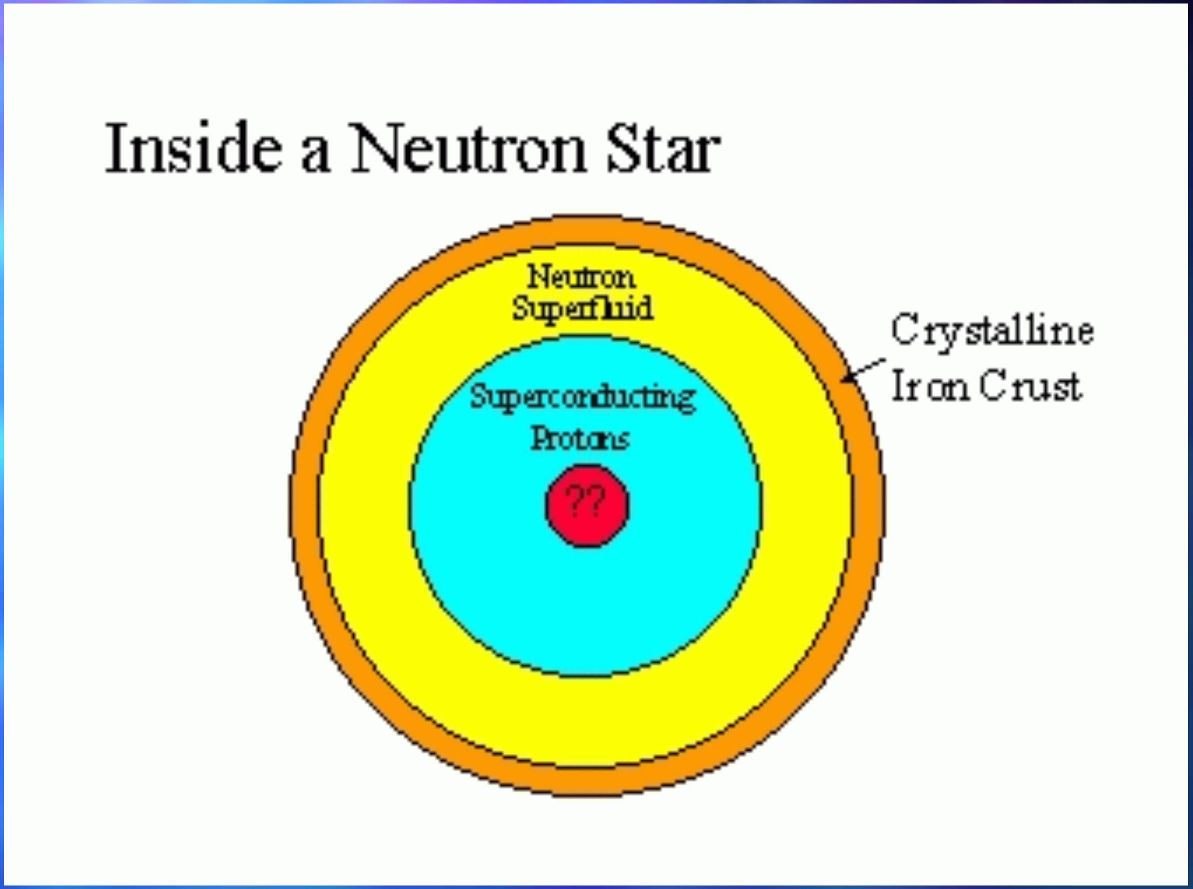
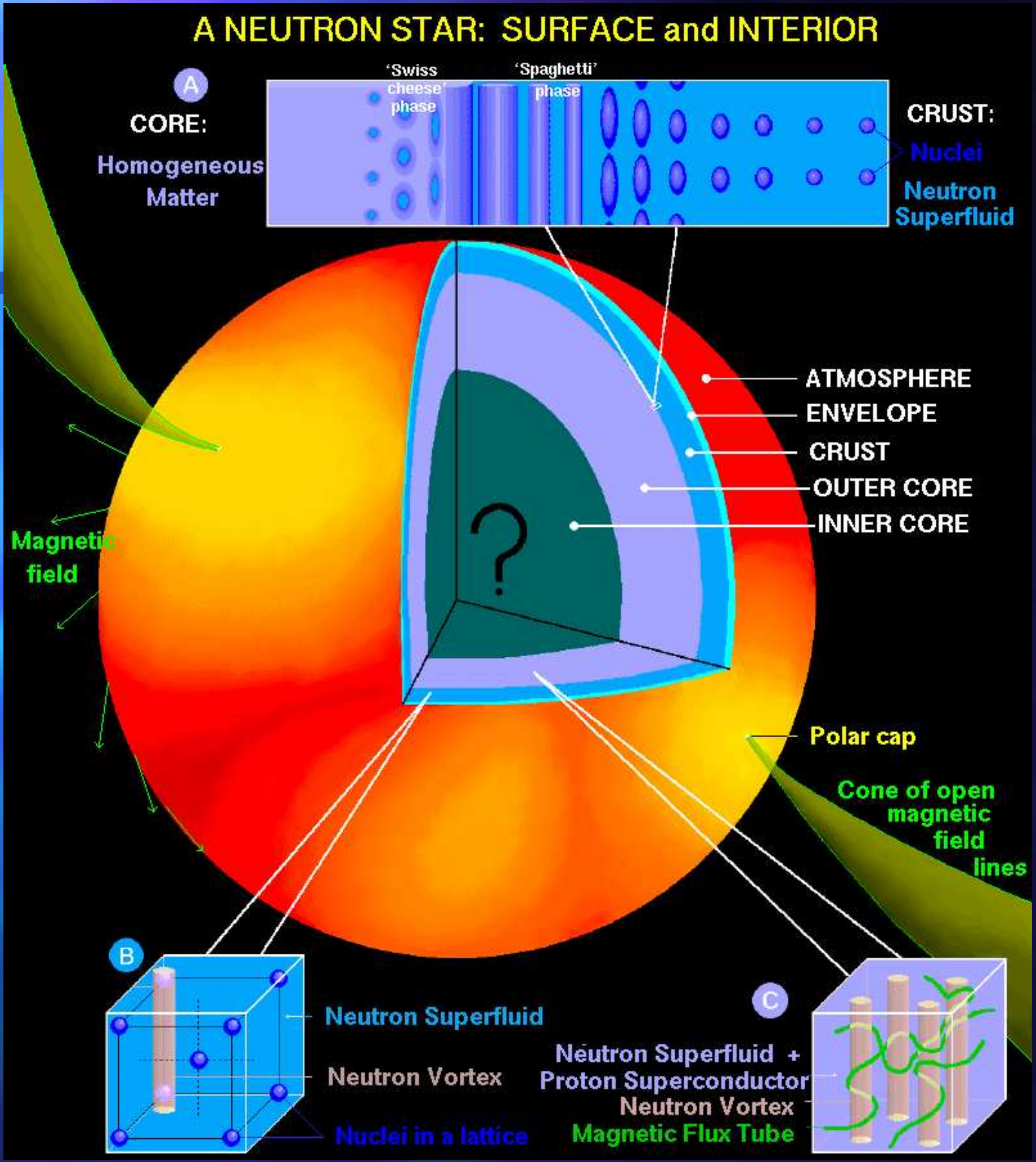
Brief history and overview of superfluidity.

In Neutron Stars (NS) both superfluidity and superconductivity are likely to be present. Indeed both the neutron and the proton components are expected to be superfluid in some density region of the NS and a charged superfluid can be identified as a superconductor. Both have a basic relevance for many phenomena that occur in NS.



Schematic view

A section (schematic) of a neutron star



Let us start with a brief survey of superfluidity , which actually was discovered after superconductivity. It was first realized in the boson ^{4}He liquid. We consider for simplicity this case since, as we will see, some of the properties of boson superfluids can be transferred to Fermion superfluids. Few years after the discovery that liquid ^{4}He at a temperature T_c of about 2.2 K undergoes a phase transition, it was realized (independently Kapitza and Meisener 1938) that below T_c the liquid has some peculiar properties, completely different from an usual classical liquid.

Schematically

1. Below T_c the viscosity of the fluid drops to a vanishing small value.

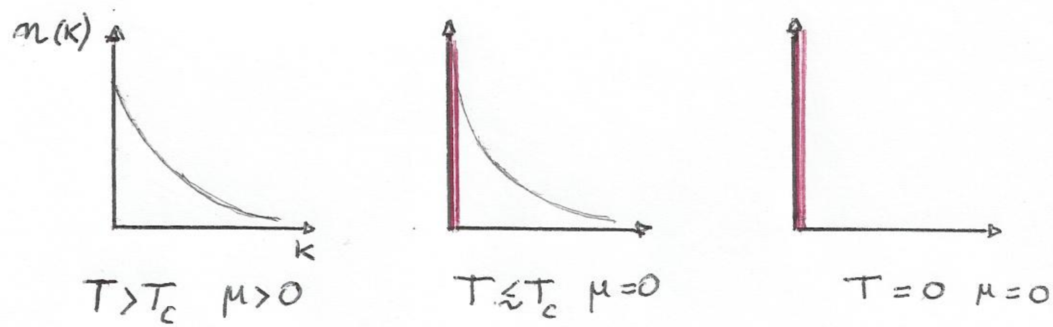
2. "Fountain effect" : heath transfer induces matter transfer.

Due to property 1 the term "supefluid" was introduced. It was London who first realized correctly that the superfluid phenomenon is associated with Bose-Einstein condensation (BEC). Nowadays perfect BEC is produced in experiments on cold atoms.

Let us consider an ideal (non interacting) gas of bosons in a large container.

$$n(k) = \frac{1}{\exp(E(k) - \mu)/T - 1} \qquad (1)$$

under the condition $\sum_k n(k) = N$, the total number of particles.



We can imagine a macroscopic occupancy at a non-zero momentum

$$\mathbf{j} = \rho\mathbf{k}/m \qquad \textit{macroscopic current}$$

$$\phi(\mathbf{r}) = \exp(i\mathbf{kr}) \qquad \textit{common wavefunction}$$

Is the "super-current" stable ? Why the fluid below T_c is frictionless (superfluid) ?

$$\mathbf{j} = \frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) \qquad \textit{current}$$

$$\frac{\partial\rho}{\partial t} + \nabla\mathbf{j} = 0 \qquad \textit{local continuity equation}$$

$$\rho = |\psi|^2 \qquad \textit{density}$$

Let us put

$$\psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\,\mathrm{exp}(-iS(\mathbf{r},t))$$

then $\quad \mathbf{j}(\mathbf{r},t) = \frac{i\hbar}{m}\rho(\mathbf{r},t)\nabla S(\mathbf{r},t)$

$i\hbar\nabla S(\mathbf{r},t)/m \equiv$ velocity $\mathbf{v}$ at the point ($\mathbf{r}$,t).

If we assume that the boson condensate is in some wave function $\phi(\mathbf{r},t)$, the current is a macroscopic current, which is N times the one associated with a single particle. It follows that the evolution of the possible condensate described by the field $\phi(\mathbf{r},t)$ is characterized by an irrotational flow

$$\mathbf{rot}\,\mathbf{v} = \nabla \times \nabla\, S(\mathbf{r},t) = 0$$

No closed fluid stream can be present. In the initial irrotational flow, a finite value of circulation can appear only through some topological change of the flow, i.e. a discontinuity in the flow pattern. Therefore, for a smooth evolution of the condensate, an initial irrotational flow will remain irrotational.

$S(\mathbf{r},t)$ determines a "potential flow". In the long wavelength regime, the condensate can be described by Euler equation

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\,\nabla\,\mathbf{v} + \nabla\mu(\mathbf{r},t) = 0$$

where $\mu(\mathbf{r},t)$ is the local chemical potential.

Neglecting the term quadratic in $\mathbf{v}$,

$$\nabla\,(\frac{\partial}{\partial t}S(\mathbf{r},t) + \mu(\mathbf{r},t)) = 0$$

$$\frac{\partial}{\partial t}S(\mathbf{r},t) + \mu(\mathbf{r},t) = C$$

To this equation we can associate the continuity equation, with the current

$$\mathbf{j}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) = \rho(\mathbf{r},t)\nabla S(\mathbf{r},t)$$

$$\frac{\partial}{\partial t}\mu(\mathbf{r},t) + s^2 \triangle S(\mathbf{r},t) = 0$$

$$\rho\, s^2 = \partial\mu/\partial\rho$$

From Euler equation and the continuity equation one gets the standard wave equation

$$\frac{\partial^2}{\partial t^2} S(\mathbf{r}, t) = s^2 \triangle S(\mathbf{r}, t)$$

where s is the velocity of wave propagation.

- No local equilibrium

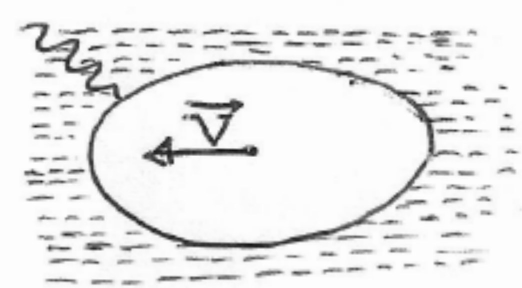
- Restoring force from particle-particle interaction

- Quantization : phonons $\quad \omega = sq$
Like photons for the Electromagnetic Field

- They exhaust the total excitation strength (to be demonstrated)

Let us consider an external body of mass M which is moving with a velocity $\mathbf{V}$ inside a superfluid.

Dissipation is present if transfer of energy occurs between the body and the fluid, i.e. if some excitations of the fluid with momentum $\mathbf{q}$ and energy $\omega(q)$ is produced by the motion of the body



Energy and momentum conservation

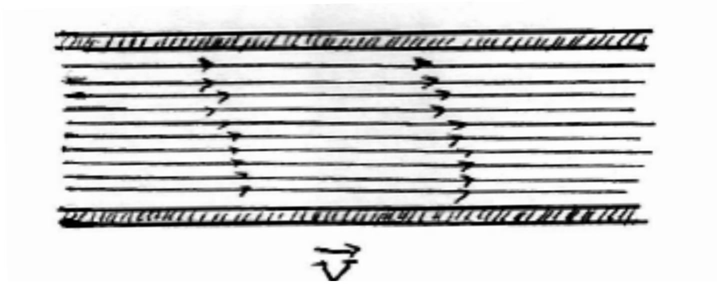
$$\begin{aligned} M\mathbf{V} &= M\mathbf{V}' + \mathbf{q} \\ \tfrac{1}{2}M\mathbf{V}^2 &= \tfrac{1}{2}M\mathbf{V}'^2 + \omega(q) \end{aligned}$$

From these equations one gets

$$q = 2M\left[\mathbf{V}\cdot\hat{\mathbf{q}} - \frac{\omega(q)}{q}\right] = 2M\left[\mathbf{V}\cdot\hat{\mathbf{q}} - s\right]$$

This last relation cannot be satisfied if $V < s$, no dissipative process is possible. For a quadratic q-dependence of $\omega(q)$, the equation can be satisfied for any value of the velocity V and the motion is damped.

The same is true if the body is at rest and the fluid is moving (Galilei transformation), in particular for a flow through a pipe



The phenomenological critical velocity is at least two orders of magnitude smaller.

•Some other type of excitations that can produce dissipation at much smaller velocity.

•These excitations must contain some rotational flow

•The overall superfluid flow is not any more irrotational everywhere

•They must correspond to some discontinuity or singularity

VORTEX !

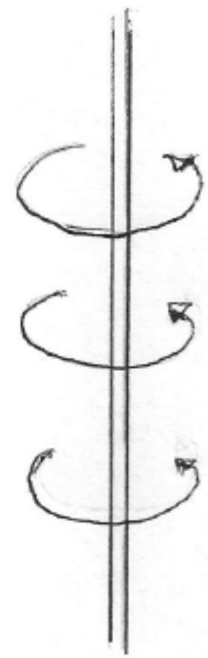
It can be described, in its simplest configuration, by a singular straight line around which a fluid circulation takes place. Around the singularity the flow is still irrotational, which means that the value of circulation is constant along any path going around the straight line

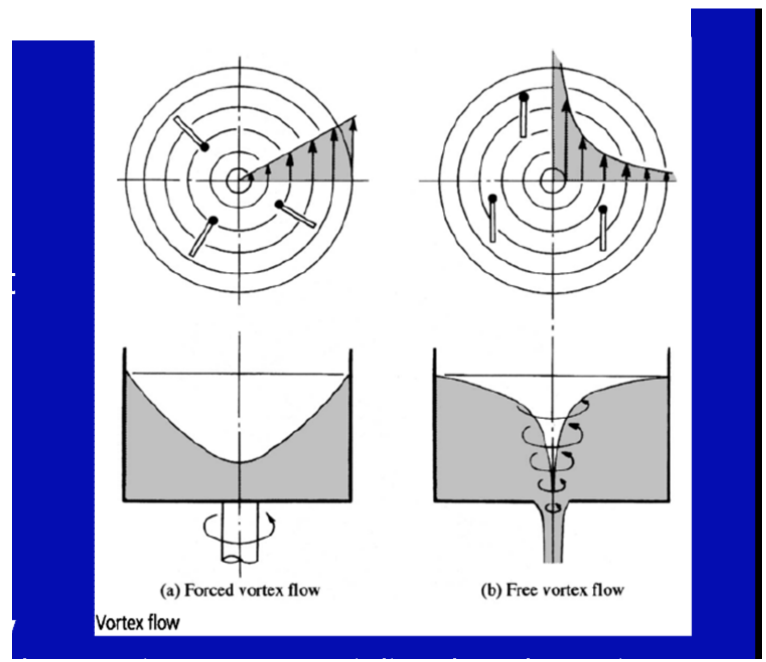
$$\mathrm{I} = \oint \mathbf{v} \cdot d\ell = \text{cost}$$

For a cylindrically symmetric vortex (and stationary) the flow velocity $\mathbf{v}(\mathbf{r})$ is tangential, and then

$$v(r) = \frac{\mathrm{I}}{2\pi r}$$

The singularity at $r = 0$ is apparent.





