Macroscopic fluctuations in non-equilibrium mean-field diffusions

Krzysztof Gawędzki
GGI, Florence, May 2014

- **Mean field** approximation has served us for over 100 years (**Curie** 1895, **Weiss** 1907) giving hints about the behavior of

  ↗ **systems with short range interactions** in high dimensions

  ↘ **systems with long range interactions** in any dimension

- Developed originally for equilibrium systems (ordered and disordered), it has been applied more recently to **nonequilibrium dynamics**

- Here, I shall employ the **Macroscopic Fluctuation Theory** of the Rome group (**Bertini**-**De Sole**-**Gabrielli**-**Jona-Lasinio**-**Landim**) to describe fluctuations around mean field approximation

- Informally, the Roman theory may be viewed as a version of **Freidlin-Wentzell large deviations theory** applied to stochastic lattice gases (zero range, SSEP, WASEP, ABC, ...)

- We shall keep a similar point of view in application to general non-equilibrium d-dimensional diffusions with mean-field coupling:

$$\frac{dx_n}{dt} = X(t, x_n) + \frac{1}{N}\sum_{m=1}^{N} Y(t, x_n, x_m) + \sum_a X_a(t, x_n) \circ \eta_{na}(t)$$

independent white noises

with $Y(x,y) = -Y(y,x)$ and $\circ$ for the **Stratonovich** convention

- Based on joint work with **F. Bouchet** and **C. Nardini**

- **Prototype model :** N planar rotators with angles θ_n and **mean field** coupling, undergoing **Langevin** dynamics

$$\frac{d\theta_n}{dt} = F - H \sin\theta_n - \frac{J}{N}\sum_{m=1}^{N} \sin(\theta_n - \theta_m) + \sqrt{2k_B T}\,\eta_n(t)$$

independent white noises

Shinomoto-Kuramoto, Prog. Theor. Phys. **75** (1986),
...... ,
Giacomin-Pakdaman-Pellegrin-Poquet, SIAM J. Math. Anal. **44** (2012)

- Close cousin of the celebrated **Kuramoto** (1975) model for synchronization (with $F \to F_n$ random and $T = 0$) whose versions were recently studied by the long-range community (papers by **Gupta-Campa-(Dauxois)-Ruffo**)

- Originally thought as a model of cooperative behavior of coupled nerve cells

- Close to models of depinning transition in disordered elastic media

- The **Shinomoto-Kuramoto** system

$$\frac{d\theta_n}{dt} = F - H \sin\theta_n - \frac{J}{N}\sum_{m=1}^{N}\sin(\theta_n - \theta_m) + \sqrt{2k_B T}\,\eta_n(t)$$

may also be re-interpreted as a classical ferromagnetic **XY** model with a mean-field coupling of planar spins $\vec{S}_n$

- $F = 0$ case (**equilibrium**):

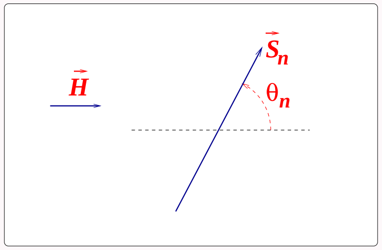
  in constant external magnetic field $\vec{H} = (H, 0)$

  $\vec{S}_n = S(\cos\theta_n, \sin\theta_n)$

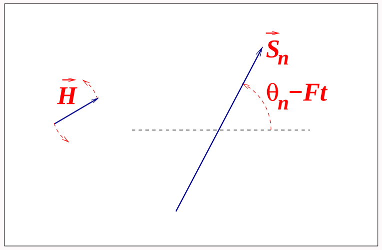
- $F \neq 0$ case (**non-equilibrium**):

  in rotating external magnetic field $\vec{H} = H(\cos(Ft), -\sin(Ft))$

  $\vec{S}_n = S(\cos(\theta_n - Ft), \sin(\theta_n - Ft))$ (i.e. spins are viewed in the co−moving frame)



$F = 0$



$F \neq 0$

- **Macroscopic quantities of interest** in the general case

$$\frac{dx_n}{dt} = X(t, x_n) + \frac{1}{N} \sum_{m=1}^{N} Y(t, x_n, x_m) + \sum_a X_a(t, x_n) \circ \eta_{na}(t)$$

- **empirical density**

$$\rho_N(t, x) = \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n(t))$$

- **empirical current**

$$j_N(t, x) = \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n(t)) \circ \frac{dx_n(t)}{dt}$$

- They are related to each other by the continuity equation:

$$\partial_t \rho_N + \nabla \cdot j_N = 0$$

- **Macroscopic Fluctuation Theory** applies to their **large deviations** at $N \gg O(1)$ around $N = \infty$ **mean field**

