The KPZ equation, its universality and multi-component KPZ

T. Sasamoto

Tokyo Institute of Technology ($\leftarrow$ Chiba university)

(Based on collaborations with H. Spohn, T. Imamura, P. Ferrari)

7 May 2014 @ GGI

1. The KPZ equation and height distribution

The KPZ equation

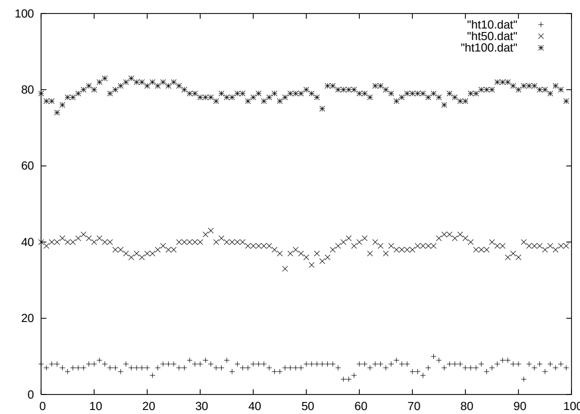
$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t > 0$

1986 Kardar Parisi Zhang

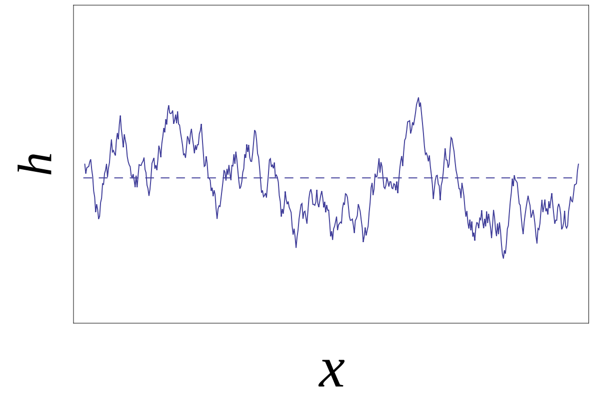
$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance
$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

The equation was introduced as a model to describe surface growth.



Scaling and KPZ universality class



Scaling (L: system size)

$$W(L,t) = \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle^{1/2}$$
$$= L^{\alpha} \Psi(t/L^z)$$

<u>For $t \to \infty$</u> $\quad W(L,t) \sim L^{\alpha}$

<u>For $t \sim 0$</u> $\quad W(L,t) \sim t^{\beta}$ where $\alpha = \beta z$

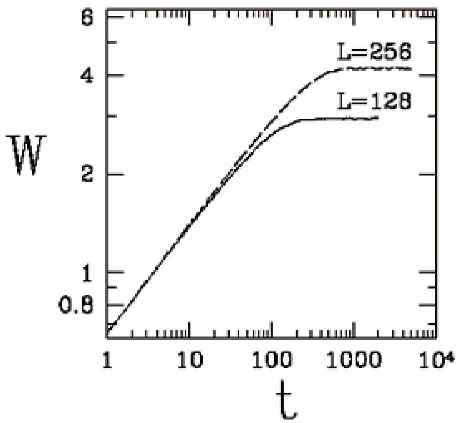


Figure 1. Interface width W versus time t for the RSOS model (Ref. [11]) in $1+1$ dimensions, in two different lattice lengths L.

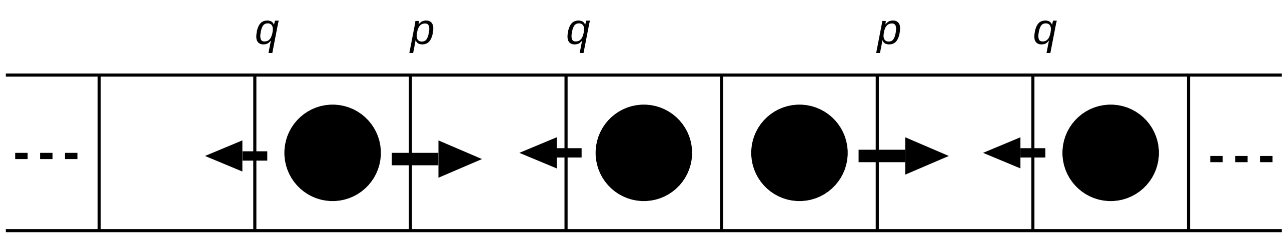
In many models, $\alpha = 1/2, \beta = 1/3$

KPZ universality class

ASEP and Conservation law

- The KPZ equation itself is not well-defined as it is and had been considered rather difficult to treat. On the other hand there are many discrete models which are known/expected to be in the KPZ universality class.

- ASEP(asymmetric simple exclusion process)



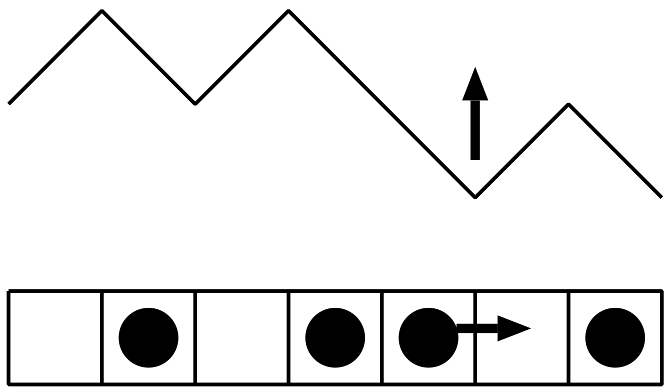
$\eta(j) = 0$ (empty at site j) or 1 (occupied).