In fact

$$\mathbf{rot}\,(\mathbf{v}(x,y)) = \mathrm{I}\,\delta(x)\delta(y)$$

Of course the singularity is an idealization and the vortex has actually a normal core. The condensate wave function ψ must be of the type

$$\psi(r,\phi) = \eta(r)\,\mathsf{exp}(in\phi)$$

Since $\psi(r,\phi)$ must be single-valued, n must be an integer. $n\hbar$ is the angular momentum of each particle along the vortex axis, and therefore circulation around the vortex is quantized

$$\oint \mathbf{p}\cdot d\mathbf{l} = m \oint \mathbf{v}\cdot d\mathbf{l} = 2\pi n\hbar$$

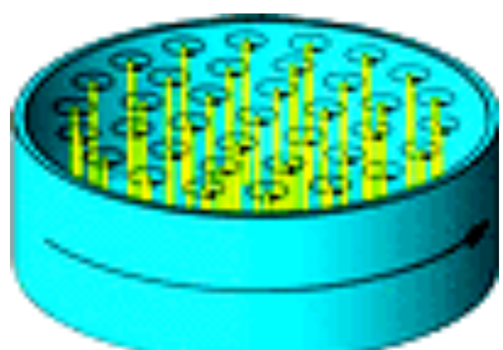
in agreement with B-S quantization.

The straight line configuration is the one with the lowest energy, because the energy is proportional to the length (as we will see). Where can a vortex end ?

- At the the border with the normal phase or the empty space.

- It can close on itself : vortex ring.

The first possibility is favoured in the case the superfluid is inside a rotating container, like in Neutron Stars



Natural vortex ring

Vortex ring from the vulcan Etna

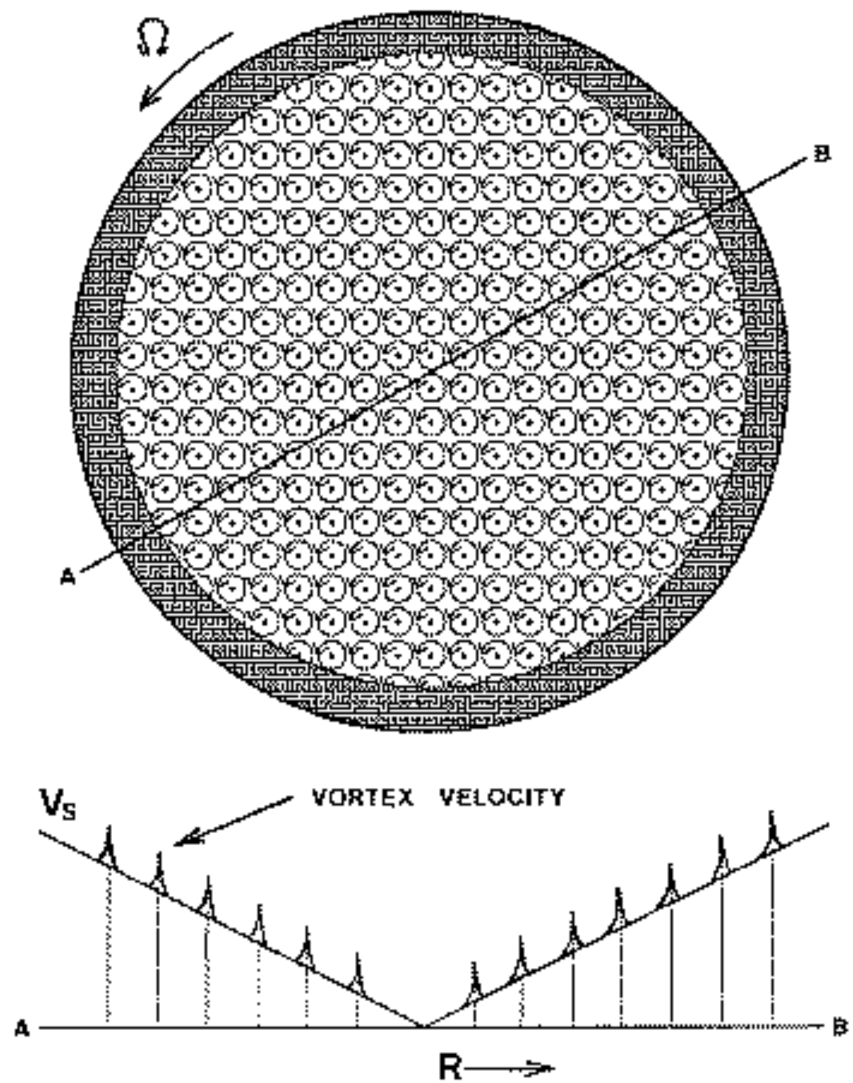
In a rotating vessel the superfluid cannot follow the rotation since this would imply the presence of a rotational flow

$$\mathbf{rot}\,\mathbf{v}(\mathbf{r}) = 2\Omega\,\hat{z}$$

The only possibility for the superfluid to follow the rotation is the formation of a certain number of vortices, co-rotating with it. The superfluid rotates as a rigid body, when it is seen at macroscopic level. However, the value of the curl vanishes everywhere except at the core of each vortex, so that

$$N = m\Omega R^2/\hbar$$

is the total number of vortices.


Ω
B
A
V_s
VORTEX VELOCITY
A
B
R

Energy and stability of vortices.

Intrinsic energy of a rotating superfluid

$$E = E_0 - \omega J_z$$

E_0 energy in the inertial reference frame

J_z total angular momentum

Weakly interacting boson gas

$$\begin{aligned} E &= \tfrac{1}{2} m \int v(r)^2 \rho_s d^3 r = \tfrac{1}{2} m\, \rho_s I^2 \int dz \frac{dr}{2\pi r} \\ &= \frac{m I^2}{4\pi}\, \rho_s\, L \ln\left(\frac{R}{a}\right) \end{aligned}$$

a vortex core radius

R cut-off distance

L vortex length

The total angular momentum of this vortex is given by $\hbar$ times the number of particles

$$J_z = \pi R^2 L \rho_s \hbar$$

The formation of the vortex will be favoured above a critical angular velocity ω_c

$$\omega_c = \frac{\hbar}{mR^2} \ln(\frac{R}{a})$$

Because the formation energy of a vortex is proportional to the square of the angular momentum, the formation of a $n = 2$ vortex is not favoured with respect to the formation of two $n = 1$ vortices. A rotating superfluid is therefore expected to contain only vortices with one quantum of vorticity.

There is an interaction among vortices. Let us consider for simplicity two vortices. The total velocity flow will be just the sum of the two flows

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

and the energy

$$E = E_1 + E_2 + \frac{2mI_1I_2}{4\pi}\rho_s \ln\frac{R}{d}$$

Vorticity of the same sign : repulsion

Vorticity of opposite sign : attraction

Similarity with the magnetic interaction between two conducting wires. A set of vortices form a triangular lattice (ideal case).

Microscopic approach

The quantization can be performed on a microscopic basis. Part of the theory of the excitations in a boson superfluid can be transferred to the fermion case.

Hamiltonian

$$H = \sum_{\mathbf{k}} \left(\frac{h^2 k^2}{2m} - \mu \right) +$$

$$\frac{g}{2V} \sum_{\{\mathbf{k}\}} a^{\dagger}_{\mathbf{k}_1} a^{\dagger}_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_4}$$

${a_{\mathbf{k}}}^{\dagger}, a_{\mathbf{k}}$: the creation and annihilation operators at momentum $\mathbf{k}$.

This Hamiltonian has to be viewed as an effective one.

Due to the interaction, even at zero temperature the condensate will be partly depleted, but we will assume that the condensate still contains a macroscopically large number of particles N_0

$$\begin{aligned} a_0^\dagger|N_0> &= |N_0+1>(N_0+1)^{\frac{1}{2}} \\ a_0|N_0> &= |N_0-1>(N_0)^{\frac{1}{2}} \end{aligned}$$

Then $a_0^\dagger$ and a_0 can be substituted by their (macroscopic) average value $\sqrt{N_0}$ and keep only the terms with the largest powers of $a_0^\dagger$ or a_0 in the interaction term of the Hamiltonian.

$$\begin{aligned} H_{int} &= \tfrac{g}{2V}[\ N_0^2 + 2N_0 \textstyle\sum_{\mathbf{k}\neq 0}(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}} a_{-\mathbf{k}}) \\ &\quad + N_0 \textstyle\sum_{\mathbf{k}\neq 0}(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}})\ ] \end{aligned}$$

Inconvenience : this interaction does not conserve exactly the number of particles. The chemical potential μ can be used as a Lagrange multiplier to conserve the average number of particles.

$$\begin{aligned} a_{\mathbf{k}} &= u_{\mathbf{k}} \alpha_{\mathbf{k}} - v_{\mathbf{k}} \alpha^\dagger_{-\mathbf{k}} \\ a^\dagger_{\mathbf{k}} &= u_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} - v_{\mathbf{k}} \alpha_{-\mathbf{k}} \end{aligned}$$

Canonical transformation : $\alpha, \alpha^\dagger$ boson operators

$$u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$$

The Hamiltonian becomes, with number density $n = N_0/V$

$$H = \frac{1}{2}Vgn^2 + \sum{}'[(\epsilon_{\mathbf{k}} + ng)v_{\mathbf{k}}^2 - ngu_{\mathbf{k}}v_{\mathbf{k}}]$$

$$+\frac{1}{2}\sum{}'[(\epsilon_{\mathbf{k}} + ng)(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) - 2ngu_{\mathbf{k}}v_{\mathbf{k}}](\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + \alpha_{-\mathbf{k}}^\dagger\alpha_{-\mathbf{k}})$$

$$+\frac{1}{2}\sum{}'[ng(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) - 2u_{\mathbf{k}}v_{\mathbf{k}}(\epsilon_{\mathbf{k}} + ng)](\alpha_{\mathbf{k}}^\dagger\alpha_{-\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}\alpha_{-\mathbf{k}})$$

The diagonalization is obtained by the vanishing of the last term. One finds

$$u_{\mathbf{k}}^2 = \frac{1}{2}[\frac{\tilde{\epsilon}_{\mathbf{k}}}{E_{\mathbf{k}}} + 1] \quad ; \quad v_{\mathbf{k}}^2 = \frac{1}{2}[\frac{\tilde{\epsilon}_{\mathbf{k}}}{E_{\mathbf{k}}} - 1]$$

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} + ng)^2 - (ng)^2} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2ng)}$$

With these parameters u and v, the Hamiltonian reads

$$H = E_0 + \frac{1}{2}\sum_{\mathbf{k}\neq 0} E_{\mathbf{k}}(\alpha_{\mathbf{k}}{}^{\dagger}\alpha_{\mathbf{k}} + \alpha^{\dagger}_{-\mathbf{k}}\alpha_{-\mathbf{k}})$$

The operators $\alpha^{\dagger}_{\mathbf{k}}\alpha_{\mathbf{k}}$ have eigenvalues 0, 1, 2

....

- E_0 : ground state energy
- $E_{\mathbf{k}}$: excitation energies of momentum $\mathbf{k}$
- $\alpha_{\mathbf{k}}|\Psi_0> = 0$: "α - vacuum"
- $\alpha^{\dagger}_{\mathbf{k}}|\Psi_0>$: excited state at energy $E_{\mathbf{k}}$

These excited are the quantized phonons. All the excitations are sets of independent phonons.

The energy $E_{\mathbf{k}}$ is linear in the momentum $\mathbf{k}$, in agreement with the macroscopic treatment. This is the "Goldstone mode", as expected in broken symmetry phase transitions. The order parameter is the condensate a_0, and the broken symmetry is the gauge invariance of the Hamiltonian

$$a_{\mathbf{k}} \; \to \; a_{\mathbf{k}} \, e^{i\alpha}$$

The order parameter and the ground state are not gauge invariant. The Goldstone mode corresponds to a slow modulation in space and time of the gauge phase.

Since there is no restoring force, at $\mathbf{k} = 0$ the energy must vanish. This is the physical basis of the general Goldstone theorem. The microscopic calculation satisfies the theorem, but it gives something more

- The velocity of the Goldstone mode is explicitly related to the particle-particle interaction.
- The system is stable only if the interaction is repulsive.

Sum rules method shows that the energy of multi phonons states is proportional to higher powers of the momentum.

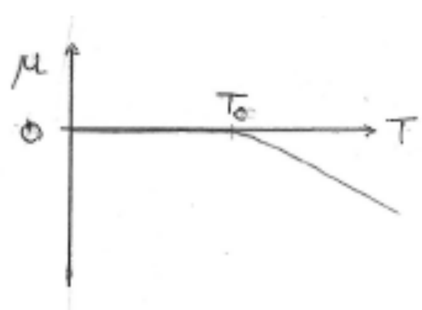
Finite temperature

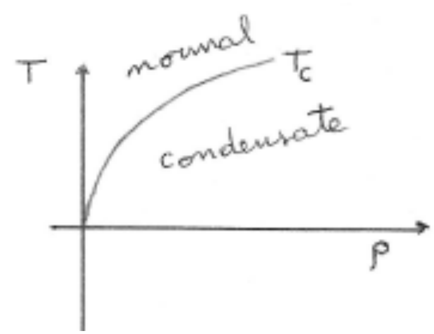
Boson condensation occurs at a given critical temperature T_C, which is a function of ρ. For a free gas at T_C the chemical potential vanishes

$$\rho = \frac{1}{2\pi^2} \int \frac{k^2\, dk}{\exp(\epsilon(k)/T_C) - 1}$$

and then

$$T_C = C\, \rho^{2/3}$$





· Gas of thermally excited quasi-particles : normal component

· Reduction of the condensate density

· Two-fluid model of superfluidity at finite temperature

· Appearance of *viscosity*, due to the normal component

· The flow of the supefluid component is still frictionless.

Fermion superfluidity and Nuclear Matter

Historically superconductivity was the first evidence of frictionless flow. It was in fact discovered in 1911 by H.K. Onnes, who found that some metals at temperature lower than few degrees Kelvin were showing zero electrical resistivity, hence the name. Up to present times it was possible to put only upper limit to the value of resistivity. For instance in some experiment an electrical current was circulating freely (i.e. no voltage) in a metallic ring for a couple of year without showing any loss of intensity.

No macroscopic occupation number is possible for fermions. How superfluidity is possible ?

· A satisfactory theory of superconductivity was found only in the late fifties, the Bardeen-Cooper-Schiffrer (BCS) theory

· Since then it was expected that liquid ^{3}He, which is a fermion system due to the nuclear spin (!), could also display superfluidity

· Only in the seventies it was indeed found that ^{3}He becomes superfluid below about 10^{-3} K .

The standard BCS theory

If the particle-particle effective interaction at the Fermi surface is attractive, the system is unstable with respect to a rearrangement of the Fermi surface with the formation of correlated pairs of particles, irrespective of the strength of the interaction

Roughly speaking the particles form bound pairs around the Fermi surface. Consider the scattering of two particles

$$T(k',k;\omega) = v(k',k)$$

$$+ \frac{1}{2\pi^2} \int_{k_F}^{k_c} k''^2 dk'' v(k',k'') \frac{1}{\omega - k''^2} T(k'',k;\omega)$$

- The total momentum of the two particles is zero (scattering around the Fermi surface). Then the relative momenta k, k' are the modulus of the (opposite) momenta of the two interacting particles.

- $k_F < k'' < k_c$

Assuming a smooth interaction, one can take $k'' = k_F$ in $v(k', k'')$. One gets

$$T(k_F, k_F; \omega) = \frac{v_F}{1 + \frac{k_F v_F}{4\pi^2} \log \left|\frac{E_c - \omega}{E_F - \omega}\right|}$$

If the interaction $v_F < 0 \rightarrow$ singularity in the vicinity of E_F , because of the divergence of the logarithm term.

- In vacuum a singularity at negative energy corresponds to a bound state
- In the medium a singularity below $2E_F$ indicates instability towards the formation of bound pairs
- The result is exact provided the effective interaction v_F is "exact"
- The "condensation" is not in a single quantum state but in a single quantum number : the pair total momentum $\mathbf{P} = 0$

The s-wave pairing in neutron matter must be in the $^1S_0$ channel

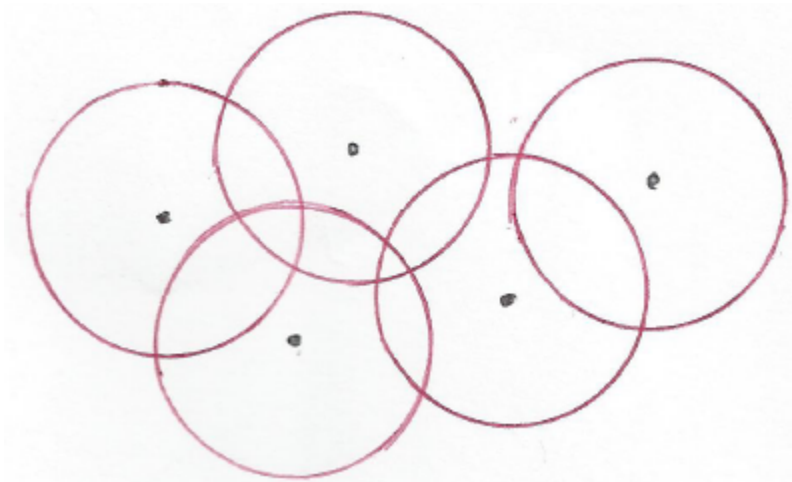
The singularity can be still present also for $\mathbf{P} \neq 0$ and the "condensation" would correspond to a flow at velocity $\mathbf{v} = \mathbf{P}/2m$. The ground state is for zero momentum.