- **Effective diffusion in the density space**

  - Substitution of the equation of motion for $\frac{dx_n(t)}{dt}$ and the passage to the **Itô** convention give:

$$j_N(t,x) = \frac{1}{N}\sum_{n=1}^{N} \delta(x - x_n(t)) \circ \frac{dx_n(t)}{dt} = j_{\rho_N}(t,x) + \zeta_{\rho_N}(t,x)$$

where

$$j_\rho = \rho\big(\widehat{X} + Y * \rho\big) - D\nabla\rho \qquad \longleftarrow \quad \textbf{quadratic in} \quad \rho$$

with

$$\widehat{X} = X - \tfrac{1}{2}\sum_a \big(\nabla\cdot X_a\big)X_a\,, \qquad D = \tfrac{1}{2}\sum_a X_a \otimes X_a$$

$$(Y * \rho)(t,x) \equiv \int Y(t,x,y)\,\rho(t,y)\,dy$$

and

$$\zeta_{\rho_N}(t,x) = \frac{1}{N}\sum_{n=1}^{N}\sum_a X_a(t,x)\,\delta\big(x - x_n(t)\big)\,\eta_{na}(t)$$

- Conditioned w.r.t. ρ_N, the noise $\zeta_{\rho_N}(t,x)$ has the same law as the **white noise** $\sqrt{2N^{-1}D(t,x)\,\rho_N(t,x)}\,\xi(t,x)$ where

$$\Big\langle \xi^i(t,x)\,\xi^j(s,y)\Big\rangle \;=\; \delta^{ij}\,\delta(t-s)\,\delta(x-y)$$

- Follows from the fact that for functionals $\Phi[\rho]$ of (distributional) densities, the standard stochastic differential calculus gives

$$\frac{d}{dt}\Big\langle \Phi[\rho_{Nt}]\Big\rangle \;=\; \Big\langle\,\big(\mathcal{L}_{Nt}\Phi\big)[\rho_{Nt}]\,\Big\rangle$$

where

$$\big(\mathcal{L}_{Nt}\Phi\big)[\rho] \;=\; -\int \frac{\delta\Phi[\rho]}{\delta\rho(x)}\,\nabla\cdot j_\rho(t,x)\,dx$$
$$+\;\frac{1}{N}\int \frac{\delta^2\Phi[\rho]}{\delta\rho(x)\,\delta\rho(y)}\,\nabla_x\nabla_y\Big(D(t,x)\,\rho(t,x)\,\delta(x-y)\Big)\,dx\,dy$$

is the generator of the (formal) diffusion in the space of densities evolving according to the **Itô** **SDE**

$$\partial_t\rho \;+\; \nabla\cdot\big(j_\rho \;+\; \sqrt{2N^{-1}D\rho}\,\xi\big) \;=\; 0$$

- $N = \infty$ **closure**

  - When $N \to \infty$, the evolution equation for the **empirical density** reduces to **Nonlinear Fokker-Planck Equation** (**NFPE**)

$$\partial_t \rho \;=\; -\nabla \cdot j_\rho \;=\; -\nabla \cdot \Big(\rho\big(\hat{X} + Y * \rho\big) - D\nabla\rho\Big)$$

    $\rightarrow$ a nonlinear dynamical system in the space of densities (autonomous or not)

  - If $Y = 0$ then the $N = \infty$ empirical density coincides with instantaneous **PDF** of identically distributed processes $x_n(t)$ and **NFPE** reduces to the linear **Fokker-Planck** equation for the latter

  - The $N = \infty$ **phase diagram** of an autonomous system with **mean-field** coupling is obtained by looking for stable stationary and periodic solutions of **NFPE** and their **bifurcations**

  - In principle, more complicated dynamical behaviors may also arise

• $N = \infty$ **phases of the rotator model**

- Stationary solutions of **NFPE** satisfy $\partial_\theta j_\rho(\theta) = 0$, i.e.

$$\partial_\theta \left(\rho(\theta)\left(F - H\sin(\theta) - J\int_0^{2\pi}\sin(\theta-\vartheta)\,\rho(\vartheta)\,d\vartheta\right) - k_B T\,\partial_\theta\rho(\theta)\right)$$

$= sin\theta\cos\vartheta - \cos\theta\sin\vartheta$

$$= \partial_\theta\left(\rho(\theta)\left(F - (H + x_1)\sin\theta + x_2\cos\theta\right) - k_B T\,\partial_\theta\rho(\theta)\right) = 0$$