Totally ASEP (TASEP): $p = 0$ or $q = 0$

Bernoulli (independent coin toss) is stationary.

Single conserved quantity (number of particles)

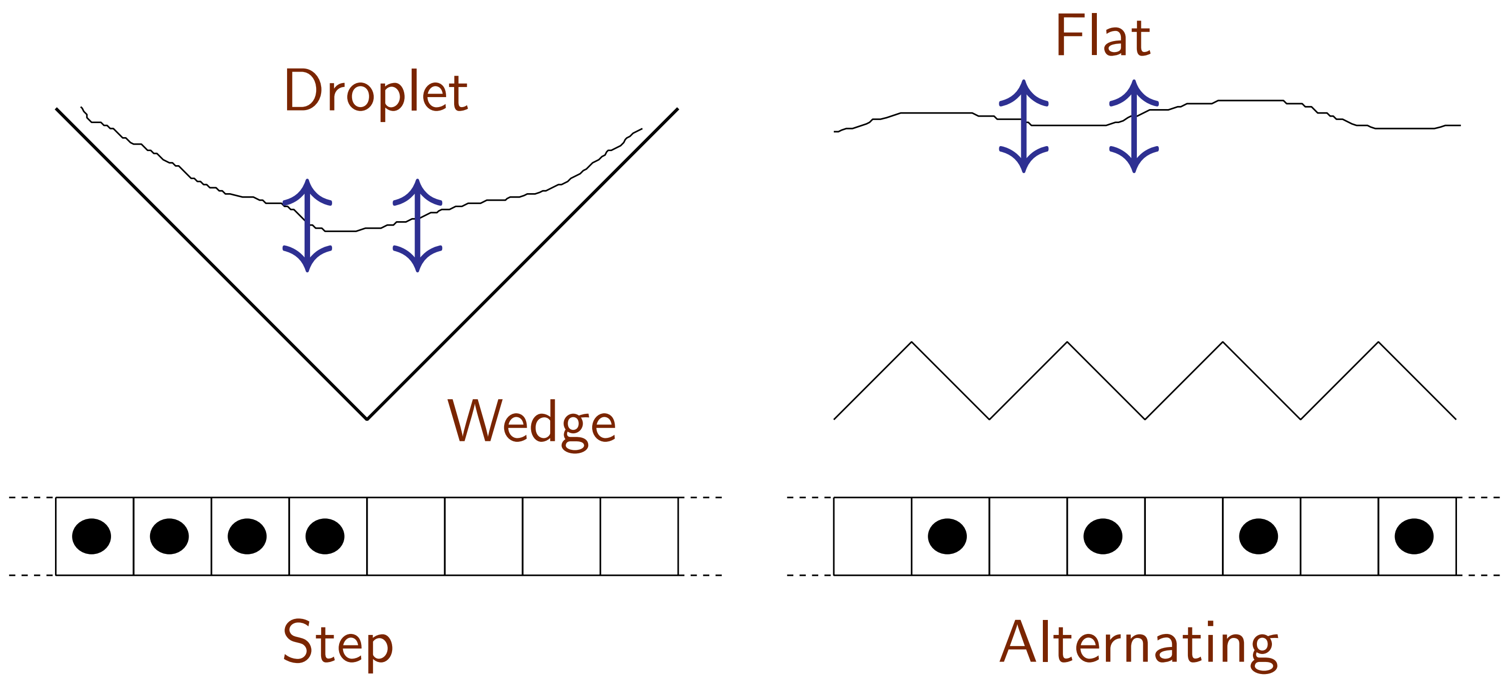
- Mapping to surface growth



- Noisy Burgers equation: For $u(x,t) = \partial_x h(x,t)$,

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \eta(x,t)$$

Two initial conditions besides stationary



Integrated current $N(x,t)$ in ASEP
$\Leftrightarrow$ Height $h(x,t)$ in surface growth

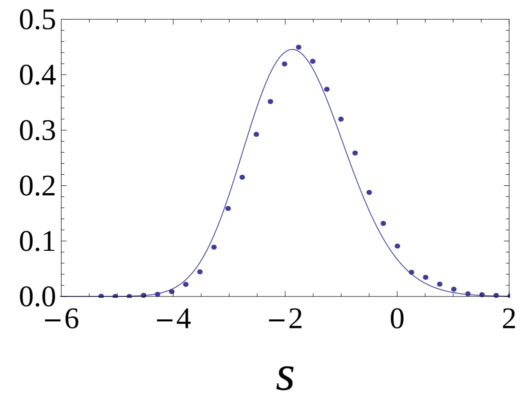
Limiting height distribution

2000 Johansson For TASEP with step i.c.

As $t \to \infty$ $$N(0,t) \simeq \frac{1}{4}t - 2^{-4/3}t^{1/3}\xi_2$$

Here $N(x=0,t)$ is the integrated current of TASEP at the origin and ξ_2 obeys the GUE Tracy-Widom distribution;

$$F_2(s) = \mathbb{P}[\xi_2 \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$



where P_s: projection onto the interval $[s,\infty)$ and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

A tentative definition of the KPZ class: Check if the one point height distribution tends to TW dist.