· The correspondence with a boson gas becomes self-evident when the pair binding is large enough to form "molecules" so tightly bound that their internal structure is irrelevant and form boson particles.

· If the interaction is not so strong, the Cooper pairs overlap and the Pauli principle could destroy the pair

$$R \approx \hbar/\sqrt{mB} >> d \approx \left(\frac{3}{4\pi\rho}\right)^{\frac{1}{3}} = \left(\frac{9}{8\pi}\right)^{\frac{1}{3}} \frac{1}{k_F}$$

However, particles are indistinguishable



"pairing field" !

The binding of the pairs in the medium can be actually even larger than in vacuum. Neutron Matter is expected to be superfluid with a binding of 1-2 MeV, but two neutrons do not form a bound state ! They form only a "virtual state" at about -60 KeV.

BCS theory

The model hamiltonian

$$H = \sum_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}} - \mu)a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}$$

$$+ \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}a^{\dagger}_{\mathbf{k}\sigma}a^{\dagger}_{-\mathbf{k}-\sigma}a_{-\mathbf{k}'-\sigma}a_{\mathbf{k}'\sigma}$$

It describes the interaction between pairs of particles at $\mathbf{P} = 0$. The pairing field is just the average of the pair operators

$$a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} \quad \rightarrow \quad < a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} >$$

• These averages are not macroscopic quantities (as in the boson case), they describe the average field acting on each pair.

We split now the pair operators in the averages and fluctuations

$$\begin{aligned} a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} &= < a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} > \\ &+ (a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} - < a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} >) \end{aligned}$$

and neglect the terms quadratic in the fluctuation. Approximate hamiltonian

$$\begin{aligned} H_{BCS} &= \sum_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}} - \mu)a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} \\ &+ \sum_{\mathbf{k}}\left(\Delta_{\mathbf{k}}a^\dagger_{\mathbf{k}\sigma}a^\dagger_{-\mathbf{k}-\sigma} + \Delta^*_{\mathbf{k}}a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma}\right) \\ &- \Delta^*_{\mathbf{k}} < a_{-\mathbf{k}-\sigma}a_{\mathbf{k}\sigma} > \end{aligned}$$

where

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} < a_{-\mathbf{k}'-\sigma}a_{\mathbf{k}'\sigma} >$$

is the average interaction of a pair with all the others.

These averages are "anomalous", they do not conserve the particle number and the ground state has not a definite number of particles

- The mean field hamiltonian includes the process of addition and removal of a pair of particles around the Fermi surface in order to form Cooper pairs

- The chemical potential μ can be used as a Lagrange multiplier to fix the average number of particles

- For N $\to \infty$, the relative fluctuations vanish

Canonical transformation

$$\begin{aligned} a_{\mathbf{k}\sigma} &= u_{\mathbf{k}}^{*}\alpha_{\mathbf{k}} + v_{\mathbf{k}}\beta_{-\mathbf{k}}^{\dagger} \\ a_{-\mathbf{k}-\sigma}^{\dagger} &= -v_{\mathbf{k}}^{*}\alpha_{\mathbf{k}} + u_{\mathbf{k}}\beta_{-\mathbf{k}}^{\dagger} \end{aligned}$$

$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \rightarrow$ the new operators α and β satisfy the fermion commutation relations.

One finds

$$\begin{aligned} |u_{\mathbf{k}}|^2 &= \frac{1}{2}\left(1 + \frac{\epsilon_{\mathbf{k}}-\mu}{E_{\mathbf{k}}}\right) \\ |v_{\mathbf{k}}|^2 &= \frac{1}{2}\left(1 - \frac{\epsilon_{\mathbf{k}}-\mu}{E_{\mathbf{k}}}\right) \end{aligned}$$

and the Hamiltonian in the diagonal form reads

$$H_{BCS} = E_0 + \sum_{\mathbf{k}} E_{\mathbf{k}}\left(\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}\right)$$

The key quantity $E_{\mathbf{k}}$ is given by

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta_{\mathbf{k}}^2}$$

$\alpha^\dagger\alpha$ and $\beta^\dagger\beta$ are positive definite. The ground state will be the state $|\Psi_0>$ such that

$$\alpha_{\mathbf{k}}|\Psi_0> = \beta_{\mathbf{k}}|\Psi_0> = 0$$

Then E_0 is just the ground state energy and the excited states, with energy $E_{\mathbf{k}}$, are

$$\alpha_{\mathbf{k}}^\dagger|\Psi_0> \quad ; \quad \beta_{\mathbf{k}}^\dagger|\Psi_0>$$

The minimal excitation energy is Δ_{k_F}, the energy spectrum has a gap. This is half the energy for breaking a pair.

The gap $\Delta_{\mathbf{k}}$ can be determined by imposing self-consistency with the original definition

$$\begin{aligned}\Delta_{\mathbf{k}} &= \\ \sum_{\mathbf{k}'} &V_{\mathbf{k}\mathbf{k}'} < (u_{\mathbf{k}'}\alpha_{\mathbf{k}'} + v_{\mathbf{k}}\beta^{\dagger}_{-\mathbf{k}'})(-v_{\mathbf{k}'}\alpha^{\dagger}_{\mathbf{k}'} + u_{\mathbf{k}'}\beta_{-\mathbf{k}'}) > \\ &= -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'}\end{aligned}$$

From the explicit expressions for u and v, one gets

$$\Delta_{\mathbf{k}} = -\frac{1}{2}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}$$

This is the " gap equation ". Due to the expression for $E_{\mathbf{k}}$, it is non-linear and in general must be solved numerically.

If the pairing gap is small with respect to the Fermi energy ("weak coupling limit"), the integrand is sharply peaked at the Fermi energy $E_F = \mu$, and one can take the gap function and the interaction at k_F, and a constant density of state. Then the integral can be done analytically

$$\Delta_0 = 2E_c \, \mathrm{exp}\left(\frac{1}{\lambda}\right)$$

$$\lambda = V_{k_F k_F}(mk_F/2\pi^2) \leftarrow density\ of\ states$$

where the integration has been restricted to a strip around the Fermi energy of width $2E_c$.

- Sensitivity to the interaction strength is exponential
- Non-analytical dependence on the interaction : phase transition, instability towards the Cooper pair formation

The average $\phi(\mathbf{k}) =< a_{-\mathbf{k}-\sigma} a_{\mathbf{k}\sigma} >$ can be clearly interpreted as the wave function of the Cooper pair. It can be written

$\phi(\mathbf{k}) = -u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}$

The wave function is sharply peaked at k_F, which shows that the pairs are formed around the Fermi surface.

- Fourier transform $\rightarrow$ pair wave function $\phi(\mathbf{r})$
- $\phi(\mathbf{r})$ is expected to display a long tail
- The size of $\phi(\mathbf{r})$ is the "coherence length" ξ

One can estimate ξ from the width δ_k in momentum of $\phi(\mathbf{k})$, that is $\delta_k \approx m\Delta_F/\hbar^2 k_F$

$$\begin{aligned} \xi &\approx \frac{\hbar^2 k_F}{m\Delta_F} \\ \frac{\xi}{d} &\approx E_F/\Delta_F \end{aligned}$$

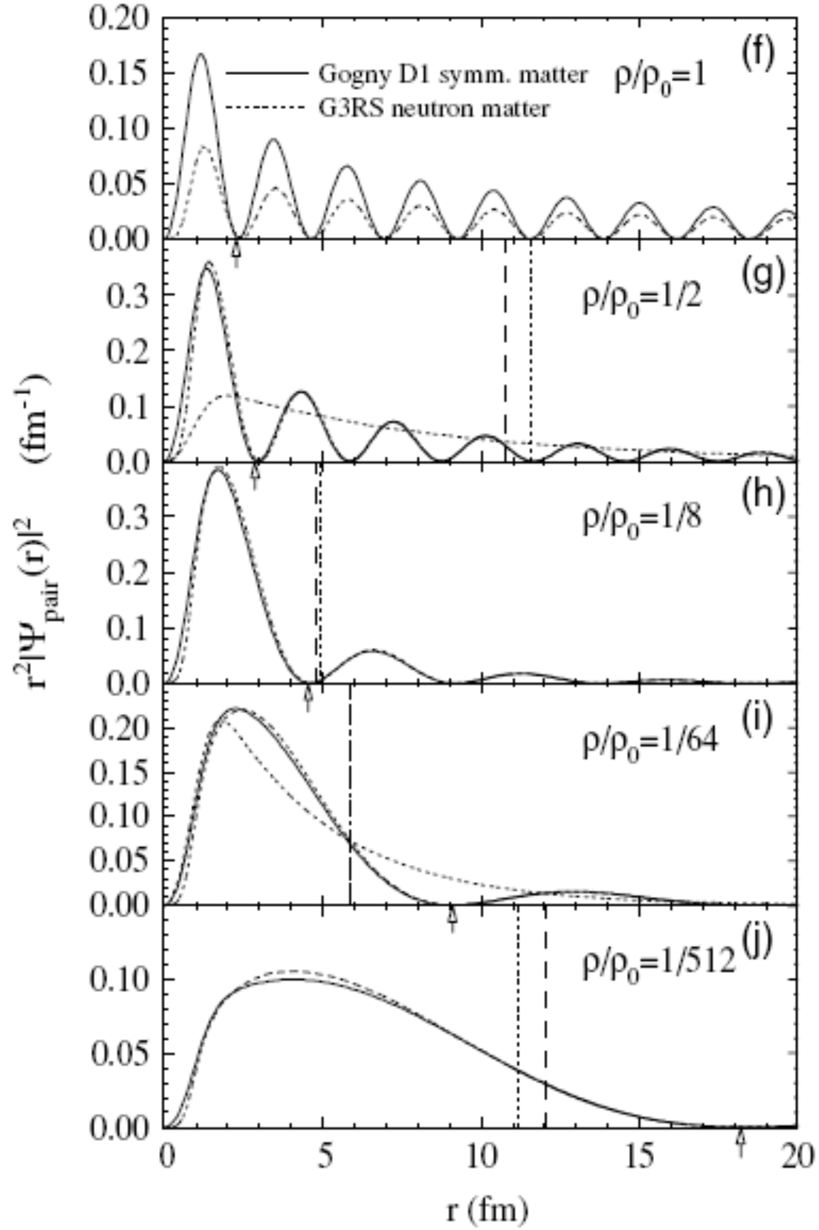
where $d \approx 1/k_F$ is the average distance between particles. The size of the Cooper pairs is larger than d whenever the gap is smaller than the Fermi energy.

The opposite limit when the pairing gap exceeds the Fermi energy should corresponds to the boson limit (BCE). However that expression is not any more valid, since then the chemical potential is drastically different from the free gas Fermi energy E_F. One can show that indeed the gap equation in this limit merges in the Schrödinger equation, where the chemical potential is negative and plays the role of the binding energy (per particle). Then two particles bound state (boson) is formed, if the potential admits it.

Calculations

in Neutron Matter

(Matsuo 2006)



Energetics

If a solution of the gap equation does exist, then the superfluid phase is lower in energy than the normal one. This is a consequence of the variational nature of the BCS solution.

Trial wave function :

$$|\Psi(\Delta) >= \Pi_{\mathbf{k}} \left[u_{\mathbf{k}} + v_{\mathbf{k}} a^{\dagger}_{\mathbf{k}\sigma} a^{\dagger}_{-\mathbf{k}-\sigma} \right] |\phi_0 >$$

ϕ_0 : vacuum state. The energy minimization on u, v produces the BCS solution. Since this set of states includes the normal one, whenever $\Delta \neq 0$ the energy of the superfluid will be lower than the normal one.

Weak coupling regime.

$$E_0 = 2\sum_{-\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu)v_{\mathbf{k}}^2 + \sum_{\mathbf{k}\mathbf{k}'}\frac{1}{4}V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}\frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}$$

The gain in energy due to the pairing interaction overcomes the increase in kinetic energy and finally the difference in energy/particle is

$$e_0^{super} - e_0^{normal} = -\frac{3}{16}\frac{\Delta_F^2}{E_F}$$

The gain in energy is much smaller than the Cooper pair binding energy, because only a fraction $\Delta_F/E_F << 1$ of the particles participates to the pairing.

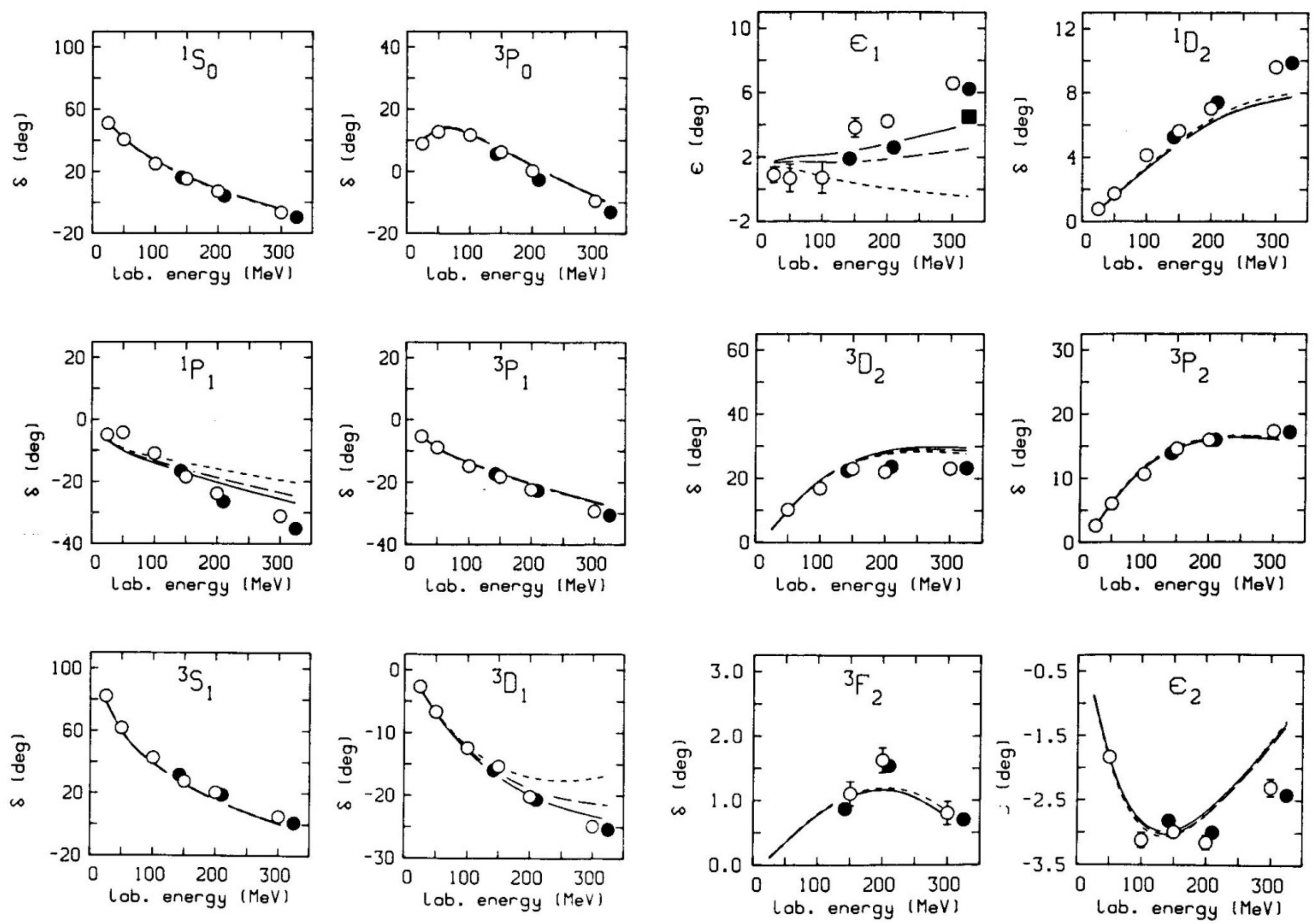
Neutron matter

The bare nucleon-nucleon (NN) interaction is complex. It is in general convenient to project the interaction in a given $(lSJT)$ channel

- l relative angular momentum
- S total spin
- J total angular momentum
- T total isospin

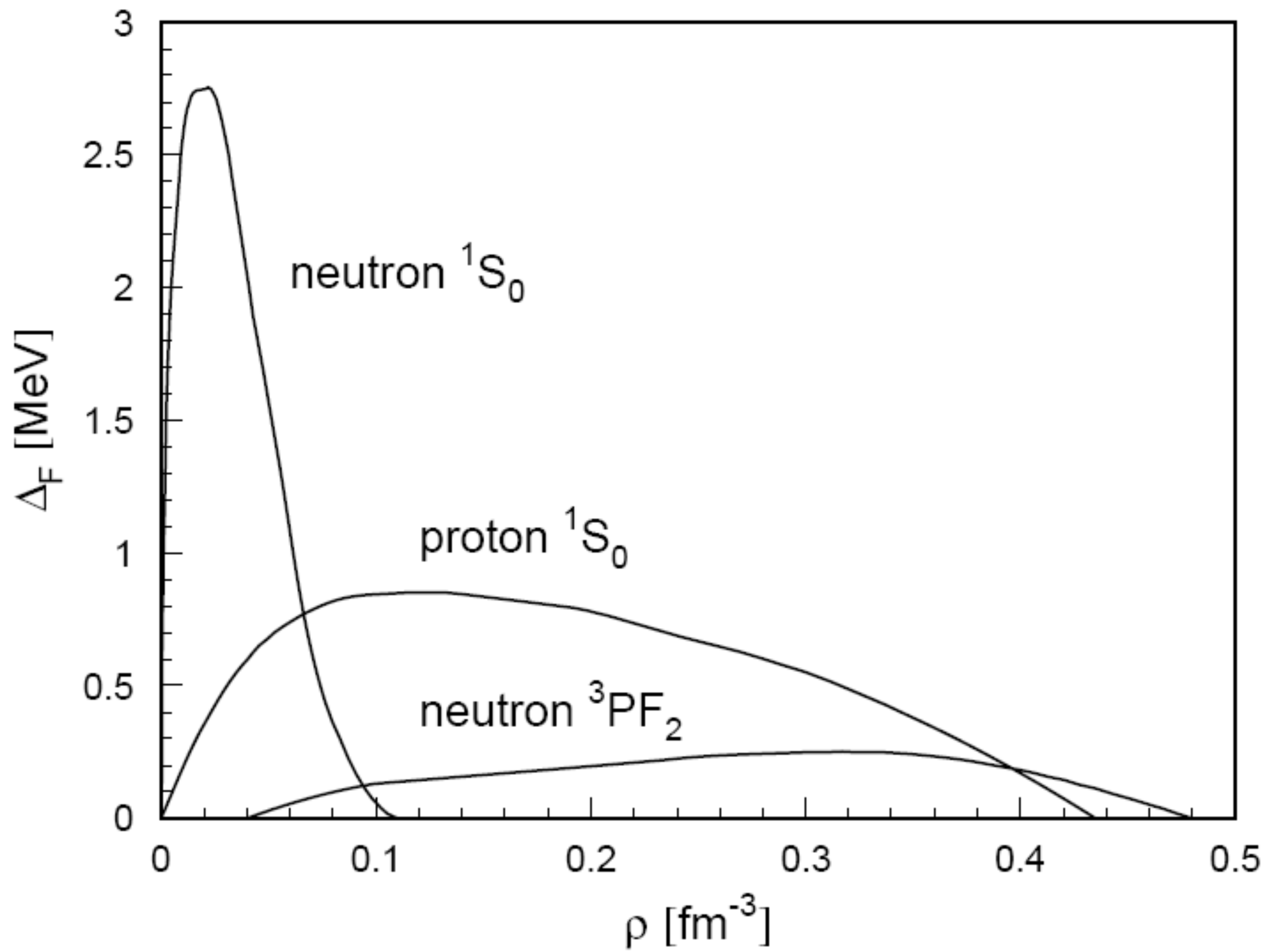
These are conserved quantum numbers in free space, but in the pairing process the channels can be coupled.

The neutron matter in NS can be expected to be superfluid on the basis of the experimental phase shifts



Taking the bare interaction as the effective pairing interaction, one can look for which channels the interaction is attractive at the relevant momentum. The Fermi momenta in NS ranges from vanishing small values to $2.5 - 3.0$ fm^{-1}. At low momenta the only attractive channel is the $^1S_0$ one. This suggests that s-wave pairing can be present at the low density typical of the inner crust and of the outer core. In the inner core the main possible pairing channel is the $^3P_2 -^3 F_2$. For the proton component the $^1S_0$ channel can be active.

Solving the gap equation with the bare NN interaction (Argonne v_{18})



The pairing gap is not much smaller than the Fermi energy, and we are neither in the weak coupling limit nor in BEC limit, which is the difficult regime !

For pairing in the $^3P_2 - ^3F_2$, the anomalous mean field κ is not rotationally invariant. The coupling scheme is more complex

$$\kappa(k)^M = < \left(a_{\mathbf{k}} a_{-\mathbf{k}}\right)^{lSJM} >=$$
$$\int d\Omega_{\hat{\mathbf{k}}} Y_{lm}(\Omega_{\hat{\mathbf{k}}})^* \sum_{\sigma\sigma' S_z m} < a_{\mathbf{k}\sigma} a_{-\mathbf{k}\sigma'} >$$
$$C(\tfrac{1}{2}\sigma\tfrac{1}{2}\sigma'|SS_z)C(SS_z lm|JM)$$

where $S = 1$, $l = 1$, $J = 2$, and $|M| \leq 2$. Then gap is a combination of M projections, and the ground state is the one which minimizes the energy.

Gapless superfluidity

The gap depends on the direction of $\mathbf{k}$, so the ground state will be in general anisotropic. It is not well established which M combination is the lowest in energy. Taking an average over direction, the different M are decoupled, as in the results of the figure. In non s-wave channels in general the pairing gap has nodes in some directions. There is not a fully forbidden energy region above the ground state energy, namely the excitation energy can vanishes in some direction. This case is called "gapless superfluidity".

The excitation spectrum

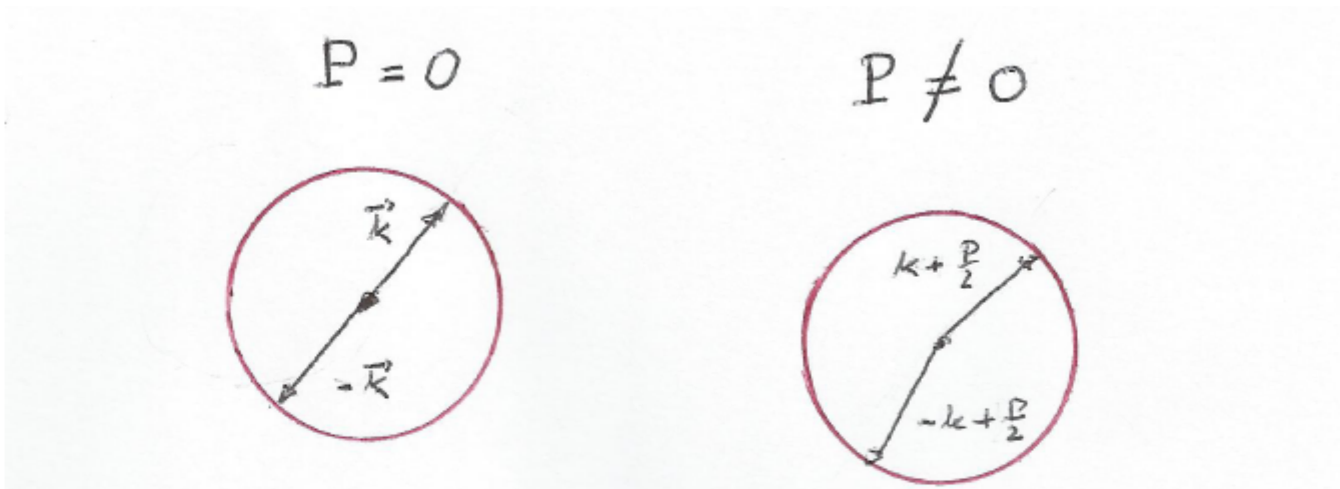
In the $^1S_0$ superfluid the breaking of a pair requires an energy 2Δ. This excitation corresponds to a two quasi-particles excitation.

$$(H_{BCS} - E_0)\, \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'} |\Psi_0^{BCS}>$$

$$= \; (E_{\mathbf{k}} + E_{\mathbf{k}'})\, \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'} |\Psi_0^{BCS}>$$

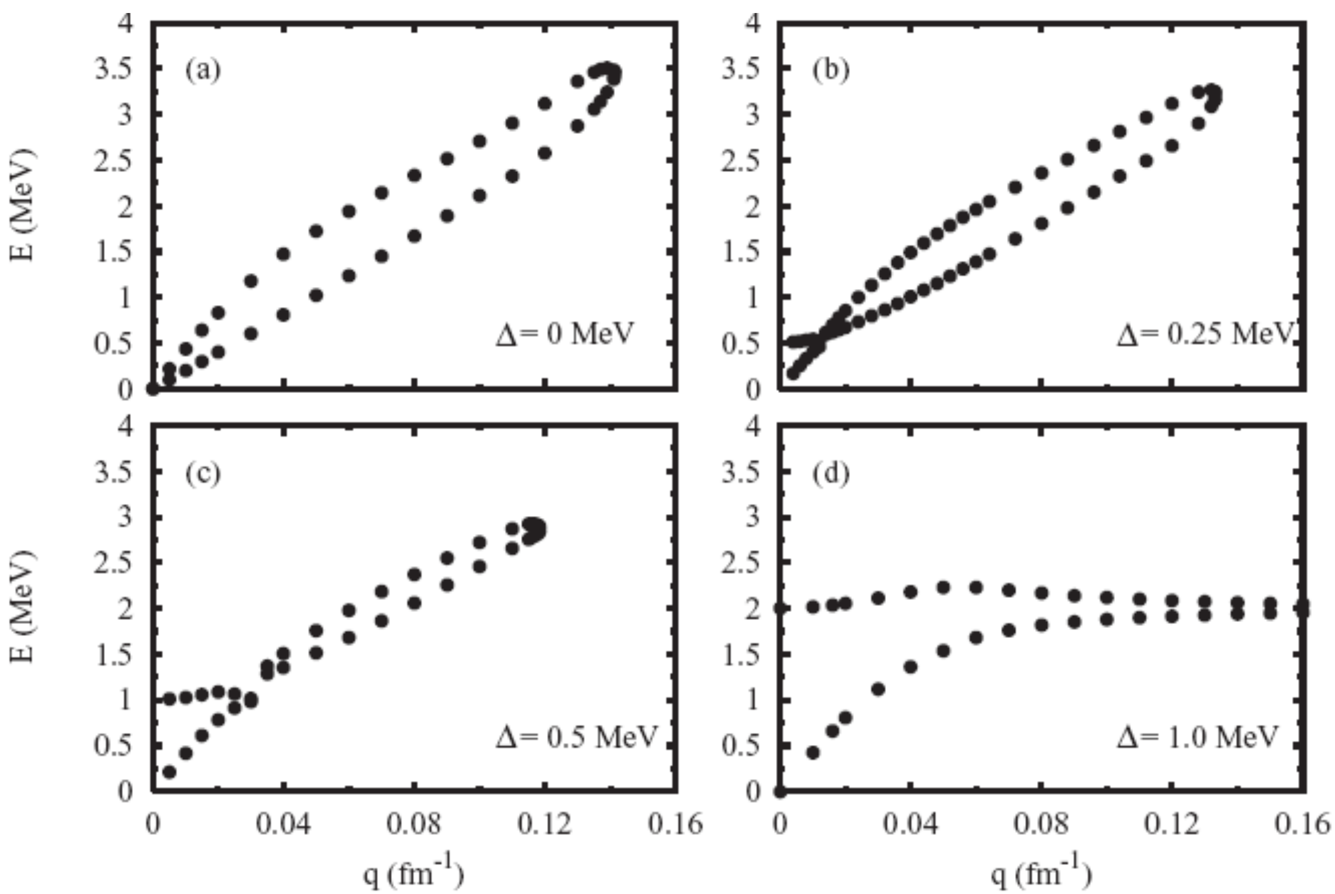
For $|\mathbf{k}| = |\mathbf{k}'| = k_F$, one has $E_{\mathbf{k}} + E_{\mathbf{k}'} = 2\Delta_F$, the minimal excitation energy. One quasi-particle excitation is the addition of an unpaired particle to the BCS ground state.

Besides this breaking pair excitation, we must expect also the Goldstone mode. Gauge invariance is broken below the transition point. The order parameters is just the pairing field $\kappa(\mathbf{k}) = < a_{\mathbf{k}} a_{-\mathbf{k}} >$, which is non-zero in the superfluid phase. Physically speaking the two modes corresponds to the fluctuations of the pairing gap around its ground state value, one for the modulus and the other for the phase. The superfluid flow associated to the Goldstone mode are produced by pairs of particles with momenta $\mathbf{k} + \mathbf{q}/2$ and $-\mathbf{k} + \mathbf{q}/2$



Different relative momentum $\mathbf{k}$ of the pairs can contribute and they can be coupled together by the interaction. The relation between energy and momenta is dependent on the interaction and a microscopic approach is necessary to find the mode velocity.

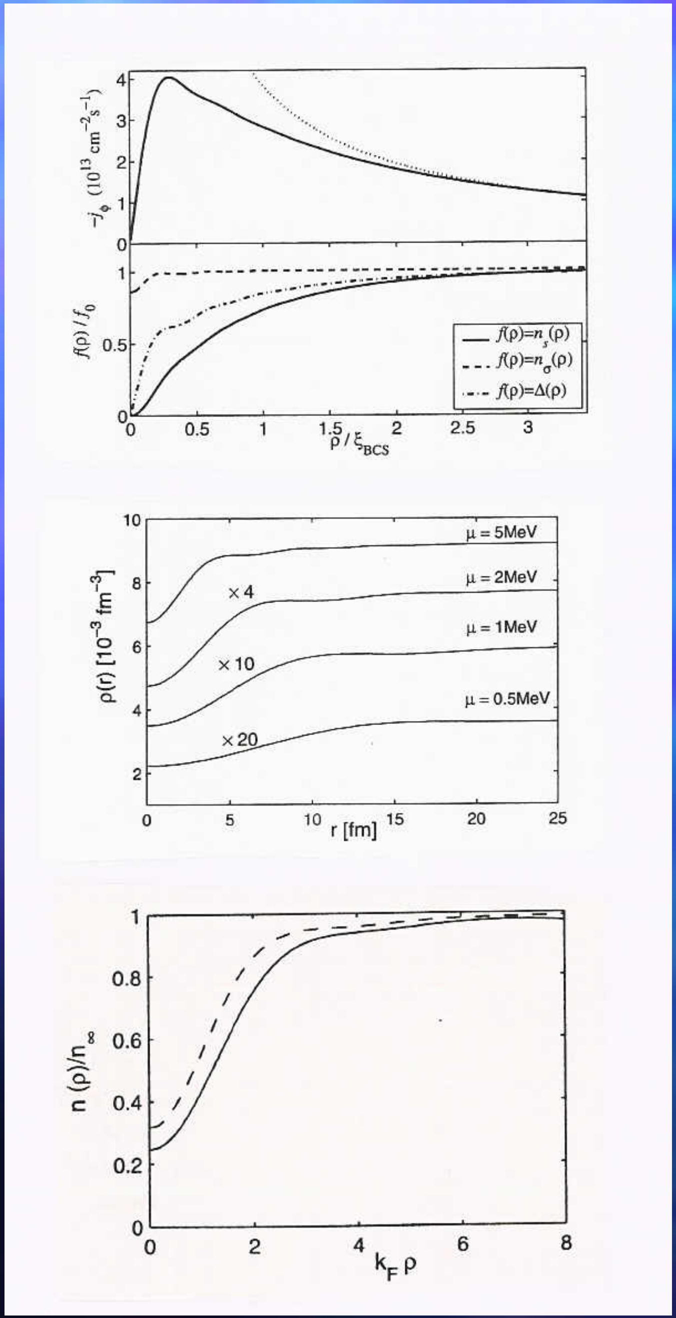
The two modes are actually coupled, since phase and density are coupled. The pair-breaking mode above $2\Delta_F$ and the Goldstone mode can be identified in the microscopic calculations reported in the figure



Besides these elementary excitations, vortices can be present in the superfluid matter, as in the case of ^{4}He. This is particularly relevant in rotating NS, where vortex structures must be present. One has to keep in mind that the condensate wave function is the center of mass wave function of the Cooper pairs. At the center of the vortex the transition to normal matter can be produced by Cooper pairs breaking.

Microscopic calculations, within the BCS scheme, reveal also that the density tends to be reduced in the vortex core.