Generalizations

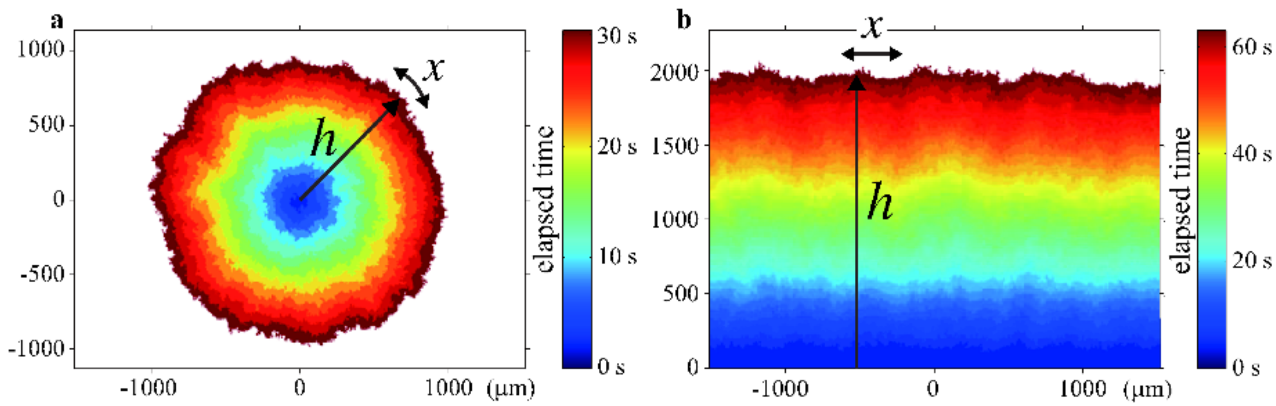
Current fluctuations of TASEP with flat initial conditions: GOE TW distribution

More generalizations: stationary case $\cdots$ $\boldsymbol{F_0}$ distribution, multi-point fluctuations: Airy process, etc

Experimental relevance?

What about the KPZ equation itself?

Takeuchi-Sano experiments



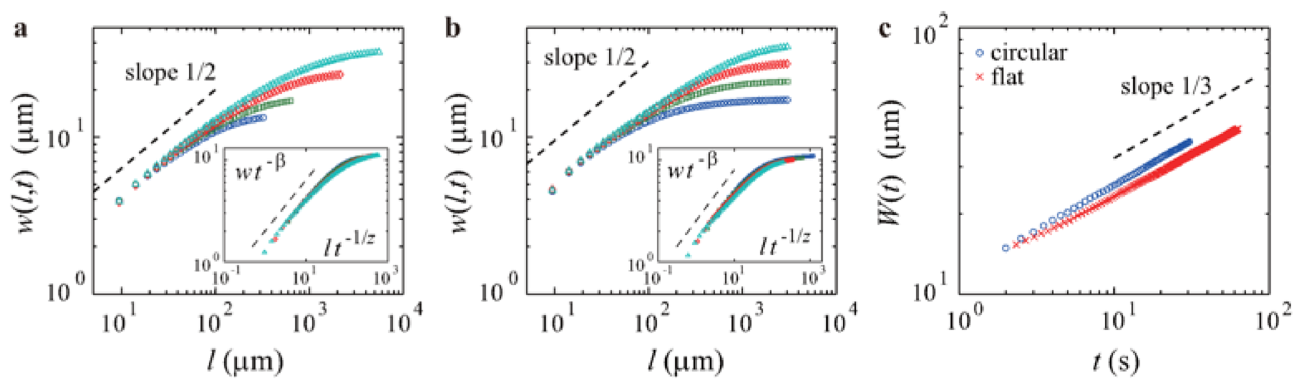


Figure 2 | **Family-Vicsek scaling.** a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