The structure of a single vortex line



Nygaard et al., PRL 90 (2003)
Bogoliubov-De Gennes equations (BCS)

Yu & Bulgac, PRL 90 (2003)
Energy functional

Bulgac and Yu, PRL 91 (2003)
En. funct. inspired by Monte-Carlo calculations.
Very low density, maybe not relevant for NS (rather for trapped atomic gas)

Finite temperature and the phase transition

The gap equation can be extended to finite temperature T. The modification is only in the occupation numbers. For the $^1S_0$ case

$$\Delta_{\mathbf{k}}(T) = -\tfrac{1}{2}\textstyle\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}(1 - 2f(E_{\mathbf{k}'}/T))\frac{\Delta_{\mathbf{k}'}(T)}{E_{\mathbf{k}'}}$$

$$f(E_{\mathbf{k}}/T) = \frac{1}{\exp(E_{\mathbf{k}}/T)+1} \quad \textit{q.p. occupation number}$$

Notice that, as $T \to 0$

$$1 - 2f(E_{\mathbf{k}}/T) \equiv \tanh(E_{\mathbf{k}}/T) \to 1$$

As the temperature increases the factor $\tanh(E/T)$ decreases. No solution for $T \geq T_c$: phase transition to the normal state. The Cooper pairs are thermally broken, and at T_c the pairing field is too weak to be supported by the interaction. Then T_c must be close to 2Δ. **Here we assume that all pairs have still zero total momentum**. To estimate T_c we put $\Delta = 0$, $E_{\mathbf{k}} = |\epsilon_{\mathbf{k}} - \mu|$ in the gap equation

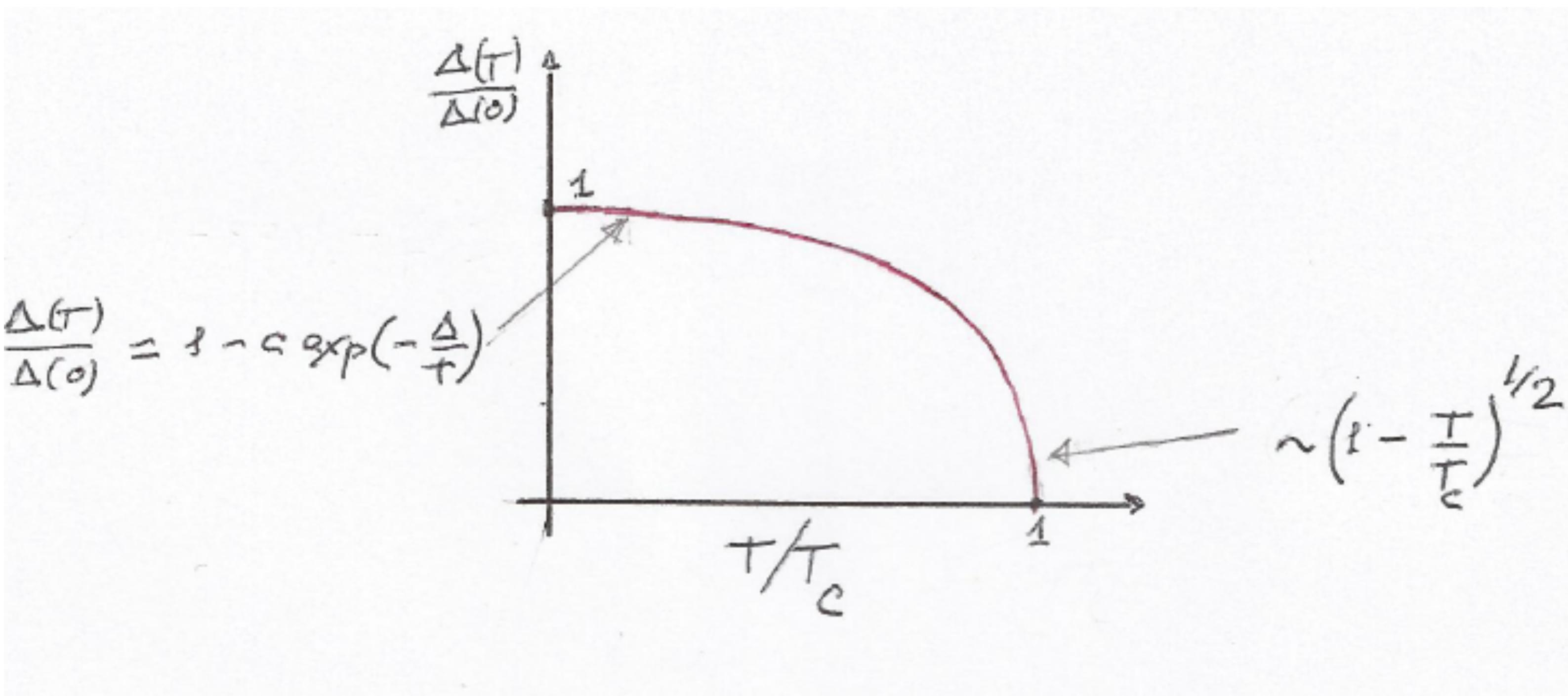
$$1 = -\tfrac{1}{2}\textstyle\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \tanh(E_{\mathbf{k}'}/T_c))/E_{\mathbf{k}'}$$

$$T_c \approx 1.764\,\Delta_F \qquad \textit{weak} \quad \textit{coupling}$$

The full temperature dependence of the gap, can be obtained only numerically. It can be well approximated by

$$\frac{\Delta(T)}{\Delta(0)} = \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}}$$

A more precise trend is reported in the figure



Specific heath capacity

Within BCS theory it is possible to derive the thermodynamic properties of a fermion superfluid, in particular the specific heath capacity, relevant for NS cooling

- Thermal pair breaking : Boltzmann factor $\exp(-\Delta_F/T)$
- Gas of Goldstone quanta : T^3 contribution for $T << \Delta_F$
- For $T \sim T_c$ large contribution of the pair-breaking mode to the neutrino emission, and thus to cooling.

Proton superconductivity

- Protons in NS can be superfluid

- They form a superconductor in the homogeneous matter

First consider NON-superfluid protons.

· Ohmic regime : $\mathbf{j} = \sigma \mathbf{E}$

The conductibility σ is dominated by the electrons, which are not superfluid.

· Maxwell eq. : $\mathbf{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$

In the magneto-hydrodynamical limit the time derivative term is neglected. From the second Maxwell equation

$$
\begin{aligned}
\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} &= -\,\mathbf{rot}\mathbf{E} = -\frac{c}{4\pi\sigma}\mathbf{rot}\,(\mathbf{rot}\,\mathbf{H}) \\
&= -\frac{c}{4\pi\sigma}\left[-\nabla^2\mathbf{H} + \nabla(\nabla\cdot\mathbf{H})\right]
\end{aligned}
$$

Since $\nabla\cdot\mathbf{H} = 0$, the evolution equation of $\mathbf{H}$ becomes

$$
\frac{\partial \mathbf{H}}{\partial t} = \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{H}
$$

- Diffusion equation with $D = c^2/4\pi\sigma$

Proton superconductivity

- $\sigma \to \infty$: completely trapping of the magnetic field ?
- In some sense it is the opposite ! The magnetic field is expelled from a superconductor ! This is Meissner effect, valid for not too high magnetic field.

First of all, Ohm law must be substituted by Newton' s law

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{H}\right)$$

No stationary flow is possible if $\mathbf{E} \neq 0$. Here the derivative is the total one and all quantities are "coarse grained".

Meissner effect comes from the gauge invariance of the electromagnetic field coupled to matter. For a charged particle moving in a static magnetic field the relation between the canonical momentum $\mathbf{p}$ and the velocity $\mathbf{v}$ is

$$m\mathbf{v} = \mathbf{p} - \tfrac{q}{c}\mathbf{A},$$

$\mathbf{A}$: vector potential, q : charge. The hamiltonian (kinetic energy) is

$$H_B = \tfrac{1}{2m}\left(\mathbf{p} - \tfrac{q}{c}\mathbf{A}\right)^2 \quad q.m. \ : \ \mathbf{p} = -i\hbar\nabla$$

For a Cooper pair $m = 2M$ and $q = 2e$.

This form of the hamiltonian is gauge invariant. At quantum mechanical level, if the wave function is multiplied by a phase and to the vector potential is added a corresponding gradient term

$$\psi \;\rightarrow\; e^{i\phi}\,\psi \qquad ; \qquad \mathbf{A} \;\rightarrow\; \mathbf{A} + \frac{\hbar c}{q}\nabla\phi$$

the Schrödinger equation for H_B remains unchanged. The phase contribution coming form the momentum operator is cancelled by the additional term of the vector potential. This invariance implies the local conservation of current.

In a neutral superfluid the current is given by

$$\mathbf{j} = \frac{\hbar}{m} n_s \nabla S \quad ; \quad S = phase\ of\ the\ condensate$$

where n_s is the superfluid density, and the flow is irrotational. For a superconductor under a magnetic field this expression must be modified. This expression for the current is not gauge invariant as it should be. As suggested by the form of the hamiltonian, we add a term proportional to the vector potential, and the electric current becomes

$$\mathbf{j} = \frac{e\hbar}{m} n_s \left(\nabla S - \frac{2e}{\hbar c} \mathbf{A} \right)$$

The flow is not any more irrotational

$$\mathbf{rot}\ \mathbf{j} = -\frac{2e^2 n_s}{mc}\,\mathbf{H}$$

$\mathbf{H}$ is the magnetic field. This is the second constituent equation of the theory of superconductivity, as first developed by London. It shows that the rotational part of the flow is produced by the magnetic field (+ possible vortices). What is irrotational is the flow of the momentum $\mathbf{p}$

$$\mathbf{rot}\ \mathbf{p} = \frac{m}{n_s e}\,\mathbf{rot}\left(\mathbf{j} + \frac{2e^2 n_s}{mc}\mathbf{A}\right) = 0$$

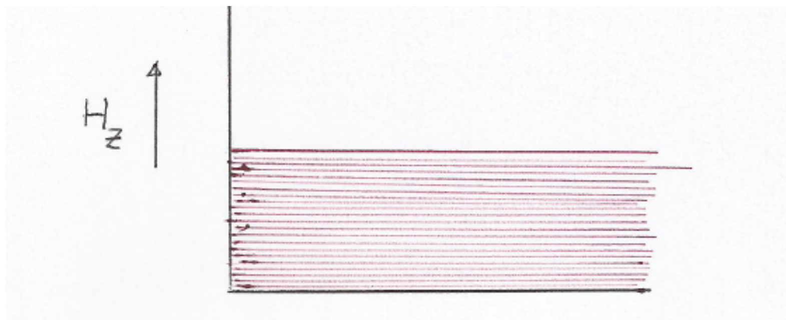
Meissner effect : in some superconductors an applied static magnetic field is expelled from the interior, provided the applied intensity is below a critical value, above which superconductivity is destroyed. This effect can be explained from the London' s equation. Taking the curl of the second Maxwell equation, one gets

$$\mathbf{rot}\,(\mathbf{rot}\,\mathbf{H}) = -\nabla^2\mathbf{H} = -\frac{8\pi e^2 n_s}{mc^2}\,\mathbf{H}$$

This elliptic differential equation fixes the magnetic field configuration.

One dimension : $H_z = H_z(x)$, $H_z(x) = H_z^o$ if $x < 0$

$$\frac{d^2 H_z}{dx^2} = \frac{8\pi e^2 n_s}{mc^2} H_z$$



Inside the sample (assuming a constant density) the field H_z decays exponentially

$$H_z = H_z^o \exp(-x/\lambda)$$

$$\lambda = \left(\frac{mc^2}{8\pi e^2 n_s}\right)^{\frac{1}{2}}$$

λ is the *London penetration depth* : the magnetic field cannot penetrate the superconducting sample more than λ.

The origin of the effect is on the possibility of persistent currents. According to the Maxwell equations a current $\mathbf{j}$ must be present inside the sample along the y direction

$$j_y = \frac{c}{4\pi}\frac{\partial H_z}{\partial x} = -\frac{c}{4\pi\lambda} H_z^o \exp(-\frac{x}{\lambda})$$

This current produces a magnetic field that cancels the applied magnetic field inside the sample. The superconductor is a perfect diamagnetic material.

Above a critical magnetic field superconductivity disappears. This can be seen from energy balance considerations.

- Condensation energy/particle :

$$e_{cond} = -\frac{3}{16}\frac{\Delta_F^2}{E_F}$$

- Magnetic energy density : $E_H = H^2/8\pi$

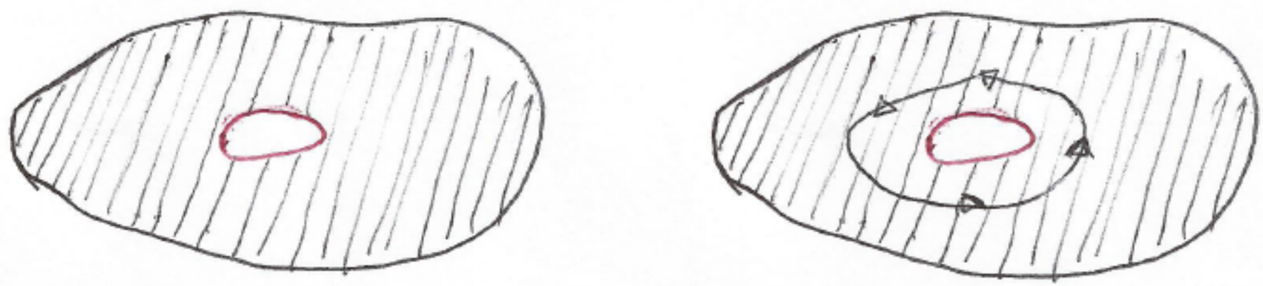
At small H, the superconducting phase is favoured, the magnetic field is excluded. As the magnetic field increases, the energy cost for the exclusion increases, up to the value H_c where normal phase becomes favoured.

The value of H_c is obtained by equating the two energies

$$H_c = \left(\frac{mk_F\Delta^2}{\pi\hbar^2}\right)^{\frac{1}{2}} = \left(4\pi N(0)\Delta^2\right)^{\frac{1}{2}}$$

where $N(0)$ is the density of state. The critical magnetic field depends on temperature through the pairing gap. The magnetic field exclusion is reversible as the temperature crosses the critical value at a given external field.

According to the shape of the sample, the magnetic field can produce regions of normal phase inside the sample before the suppression of superconductivity, which is usually indicated as "intermediate phase". This is not so relevant for NS, where the bulk part is dominating. However some superconductors allow the formation of flux tubes that passes through the matter, below the critical value, independently of the shape of the sample. Before discussing this relevant issue, it is convenient to consider the case of a superconducting sample that has a "hole".



The circulation of the momentum along a circuit that encloses the hole

$$I = \oint \mathbf{p} \cdot d\mathbf{l}$$

is independent of the circuit because the momentum vector field is irrotational. According to the expression of the canonical momentum, one has

$$I = \frac{m}{2en_s} \oint \mathbf{j} \cdot d\mathbf{l} + \frac{2e}{c}\Phi$$

Φ is the flux of the magnetic field through a surface S impinging on the circuit

$$\Phi = \oint \mathbf{A} \cdot d\mathbf{l} = \int_S \mathbf{H} \cdot d\mathbf{S}$$

The current is concentrated at the (internal) border of the hole. It decays exponentially inside the superconductor and therefore the momentum circulation is given only by the flux of the magnetic field.

By quantization (Bohr-Sommerfeld quantization or the usual "phase argument") the flux of the magnetic field through the hole must be quantized

$$\Phi = \frac{c}{2e} n h$$

Thus the flux going through the hole must be multiple of the "flux quantum" Φ_0

$$\Phi_0 = \frac{hc}{2e} \approx 2.07 \times 10^{-7} \text{ Gauss cm}^2$$

This is the flux that can be trapped in the hole, even in absence of an external applied magnetic field, as produced by a persistent super-current flowing just near the internal edge of the hole.
Let us now consider, instead of the hole, the normal core of a superfluid vortex through which the magnetic field can penetrate.

The flow is not any more irrotational, but the momentum circulation I must be still a multiple of h. The circulation I, following London, is called "**fluxoid**", and the corresponding physical configuration "**flux tube**". Both the current circulation and the flux of magnetic field are not constrained by quantization, in principle they can have arbitrary values. It is only the fluxoid that is quantized.

Type II superconductors

In some superconductors one finds a first critical strength H_{c1}, above which flux tubes start to appear. Their number increases as the magnetic strength increases. At a second critical field H_{c2} the whole system makes a transition to the normal state. These supeconductors are called of type II. The others, where no flux tubes appear, are called of type I.

It turns out that the discrimination between type I and type II superconductors is determined by the ratio κ between the London penetration depth λ and the correlation length ξ, which characterizes the Cooper pair size. If this ratio is larger than 1, then we have a type II superconductor. **Proton matter in NS** is expected to be of type II, and therefore nuclear matter inside a standard NS is expected to contain flux tubes.

In fact, after some manipulations, one finds

$$\kappa = \frac{\lambda}{\xi} = \frac{\pi}{2}\frac{\Delta}{E_F}\left(\frac{3\pi Mc^2}{4e^2k_F}\right)^{\frac{1}{2}} \approx 60 \times \left(\frac{\Delta}{E_F}\right)$$

M : proton mass, $k_F \approx 1$ fm^{-1}. In NS proton matter one expects Δ/E_F of the order $0.1 - 0.2$, and therefore κ should be substantially larger than 1. However these estimates contain large uncertainties, and the possibility of type I superconductivity cannot be definitely excluded.

The quantization condition for a flux tube can be written

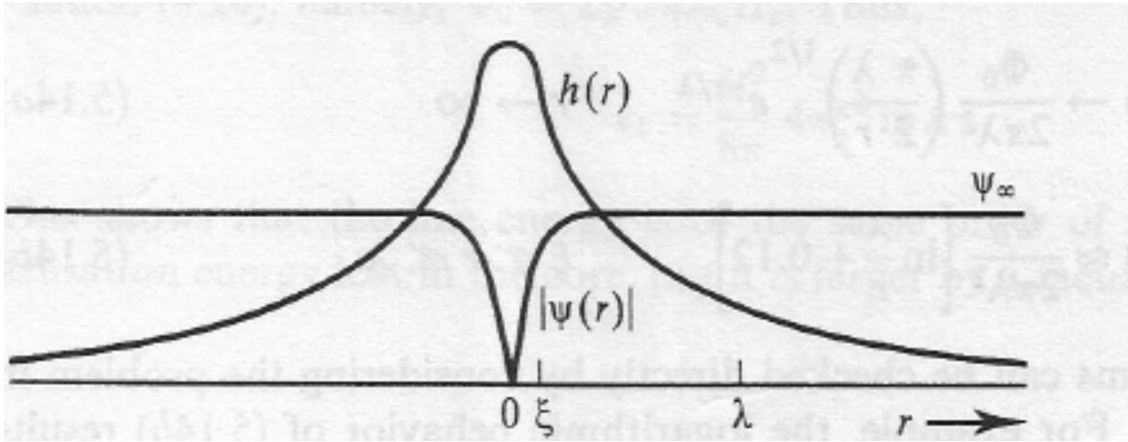
$$I = \oint \mathbf{p} \cdot d\mathbf{l} = m \oint \mathbf{v} \cdot d\mathbf{l} + \frac{2e}{c}\Phi = n\,h$$

the integration is along a close path enclosing the flux tube and Φ is the flux of the magnetic field across the surface enclosed by the path. The circulation of $\mathbf{v}$ is not any more independent of the path, and so is the flux Φ. However their sum is a constant and independent of the path. The lowest energy is obtained by considering the lowest possible kinetic energy.

It can be seen that this implies an exponentially small velocity field $\mathbf{v}$ at large distance. Taking a path at large distance, one can conclude that the flux Φ must be again a multiple of the flux quantum

$$\Phi = n\,\Phi_0$$

The lowest stable configuration will be for $n = 1$. The flux tube is a 'magnetic vortex', i.e. a vortex induced by the magnetic field. Around the flux tube the magnetic field and the superconducting density behave like in the figure.



The magnetic field and the velocity field are maximum at the center of the core and decreases exponentially at large distance. The density recovers from zero to the bulk value in a distance of the order of λ. Each flux tube carries a flux quantum Φ_0.

One can estimate the density of the flux tubes in the NS matter by considering a typical magnetic field strength $H \approx 10^{12}$ Gauss. In a surface of 1cm^2 the number N of flux tubes turns out to be of the order

$$N = \frac{H}{\Phi_0} \approx 10^{19}/\mathrm{cm}^2$$

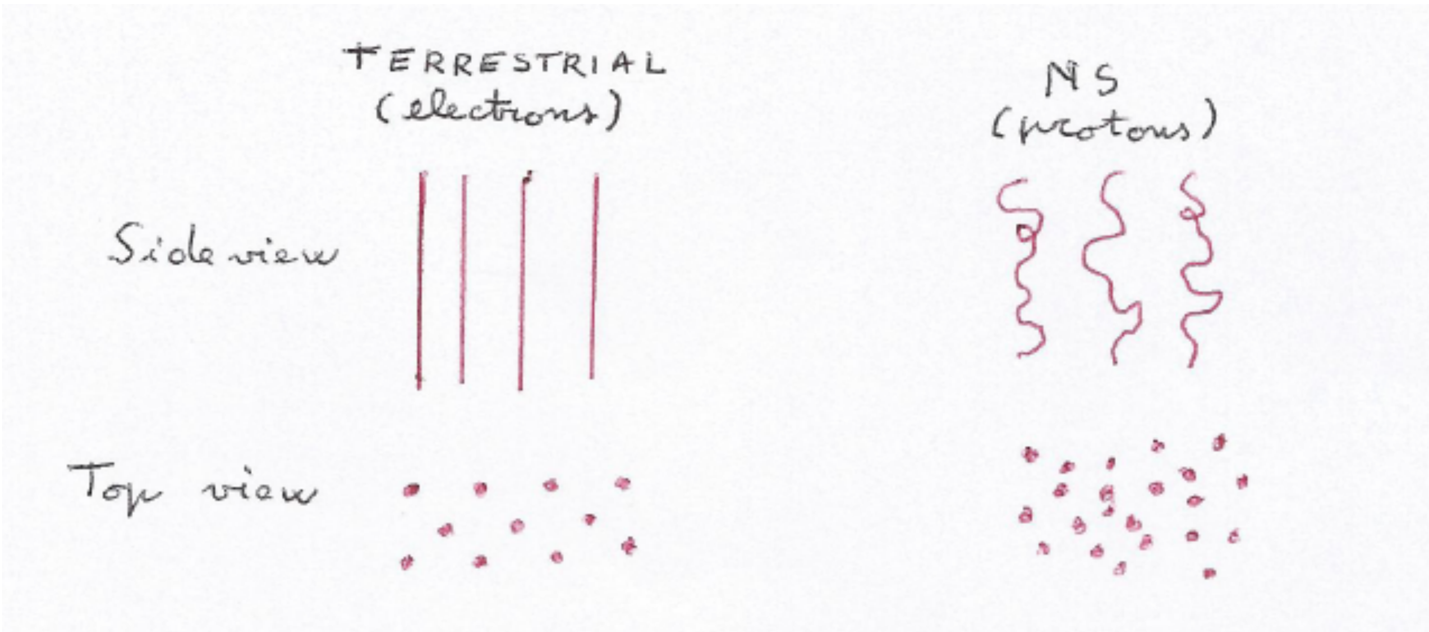
Their average distance at a typical proton density is of the order of 10^6 fm, while the penetration depth is about 30 fm. The flux tubes can be then considered isolated from each other.

This density of flux tubes has to be compared to the expected density of rotational vortices. The quantum of vorticity is $2\pi\hbar/2M$, so that the number of vortices per cm^2 can be estimated as

$$\frac{N_r}{S} \approx \frac{\omega M}{2\pi\hbar} \approx \frac{10^4}{T(\text{sec})} \text{ vortices/cm}^2$$

Even for the fastest pulsars with $T \approx 10^{-3}$sec, this density is overwhelming much smaller than the flux tubes density. The rotational vortices are embedded in a tangle of flux tubes.

The transition to the normal phase will happen when the average distance between flux tubes is of order of λ. This requires a magnetic field strength $H \approx 10^{17}$ Gauss, which is hardly reached even in magnetars. In a terrestrial type II superconductor (electrons) the vortices, at high enough density, form a triangular lattice (due to the repulsion among them). This is not the case of NS proton matter, where they are expected to be "curly" and randomly distributed.



Flux tubes are formed in NS when the temperature decreases below the critical temperature and the critical magnetic filed. However the formation is affected by convective or turbulent motion of the matter.

Effective mass and entrainement

In a Fermi gas, like the neutron and proton matter in NS, the particle interact among each others. One can assume that on each particle is acting a mean field $U(\mathbf{k})$ produced by all the other particles. For a hydrodynamical flow this can be a good starting point. The mean field is in general momentum dependent and therefore the energy of a particle will be

$$\epsilon(\mathbf{k}) = \frac{\hbar^2\mathbf{k}^2}{2m} + U(\mathbf{k})$$

The particle dynamics is around the Fermi surface, so we can expand the energy to first order in $k - k_F$

$$\begin{aligned}\epsilon(\mathbf{k}) &= E_F + \frac{\hbar^2 k_F}{m^*}(k - k_F) \\ \frac{m}{m^*} &= 1 + \frac{m}{\hbar^2 k_F}\frac{dU}{dk}\end{aligned}$$

where the effective mass m^* was introduced. The velocity and corresponding current are

$$\begin{aligned}\mathbf{v} &= \frac{\partial \epsilon}{\partial \mathbf{p}} = \frac{\hbar \mathbf{k}}{m^*} \\ \mathbf{j} &= \rho \mathbf{v}\end{aligned}$$

where $|\mathbf{k}| = k_F$ and ρ is the density

The momentum dependence of U is due to the momentum dependence of the effective quasi-particles interaction at the Fermi surface, the basis of Landau theory of Fermi liquid. If the system is superfluid, the current becomes

$$\mathbf{j} = (\rho - \rho_S)\mathbf{v}_N + \rho_S \mathbf{v}_S$$

where $\mathbf{v}_N$ is the velocity of the normal component, and the index $'S'$ stands for superfluidity.

For the two component NS matter the effective mass has a non-diagonal term, due to the neutron-proton interaction, i.e. the neutron and proton currents are coupled

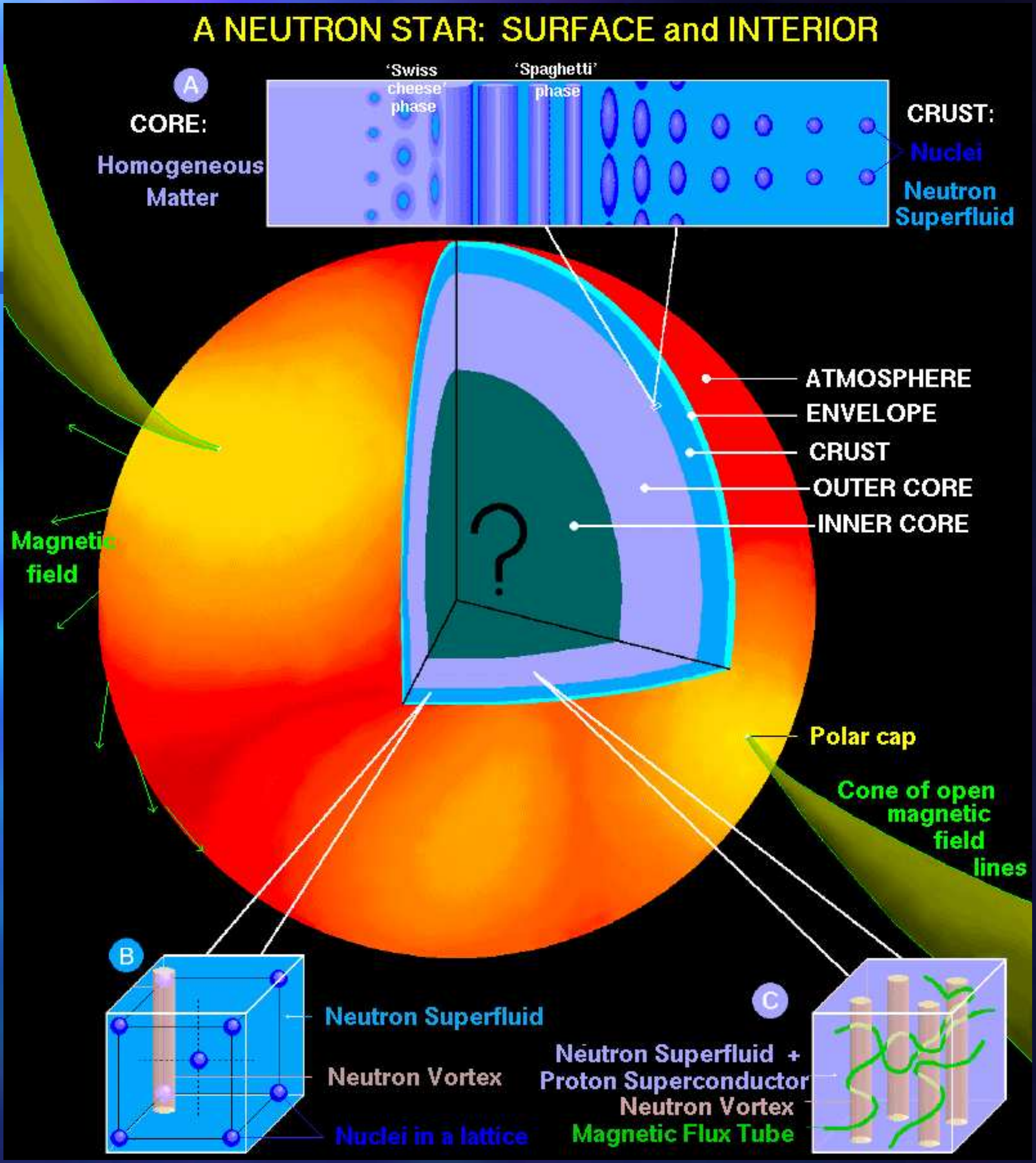
$$
\begin{aligned}
\mathbf{j_n} &= (\rho - \rho_{nn} - \rho_{np})\mathbf{v}_N + \rho_{nn}\mathbf{v}_{Sn} + \rho_{np}\mathbf{v}_{Sp} \\
\mathbf{j_p} &= (\rho - \rho_{nn} - \rho_{pn})\mathbf{v}_N + \rho_{pn}\mathbf{v}_{Sp} + \rho_{pn}\mathbf{v}_{Sn}
\end{aligned}
$$

A neutron current "drags" a proton current. A striking consequence of this "entrainement" is that also the rotational vortices carry a magnetic flux.

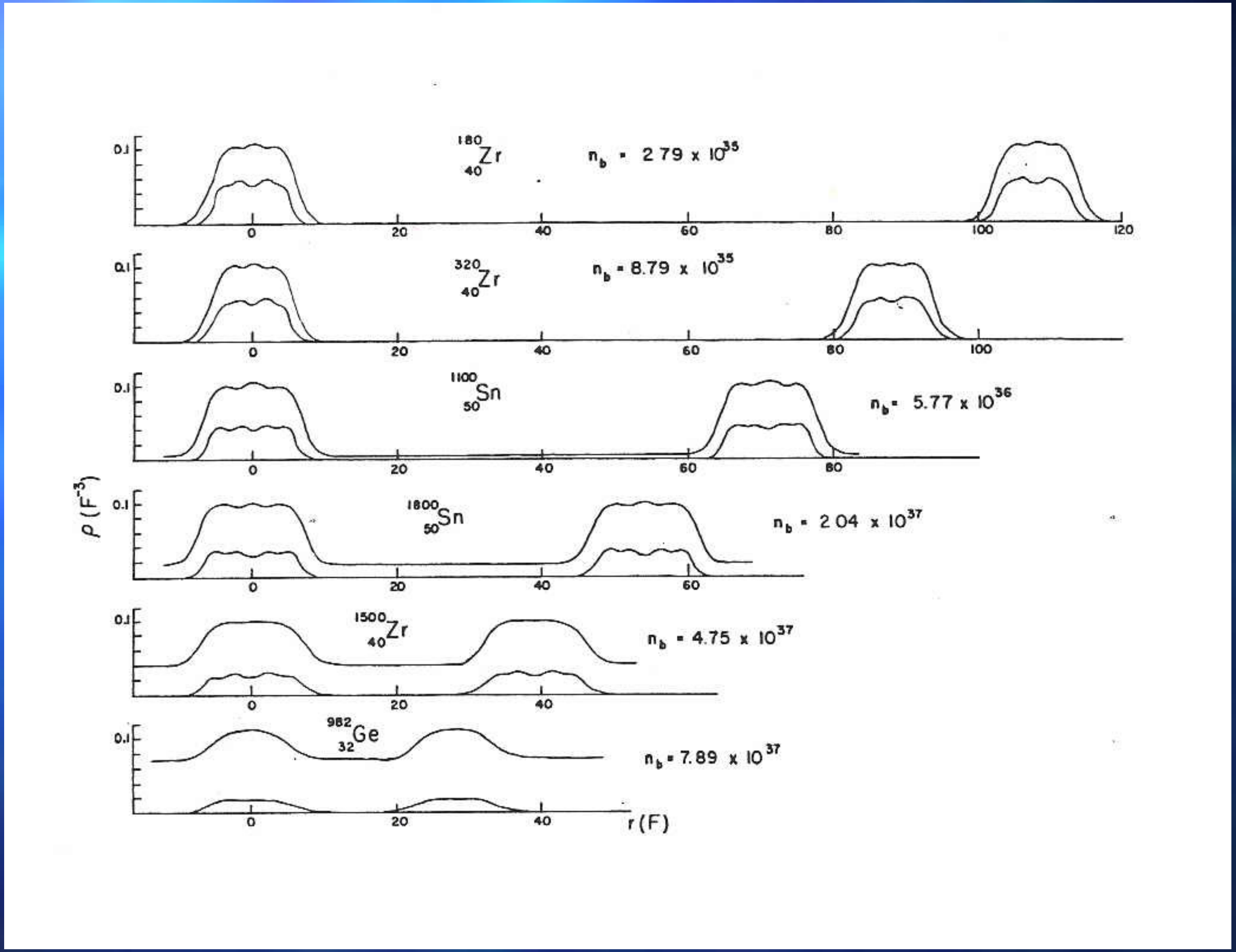
Superfluidity in the Crust

The internal crust of a Neutron Stars is a non-homogeneous matter composed by a lattice of nuclei, a neutron gas and electrons. At finite temperature a small fraction of unbound protons are also present. In the transition region between the crust and the core the nuclear lattice could be replaced by an irregular structure (pasta phase).