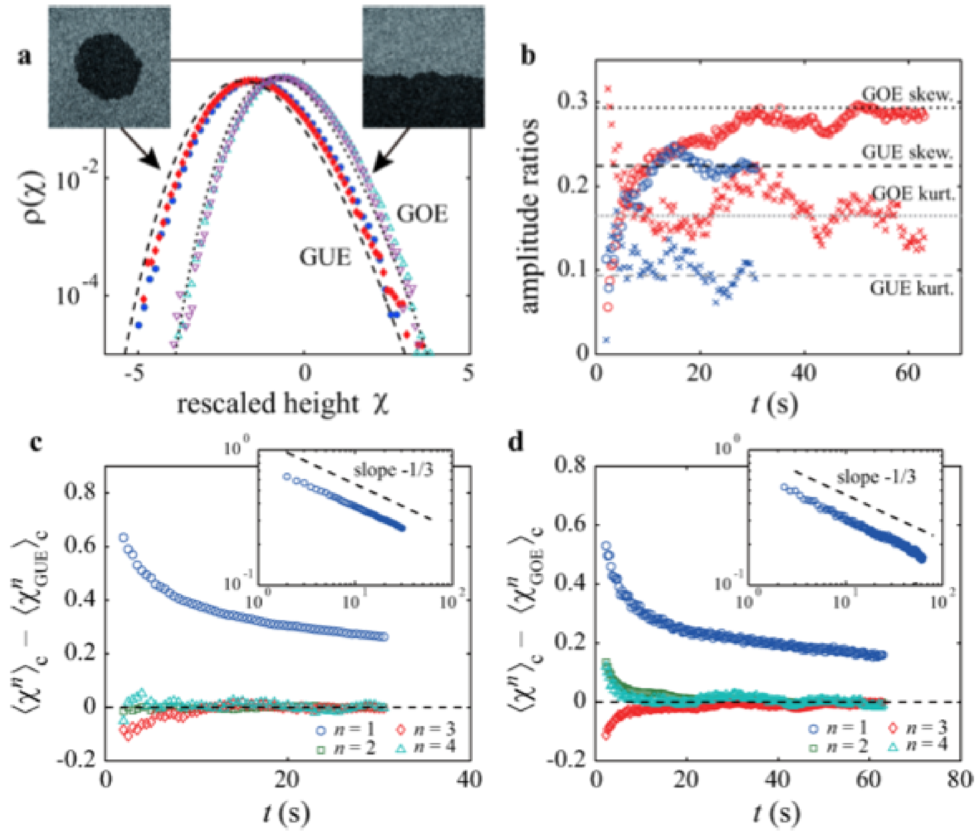


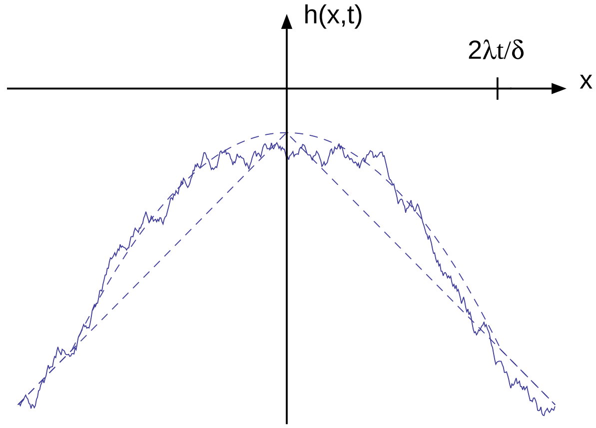
Figure 3 | Universal fluctuations. a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle \chi^n \rangle_c$ and the corresponding TW distributions $\langle \chi^n_{GUE} \rangle_c$ for the ciruclar interfaces (c) and $\langle \chi^n_{GOE} \rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

See Takeuchi Sano Sasamoto Spohn, Sci. Rep. 1,34(2011)

Exact solution for the KPZ equation

2010 Sasamoto Spohn, Amir Corwin Quastel

Narrow wedge initial condition



Macroscopic shape is

$$h(x,t) = \begin{cases} -x^2/2t & \text{for } |x| \leq t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta \end{cases}$$

which corresponds to taking the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1$$

Distribution

$$h(x,t) = -x^2/2t - \frac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (2t)^{1/3}$.

The distribution function of ξ_t

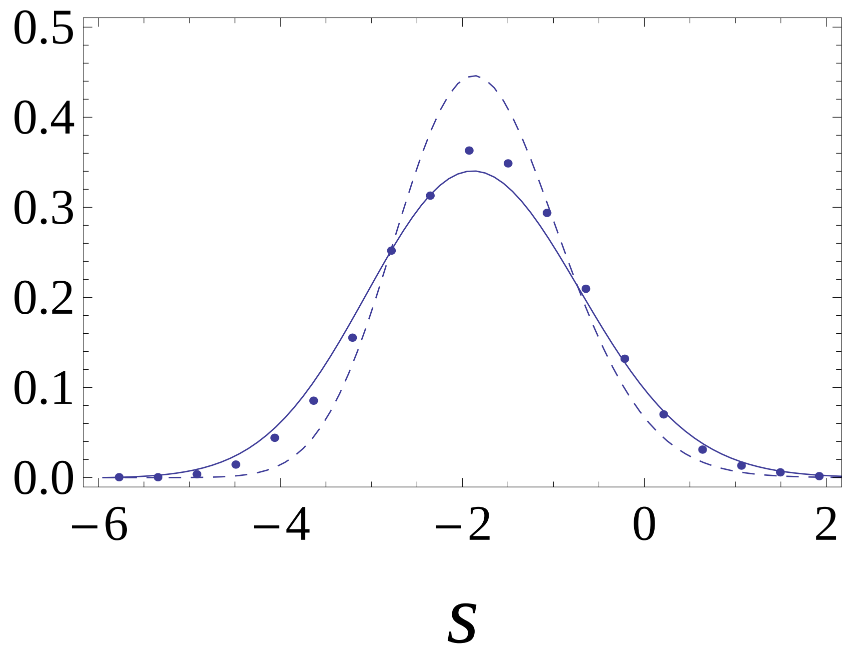
$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big]$$
$$\times\big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t \lambda} - 1}$$

- In the large t limit, F_t tends to F_2. (KPZ eq is in KPZ class!)

Finite time KPZ distribution and TW



—: exact KPZ density $F_t'(s)$ at $\gamma_t = 0.94$

– –: Tracy-Widom density ($t \to \infty$ limit)

•: ASEP Monte Carlo at $q = 0.6$, $t = 1024$ MC steps

Cole-Hopf transformation

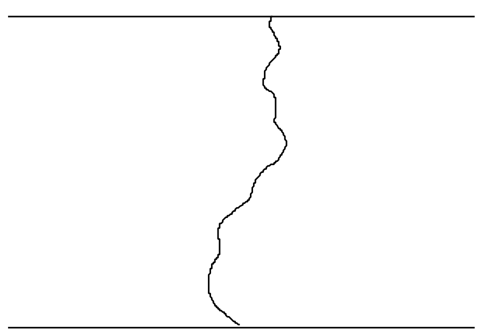
If we set

$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.



Narrow wedge corresponds to pt-to-pt polymer.

Replica method

For a system with randomness, e.g. for random Ising model,

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

where i is site, $s_i = \pm 1$ is Ising spin, J_{ij} is iid random variable(e.g. Bernoulli), we are often interested in the averaged free energy $\langle \log Z \rangle$, $Z = \sum_{s_i = \pm 1} e^{-H}$.

In some cases, computing $\langle \log Z \rangle$ seems hopeless but the calculation of Nth replica partition function $\langle Z^N \rangle$ is easier.