A section (schematic) of a neutron star



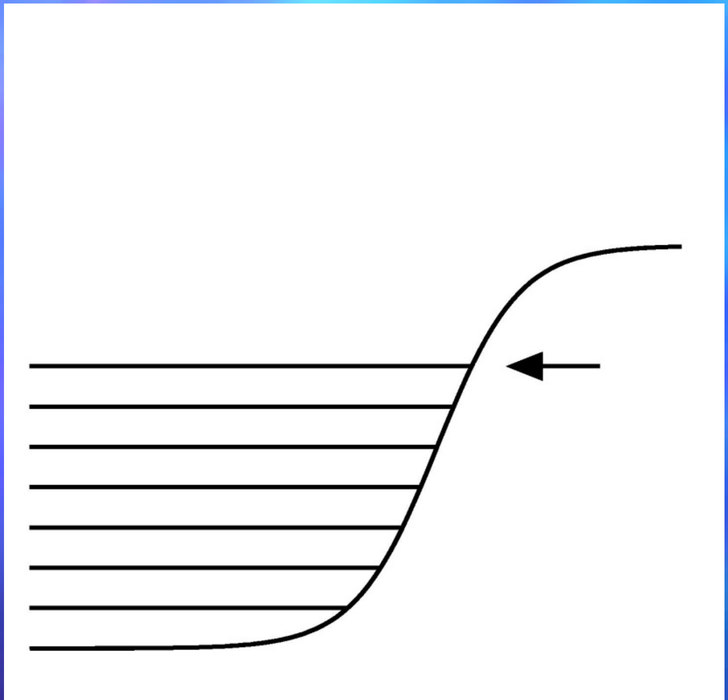
The structure of nuclei and Z/N ratio are dictated by beta equilibrium $\mu_n = \mu_p + \mu_e$

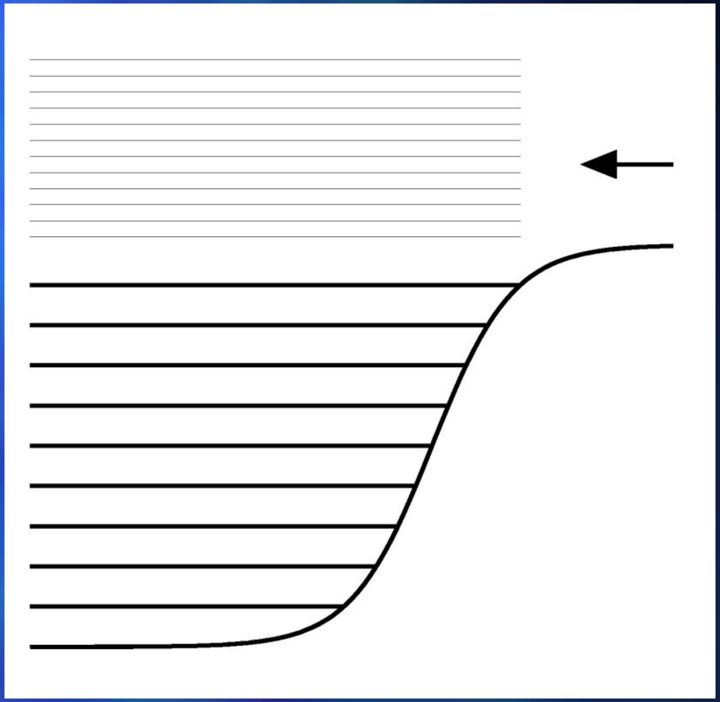


Negele & Vautherin classical paper (no pairing)

Outer Crust

Inner Crust





No drip region

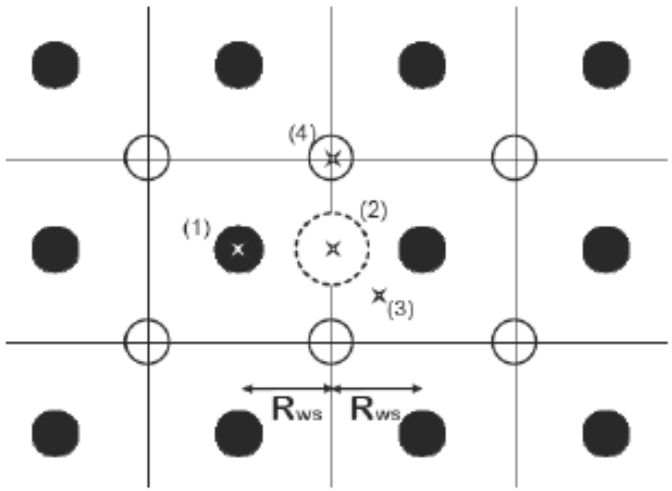
Drip region

Position of the neutron chemical potential

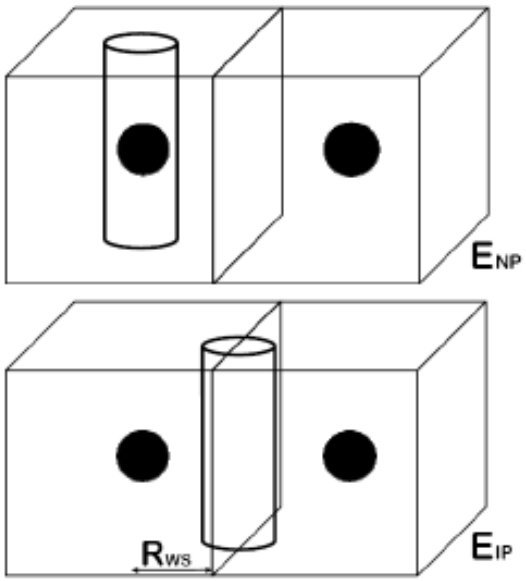
The neutron gas is probably superfluid, but in general also the neutrons bound in nuclei are superfluid. Therefore, superfluidity permeates the whole structure of the crust. The rotational vortices penetrate inside the crust, but the presence of the nuclear lattice modifies their dynamics. Indeed a vortex can be "pinned" by the lattice.

The processes of pinning and de-pinning is believed to be at the origin of the glitch phenomenon.

Possible positions of a vortex in the crust



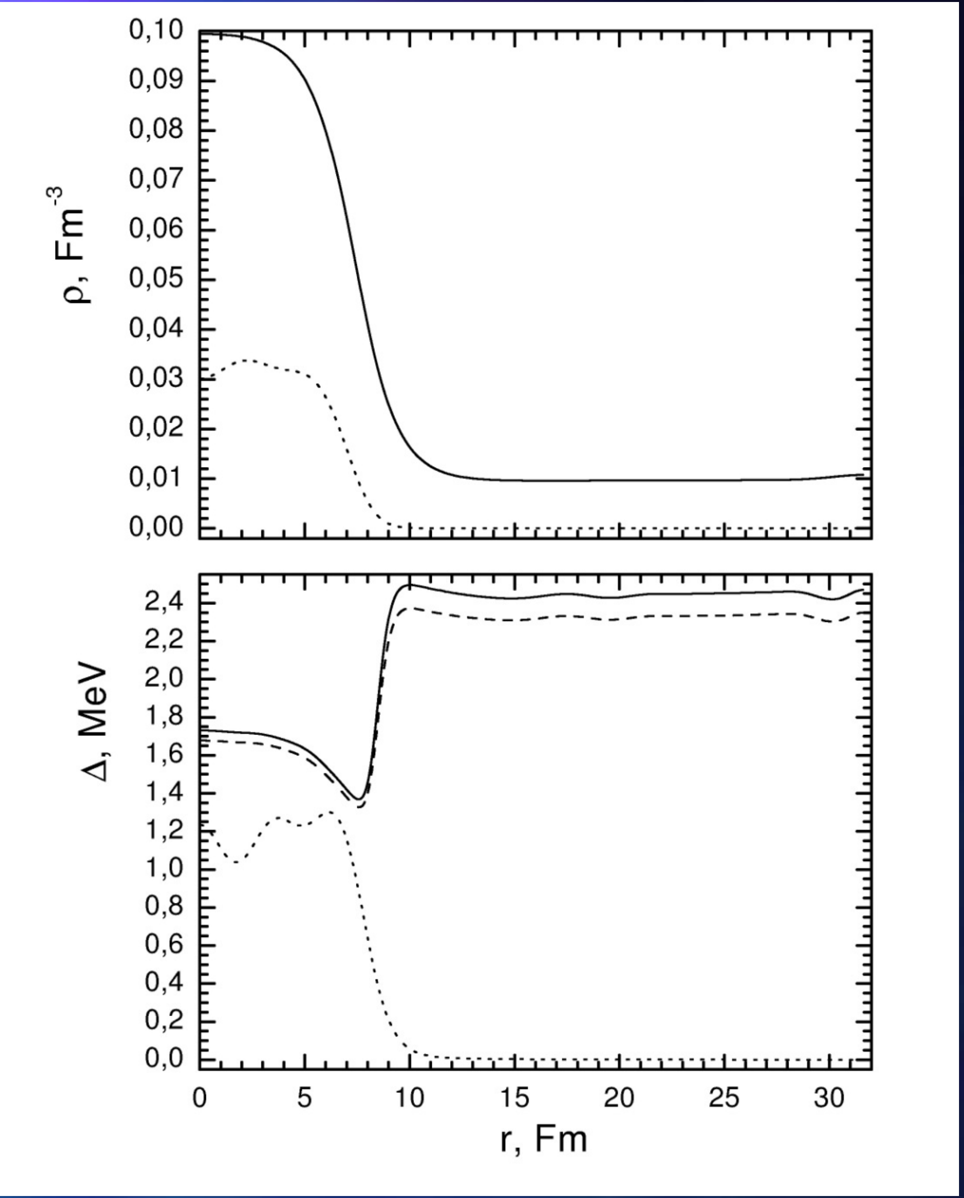
Top view



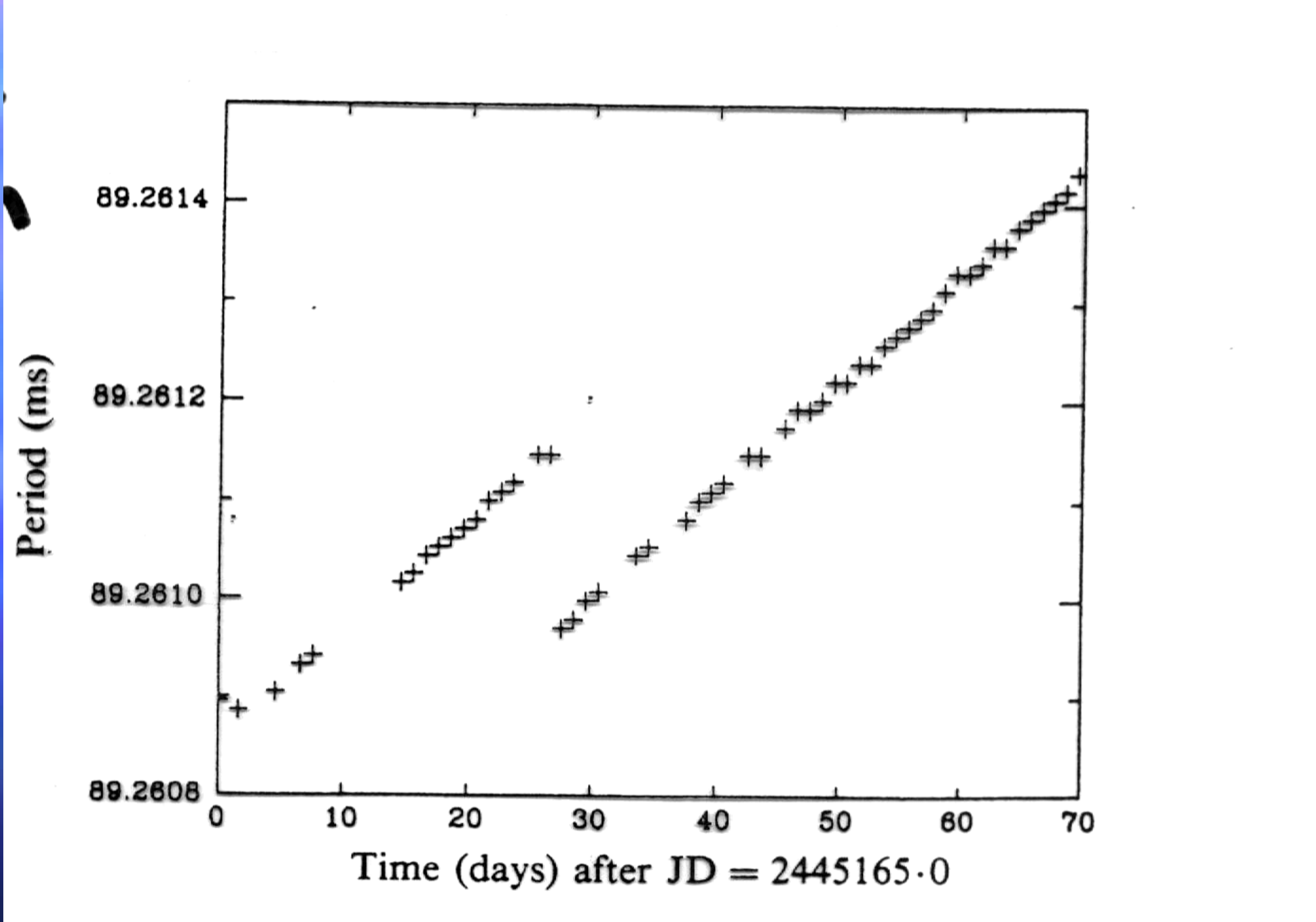
Side view

Neutron and proton density profiles

Neutron and proton pairing profiles

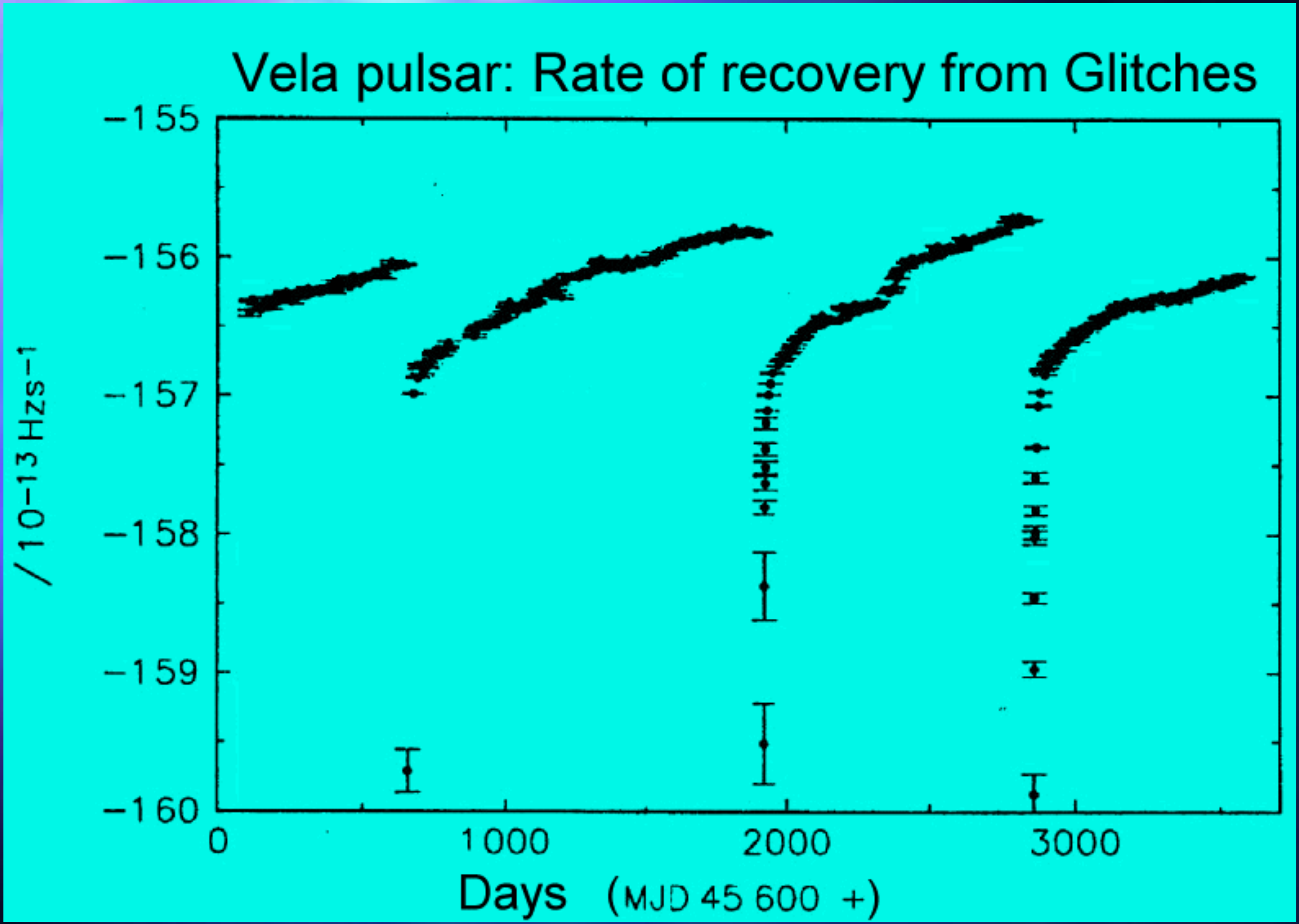


Density = 1/10 saturation density , Wigner-Seitz cell

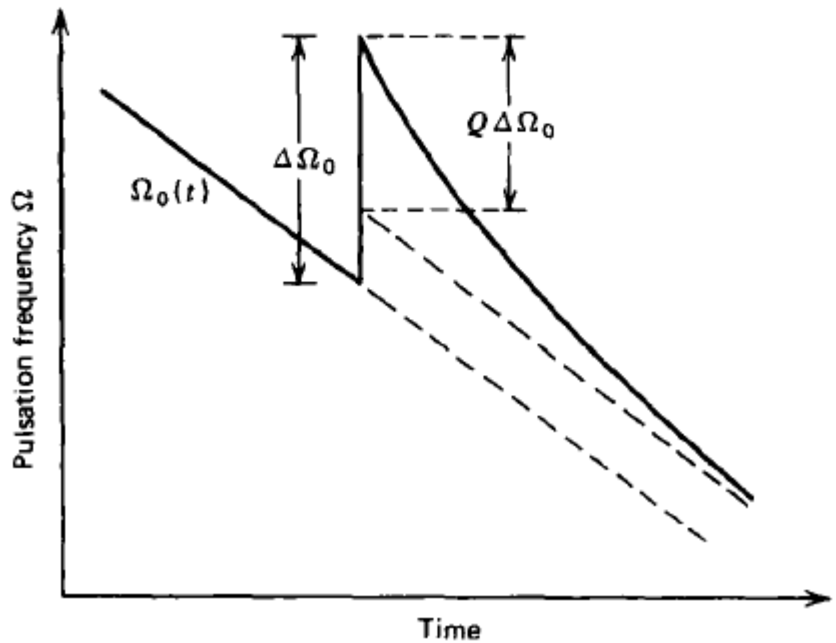


A close view at a glitch

The “regular” increase of the period is mainly due to dipolar radiation



A set of glitches in the Vela pulsar
(Period time derivative)



$$\Omega(t) = \Omega_0(t) + \Delta\Omega_0\left[Qe^{-t/\tau} + 1 - Q\right].$$

The observed recovering time τ is of the order of months or years. This long time is evidence of a very low friction between crust and core, which points in the direction of a superfluid component.