In replica method, one often resorts to the following identity

$$\langle \log Z \rangle = \lim_{N \to 0} \frac{\langle Z^N \rangle - 1}{N}.$$

For KPZ: Feynmann-Kac and δ Bose gas

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for pt-to-pt case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

Remember $h = \log Z$. We are interested not only in the average $\langle h \rangle$ but the full distribution of h. Here we compute the generating function $G_t(s)$ of the replica partition function,

$$G_t(s) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N \frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$. This turns out to be written as a Fredholm determinant. We apply the inversion formula to recover the p.d.f for h. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$.

- By considering discrete model like q-boson ZRP and ASEP, one can make this replica analysis rigorous.

2. Stationary 2pt correlation

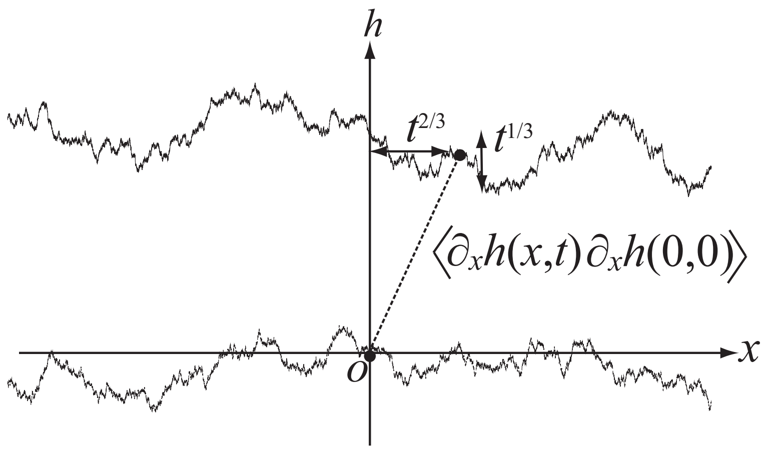
Not only the height/current distributions but correlation functions show universal behaviors.

- For the KPZ equation, the Brownian motion is stationary.

$$h(x, 0) = B(x)$$

where $B(x), x \in \mathbb{R}$ is the two sided BM.

- Two point correlation



Scaling limit

- The limiting two-point correlation function was first computed (for PNG) by Prähofer Spohn (2002).
- For TASEP (Ferrari Spohn (2002)) with density 1/2,

$$S(j,t) = \langle \eta(j,t)\eta(0,0)\rangle - \frac{1}{4}$$
$$\sim C_1 t^{-2/3} g''(C_2 j/t^{2/3})$$

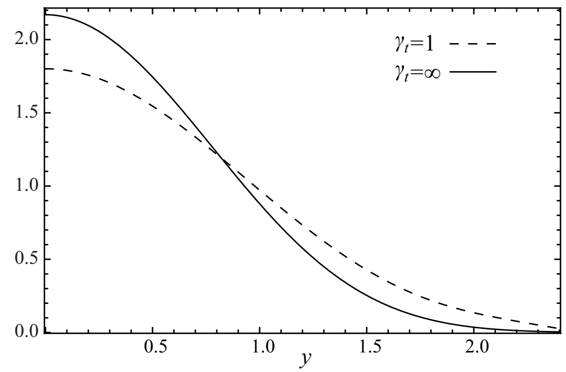
- The KPZ equation case was studied by Imamura TS (2012).

$$\langle \partial_x h(x,t) \partial_x h(0,0)\rangle = \frac{1}{2}(2t)^{-2/3} g_t''(x/(2t)^{2/3})$$

$$\lim_{t\to\infty} g_t''(x) = g''(x)$$

Scaled KPZ 2-pt function

Figure from exact formula



Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t := (\frac{t}{2})^{\frac{1}{3}} = 1$.
The solid curve is the scaling limit $g''(y)$.

- This scaled KPZ 2-pt function is expected to appear in various systems with a single conservation law.

More and more developments

- Other initial and boundary conditions. Flat, half space, etc
- Multi-point distributions
- Other models. q-boson zero range, Interacting Brownian particles with oblique reflection...
- Connections to integrable systems. Quantum Toda, Macdonald $\cdots$
- Simulations in 2D for distributions. Showing geometry dependence.

3. Multi-component KPZ

1D Hamiltonian dynamics

$$H = \sum_j \left(\frac{1}{2} p_j^2 + V(q_{j+1} - q_j) \right)$$

FPU chain $$V(x) = \frac{1}{2}x^2 + \frac{1}{3}ax^3 + \frac{1}{4}bx^4$$

Hard-point particles Alternating mass, shoulder potential

$$V(x) = \infty \ (0 < x < \frac{1}{2}), 1(\frac{1}{2} < x < 1),\ 0(x > 1)$$

There are three conserved quantities.

$\cdots$ Connection to KPZ!?

Conjecture by Henk

Beijreren 2011

- The scaled KPZ 2-pt function would appear in rather generic 1D fluid systems. The first observation was based on mode-coupling approximation.

  Three conserved quantities. Two sound modes with velocities $\pm c$ and one heat modes with velocity $\mathbf{0}$.

- Now there have been several attempts to confirm this by numerical simulations. Mendl, Spohn, Dhar, Beijeren, $\cdots$

- The conjecture would hold also for stochastic models with more than one conserved quantities. Here we formulate the conjecture for AHR model (which has two conserved quantities) and confirm it by monte carlo simulations.

A multi-component ASEP

Arndt-Heinzel-Rittenberg(AHR) model (1998)

- Rules

$$+\,0 \xrightarrow{\alpha} 0\,+$$

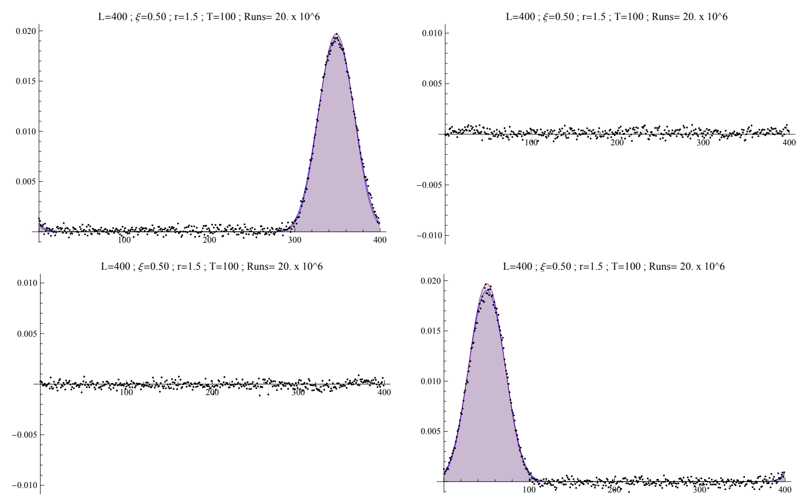
$$0\,- \xrightarrow{\alpha} -\,0$$

$$+\,- \xrightarrow{1} -\,+$$

- Two conserved quantities (numbers of $+$ and $-$ particles).
- Exact stationary measure is known (not a product measure, using matrix product techniques).

Monte Carlo simulation result

The KPZ scaling function in AHR model in a Monte Carlo simulation



For the moment it seems rather difficult to show analytically.

3.2 Nonlinear fluctuating hydrodynamics

- n-component lattice gas on $\mathbb{Z}$
- A configuration: $\eta = \{\eta(j), j \in \mathbb{Z}\}$ $\eta(j) = 0, 1, \ldots, n$
- Dynamics: $c_{l,m}^{(j)}$ rate of exchange of l, m at sites $j, j+1$
- The density of each component ρ_l, $l = 1, \ldots, n$ is conserved.
- Stationary measure specified by $\rho = (\rho_1, \ldots, \rho_n)$.
- Current:
  $j_l(\rho) = \langle \sum_{m=0}^{n} (c_{lm}^{(0)} \delta_{\eta(0),l} \delta_{\eta(1),m} - c_{ml}^{(0)} \delta_{\eta(0),m} \delta_{\eta(1),l}) \rangle_\rho$
- Two-point correlation function (Structure function):
  $S_{lm}(j,t) = \langle \delta_{\eta(j,t),l} \delta_{\eta(0,0),m} \rangle_\rho - \rho_l \rho_m$

Continuum description of large scale behaviors

- Macroscopic density: $\mu(x, t)$
- Conservation law (hydrodynamic limit):

$$\partial_t \mu + \partial_x j(\mu) = 0$$

or

$$\partial_t \mu + \partial_x (A(\mu)\mu) = 0, \quad A_{lm}(\rho) = \frac{\partial j_l(\rho)}{\partial \rho_m}.$$

- We focus on the fluctuations

$$\mu(x, t) = \rho + u(x, t)$$

- Adding noise and dissipation, $\boldsymbol{u}$ is governed by a SDE

$$\partial_t u + \partial_x (A(\rho) u - \frac{1}{2} \partial_x D(\rho) u + \xi) = 0$$

$D(\rho)$ is diffusion matrix and the space-time while noise ξ has covariance

$$\langle \xi_l(x,t) \xi_m(x',t') \rangle = B_{lm}(\rho) \delta(x - x') \delta(t - t')$$

- Stationary correlation matrix

$$\langle u_l(x) u(x') \rangle = C_{lm} \delta(x - x')$$

should be compatible with the above SDE, which requires

$$AC = CA^T$$

(T means the transpose) and

$$DC + CD = BB^T$$

- The second equality represents the fluctuation-dissipation relation. The first equality had been discussed by Tóth and Valkó, Grisi and Schütz and has turned out to be a simple consequence of the conservation laws and space-time stationarity.

Nonlinearity

For asymmetric models, one has to include the nonlinear term (second derivative of the current). Then the SDE becomes

$$\partial_t u + \partial_x (A(u)u + \frac{1}{2}\langle u, H(\rho)u\rangle - \frac{1}{2}\partial_x D(\rho)u + \xi) = 0$$

where

$$\langle u, H^{(k)}(\rho)u\rangle = \sum_{l,m=1}^{n} \frac{\partial}{\partial \rho_l}\frac{\partial}{\partial \rho_m} j^{(k)} u_l u_m$$

Heuristically this seems ok but mathematically there are various issues (well-definedness, derivation by weakly asymmetric limit, etc).

Normal modes

We switch to normal modes for which $\boldsymbol{A}$ is diagonalized. Due to $\boldsymbol{AC} = \boldsymbol{CA^T}$, both $\boldsymbol{A}$ and $\boldsymbol{C}$ are diagonalized simultaneously by a matrix $\boldsymbol{R}$:

$$\boldsymbol{RAR^{-1}} = \text{diag}, \quad \boldsymbol{RCR^{-1} = 1}$$

For the normal modes

$$\boldsymbol{\phi = Ru}$$

$$\boldsymbol{\partial_t \phi_l + \partial_x (v_l \phi_l + \langle \phi, G^l \phi \rangle - \partial_x (D\phi)_l + \sqrt{2D}\xi_l) = 0}$$

where $\boldsymbol{v_l}$ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{G}$ represents the strengths of the nonlinearilites coming from $\boldsymbol{H}$.