The Many-body problem

The theory of superfluidity / superconductivity can be formulated in its more general form in the framework of the many-body theory of fermion systems, in particular in Nuclear Matter. The challenging ambition is to predict the pairing gaps in NS matter. In the most general form of the gap equation the gap $\Delta(\mathbf{k}, \omega)$ is a function of both momentum $\mathbf{k}$ and energy ω, and it reads

$$\Delta(\mathbf{k}, \omega) = \sum_{\mathbf{k}'} \int d\omega' I(\mathbf{k}\omega, \mathbf{k}'\omega') \frac{\Delta(\mathbf{k}', \omega')}{D(\mathbf{k}', \omega')}$$

I is the irreducible pairing interaction and

$$D(\mathbf{k},\omega) = (\epsilon_k - \mu + \omega - M(\mathbf{k},-\omega))\,(\epsilon_k - \mu - \omega - M(\mathbf{k},\omega)) + \Delta(\mathbf{k},\omega)^2$$

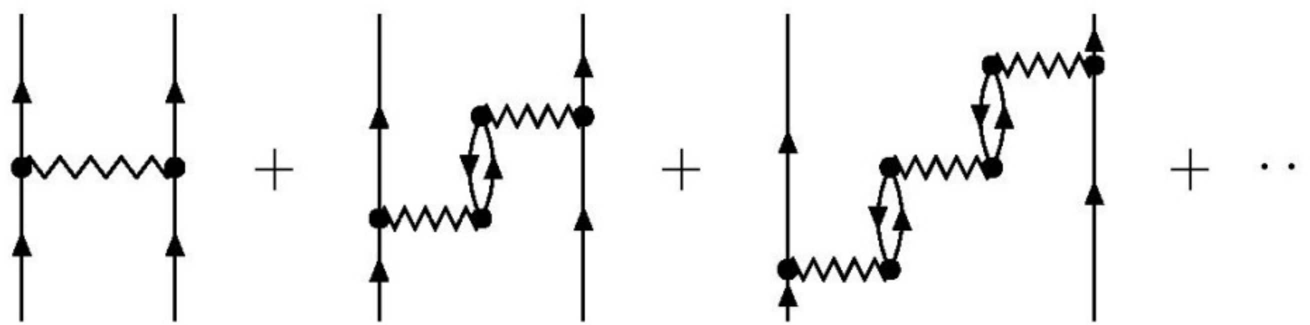
The quantity M is normal self-energy. The ratio Δ/D is the "anomalous" propagator $F(\mathbf{k},\omega)$, which is a generalization of the Cooper pair wave function. All many-body effects are included, even the direct effect of pairing on the interaction I. The formula is in principle exact. If we neglect all the energy dependences one obtains the previous BCS gap equation.

In the previous simplified formulation we took $M = 0$. The different many-body effects can be identified as follows.

- Energy dependence of the effective pairing interaction I, which implies a pairing gap dependent also on energy. The energy dependent is produced by the dynamical interaction processes in the medium.

- The effective pairing interaction is in general different from the bare NN interaction. One of the main many-body effect is the so-called "induced interaction".

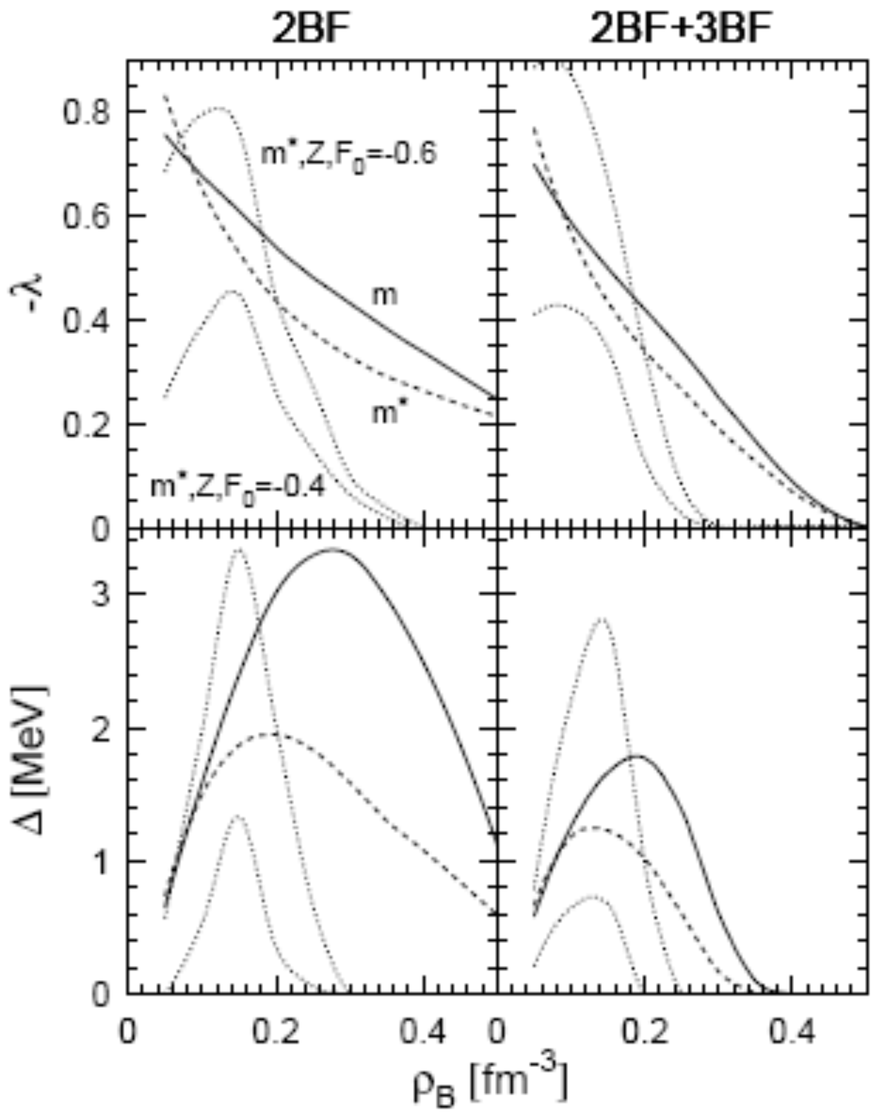
Two particles that propagate in the medium can excites the medium and exchange between them the Nuclear Matter excitation quanta



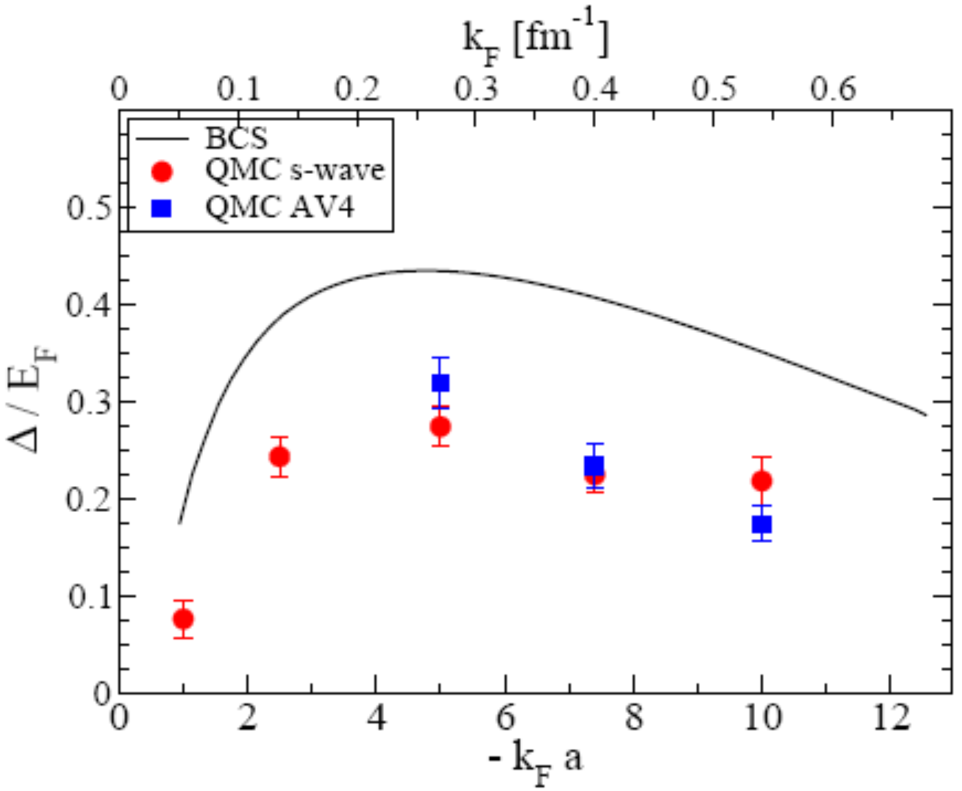
- The self-energy $M(\mathbf{k},\omega)$ introduces the effective mass m^* and the quasi-particle strength Z in the gap equation.

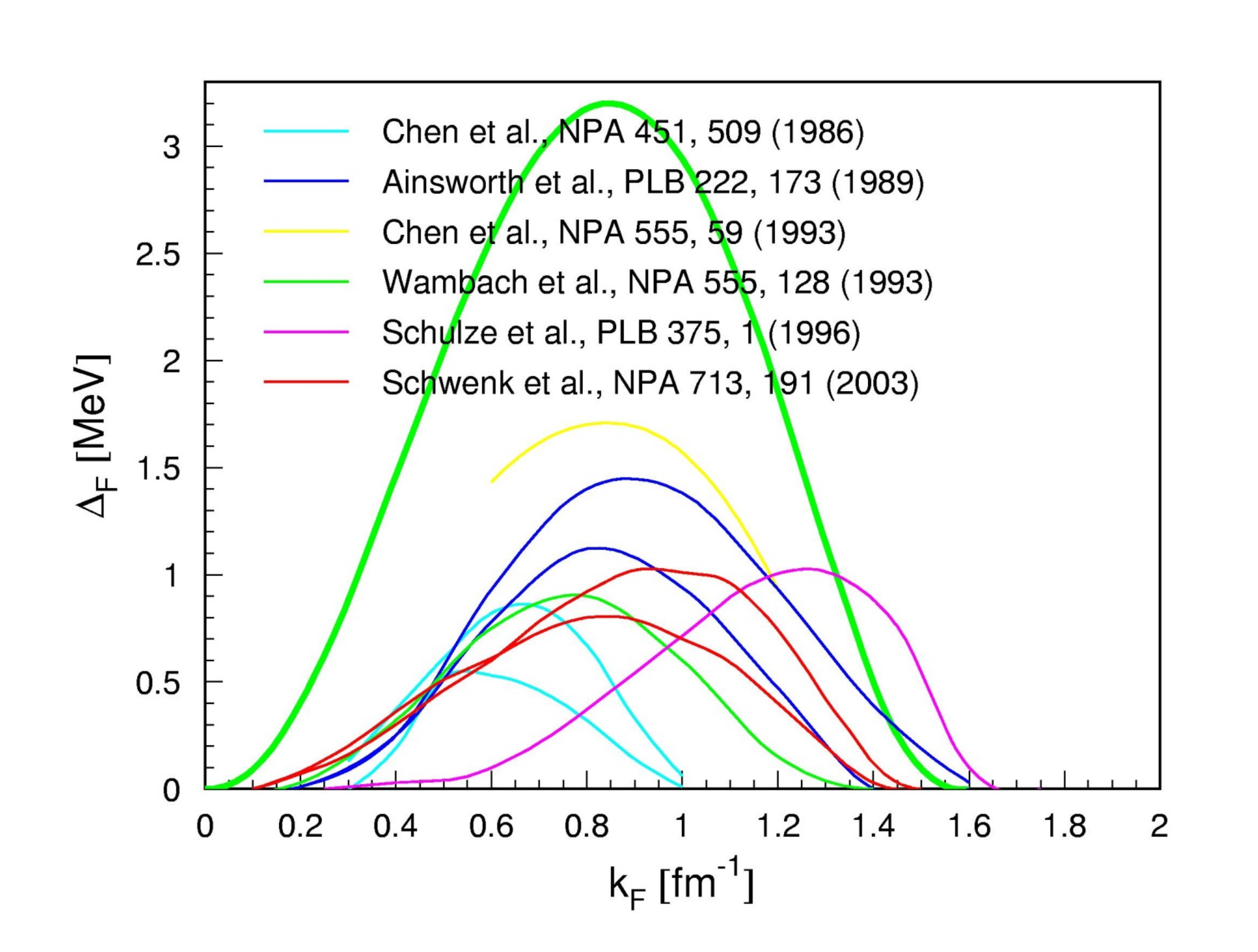
These many-body effects modify only moderately the gap equation, but due to the exponential form of the gap, it is difficult to get the desired accuracy. The pairing gap is therefore not well known. For the $^1S_0$ gap the accuracy can be estimated to be a factor of 2, just by comparing different many-body approaches. For the $^3P_2$ channel, it occurs at higher density and the accuracy can hardly be better than one order of magnitude.

Many-body effects on the proton gap



Pairing gap for the s-wave in neutron matter

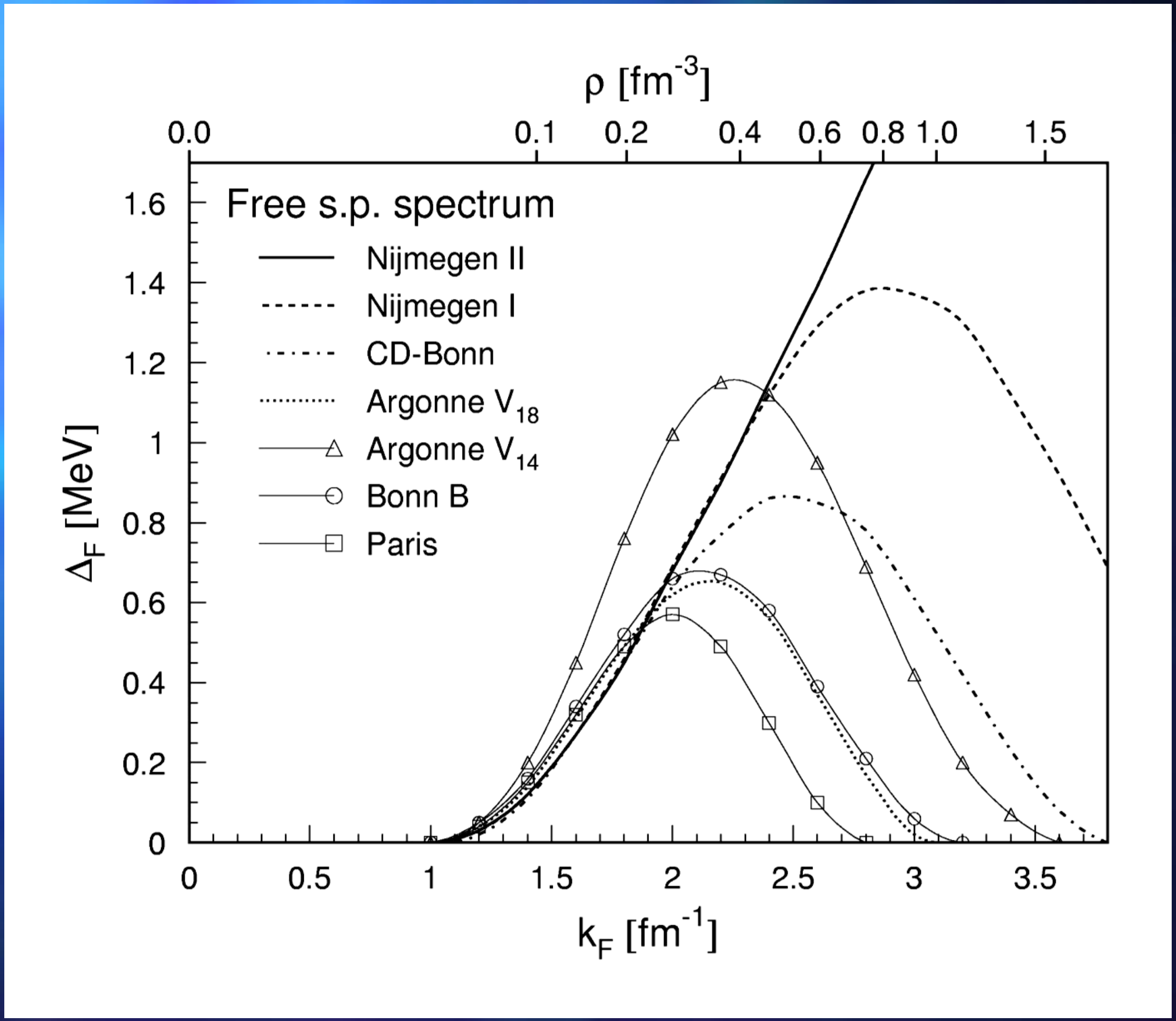




Including many-body effects

The intrinsic uncertainity on the 3P2 pairing gap due to the uncertainity on the bare NN interaction

PRC 58, 1921 (1998)



OPEN PROBLEMS

1. Strength of the 3P2 pairing
2. Density dependence of pairing
3. Proton superfluidity
4. Pinning energy
5. Effects of superfluidity on transport processes , neutrino emission and transport, ………
6. Vortices dynamics

7. Many-body effects on the gap

8. Neutron-proton excitations in the core 3P2 , “angulons”

9. Excitations in the crust (role of the lattice)

The $^3P_2$ pairing gap as a function of density calculated within the Renormalization Group method. (Schwenk & Friman, PRL 2004)