- We call this the coupled KPZ equation. It has many applications to sedimenting colloidal suspensions, crystals, magnetohydrodynamics, etc.

- The main nonlinear contribution is expected to come from $\boldsymbol{G}^{l}_{ll}$. Then the equation for each component (in normal modes) is in fact the same as the KPZ equation.

  Conjecture (a reformulation of Henk's for stochastic systems):

  In normal modes, stationary 2pt correlation is described by the scaled KPZ 2pt correlation function.

3.3 AHR model

Rates

$$+\,0 \xrightarrow{\beta} 0\,+$$

$$0\,- \xrightarrow{\alpha} -\,0$$

$$+\,- \underset{q}{\overset{1}{\rightleftharpoons}} -\,+$$

We mostly consider the case where $\alpha = \beta$ and $q = 0$.

Stationary measure by matrix product

AHR 1998, Rajewsky TS Speer 2001

Grandcanonical ensemble with fugacities ξ_+, ξ_-. The probability $P(\eta)$ that the system is in the configuration η in the stationary state is given by

$$P(\eta) = \frac{1}{Z_L(\xi_+, \xi_-)} \mathrm{Tr} \prod_{i=1}^{L} (\delta_{\eta_i +} \xi_+ D + \delta_{\eta_i -} \xi_- E + \delta_{\eta_i 0} A)$$

where D, E, A should satisfy

$$DE - qED = D + E$$
$$\beta DA = A, \quad \alpha AE = A$$

and $Z_L(\xi_+, \xi_-)$ is the normalization constant.

- For the AHR model, A, C, R can be computed explicitly.

Thermodynamic densities and currents

$$\rho_{\pm}(\xi_+,\xi_-) = \lim_{L\to\infty} \rho_{\pm,L}(\xi_+,\xi_-) = \frac{\partial}{\partial \xi_{\pm}} \log \nu(\xi_+,\xi_-)$$

$$J_{\pm}(\xi_+,\xi_-) = \lim_{L\to\infty} J_{\pm,L}(\xi_+,\xi_-) = \pm \frac{\xi_{\pm} - (\xi_{\pm} - \xi_{\mp})\rho_{\pm}}{\nu(\xi_+,\xi_-)}$$

where

$$\nu(\xi_+,\xi_-) = \frac{(\xi_- + \sqrt{\xi_+\xi_-}z)(\xi_+ + \sqrt{\xi_+\xi_-}z)}{\sqrt{\xi_+\xi_-}z}$$

$$z(\xi_+,\xi_-) = \frac{1 + \xi_- a + \xi_+ b - \sqrt{(1 + \xi_- a + \xi_+ b)^2 - 4ab\xi_+\xi_-}}{2ab\sqrt{\xi_+\xi_-}}$$

A and C

$$A = \begin{bmatrix} \frac{\partial J_+}{\partial \xi_+} & \frac{\partial J_+}{\partial \xi_-} \\ \frac{\partial J_+}{\partial \xi_+} & \frac{\partial J_+}{\partial \xi_-} \end{bmatrix} \begin{bmatrix} \frac{\partial \rho_+}{\partial \xi_+} & \frac{\partial \rho_+}{\partial \xi_-} \\ \frac{\partial \rho_+}{\partial \xi_+} & \frac{\partial \rho_+}{\partial \xi_-} \end{bmatrix}^{-1} .$$

One can also write the correlation matrix C as derivatives of ρ,

$$C_{++} = \xi_+ \frac{\partial \rho_+}{\partial \xi_+}, \; C_{+-} = C_{-+} = \xi_- \frac{\partial \rho_+}{\partial \xi_-}, \; C_{--} = \xi_+ \frac{\partial \rho_+}{\partial \xi_+}.$$

Explicite R

If we take

$$R^{(0)-1}$$

$$= \begin{bmatrix} b\xi_+ - ab\sqrt{\xi_+\xi_-}z(\xi_+,\xi_-) & b\xi_+ - \sqrt{\xi_+\xi_-}/z(\xi_+,\xi_-) \\ a\xi_- - \sqrt{\xi_+\xi_-}/z(\xi_+,\xi_-) & a\xi_- - ab\sqrt{\xi_+\xi_-}z(\xi_+,\xi_-) \end{bmatrix}$$

where

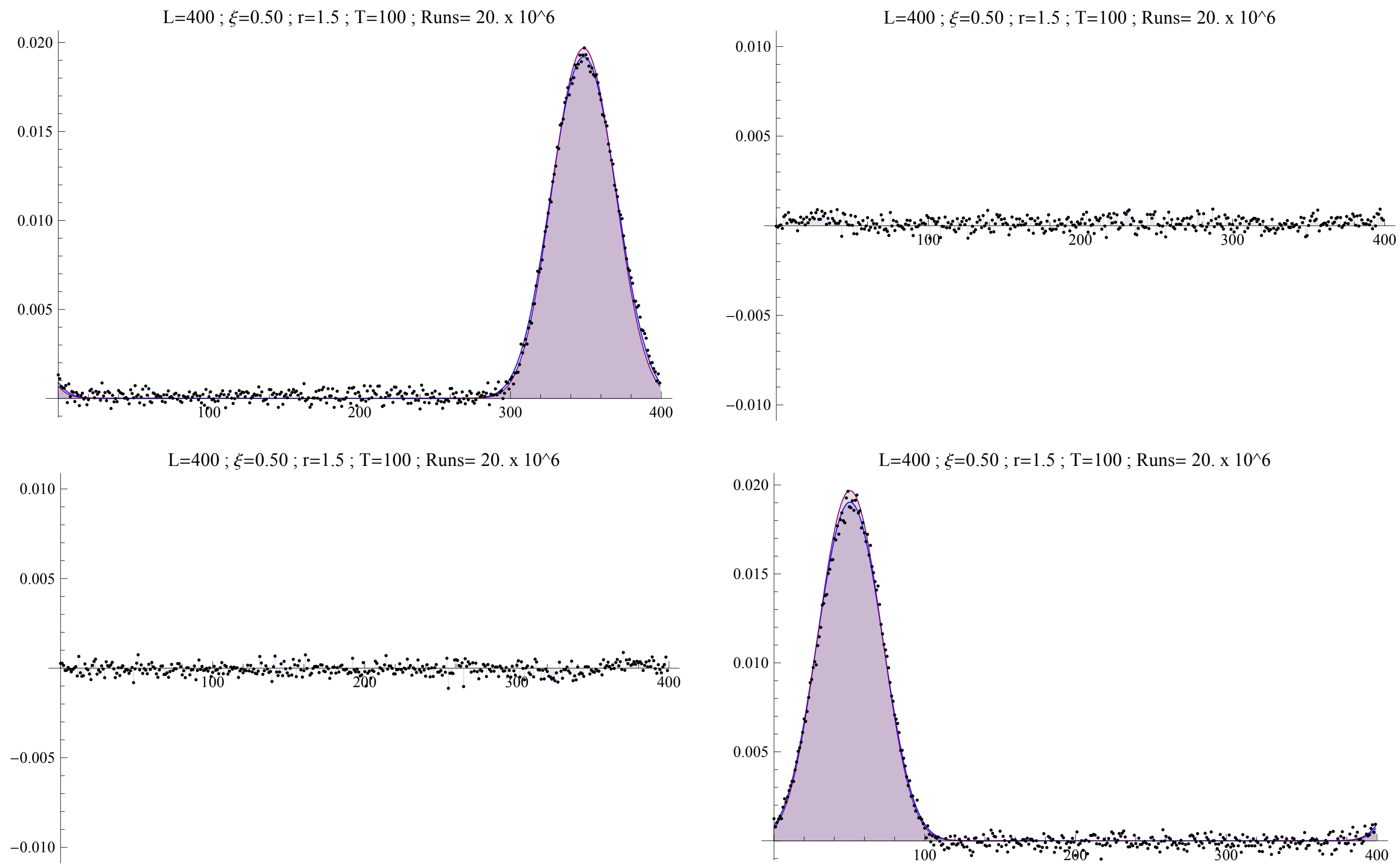
$$a = -1 + (1-q)/\alpha, b = -1 + (1-q)/\beta$$

it holds

$$R^{(0)}CR^{(0)*} = \mathrm{diag}(d_1, d_2)$$

The matrix R is obtained easily by using $R^{(0)}$ and d_1, d_2.

Simulation results



Note that the switching to the normal modes is important.

3.4 Discussions

Convergence to the limiting shape

- For AHR, the convergence to the limiting shape is fast.
- The numerical simulations of the anharmonic chain show slow decay to the limiting shape (shoulder potential seems faster).
- The difference seems to be coming from the fact that for AHR $\boldsymbol{G^1_{22} = G^2_{11} = 0}$.

An argument based on the mode-coupling

For the normal modes in the mode coupling approximation, the two point function is approximated to be diagonal,

$$S^{\sharp\phi}_{\alpha\beta}(x,t) = \delta_{\alpha\beta} f_\alpha(x,t)$$

and f_α satisfies

$$\partial_t f_\alpha(x,t) = (-c_\alpha \partial_x + D_\alpha \partial_x^2) f_\alpha(x,t) + \int_0^t ds \int_{\mathbb{R}} dy f_\alpha(x-y, t-s) \partial_y^2 M_{\alpha\alpha}(y,s)$$

where

$$M_{\alpha\alpha}(x,t) \simeq 2(G^\alpha_{\alpha\alpha})^2 f_\alpha(x,t)^2 + \sum_{\beta=1,\beta\neq\alpha}^{n} 2(G^\alpha_{\beta\beta})^2 f_\beta(x,t)^2 = M^0_{\alpha\alpha}(x,t) + M^1_{\alpha\alpha}(x,t)$$

For AHR the correction from the second term is absent.

Summary

- Some exact solutions have been obtained for the 1D KPZ equation.
- The KPZ universality seems relevant for larger (than previously thought) class of systems. In particular the scaled KPZ 2pt function is expected to describe large time asymptotics of normal modes of systems with more than one conserved quantities.
- For multi-component lattice gases, we have argued that their large scale behaviors are described by the non-linear fluctuating hydrodynamics and that the scaled KPZ 2pt function would be observed for their normal modes. We have provided a Monte Carlo simulation evidence for AHR model.

The agreement with the theory is very good (better than for hamiltonian dynamics). This seems to be related to the fact that for AHR, $\boldsymbol{G^2_{11} = G^1_{22} = 0}$.

- It is challenging to try to "prove" these more mathematically. Studying Hamiltonian systems analytically seems very difficult for the moment. There would be several things one can do for stochastic systems. It would be useful to find a stochastic model which mimics Hamiltonian dynamics (2 sound modes and 1 heat mode) and is hopefully also tractable.

- If you are interested in recent developments on KPZ, please consider attending a workshop on KPZ (2014/8/20-23) in Kyoto!