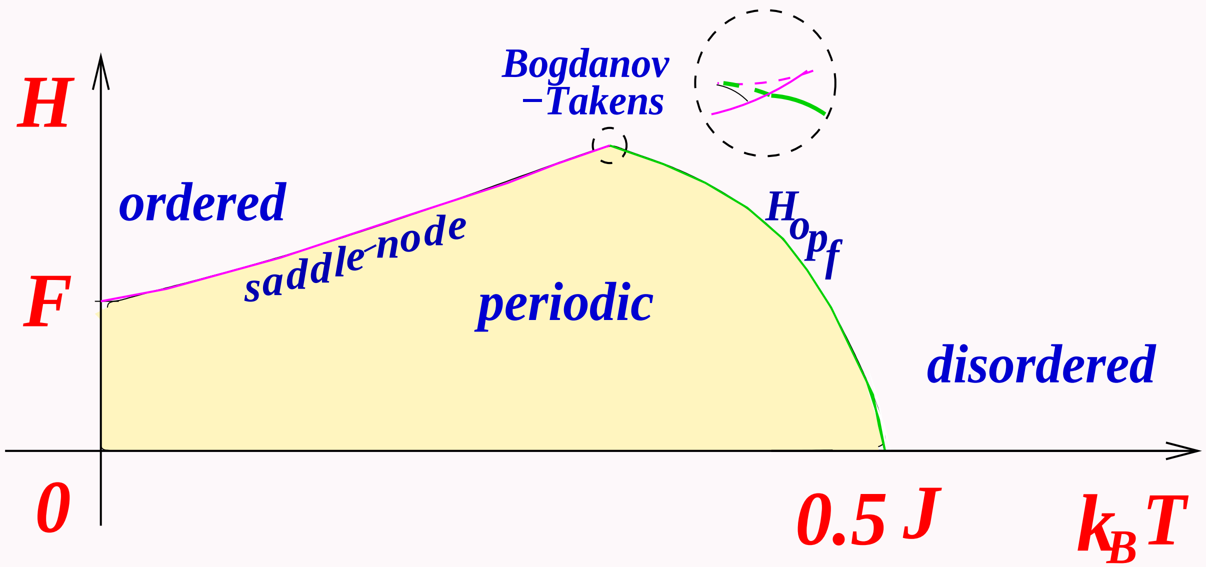
with $\quad x_1 = J\int_0^{2\pi}\cos\vartheta\,\rho(\vartheta)\,d\vartheta\,, \qquad x_2 = J\int_0^{2\pi}\sin\vartheta\,\rho(\vartheta)$

and the solution

$$\rho(\theta) = \frac{1}{Z}\,\mathrm{e}^{\frac{F\theta + (H+x_1)\cos\theta + x_2\sin\theta}{k_B T}}\int_\theta^{\theta+2\pi}\mathrm{e}^{-\frac{F\vartheta + (H+x_1)\cos\vartheta + x_2\sin\vartheta}{k_B T}}\,d\vartheta$$

- The coupled equations for 2 variables x_1, x_2 may be easily analyzed

- $N = \infty$ **phase diagram** for the **rotator model** for $F \neq 0$ (**Shinomoto**-**Kuuramoto** 1984, **Sakaguchi**-**Shin**.-**Kur**. 1986, ...)



- For $H = 0$ the periodic phase coincides with the ordered low-temp. equilibrium phase viewed in the co-rotating phase

- When $F \searrow 0$ the periodic phase reduces to the equilibrium disordered phase at $H = +0$

- Global properties of the **NFPE** dynamics for the **rotator model** have been recently studied by **Giacomin** and collaborators

- **Fluctuations for N large but finite**

  - Formally, domain of applications of the small-noise **Freidlin-Wentzell large deviations theory**

  - In **Martin-Rose-Siggia** formalism, the joint **PDF** of empirical density and current profiles is

$$
\begin{aligned}
\Big\langle \delta\big[\rho-\rho_N\big]\,\delta\big[j-j_N\big]\Big\rangle \;&=\; \Big\langle \delta\big[\partial_t\rho+\nabla\cdot j\big]\,\delta\big[j-j_\rho-\zeta_\rho\big]\Big\rangle \\
&=\; \Big\langle \delta\big[\partial_t\rho+\nabla\cdot j\big]\int \mathrm{e}^{\,iN\int a\cdot(j-j_\rho-\zeta_\rho)}\,\mathcal{D}a\Big\rangle \\
&=\; \delta\big[\partial_t\rho+\nabla\cdot j\big]\int \mathrm{e}^{\,iN\int a\cdot(j-j_\rho)\,-\,N\int a\cdot\rho D\,a}\,\mathcal{D}a \\
&\sim\; \delta\big[\partial_t\rho+\nabla\cdot j\big]\;\mathrm{e}^{-\frac{1}{4}N\int(j-j_\rho)(\rho D)^{-1}(j-j_\rho)} \;\sim\; \mathrm{e}^{-N\,\mathcal{I}[\rho,j]}
\end{aligned}
$$

  where the **rate function(al)**

$$
\mathcal{I}[\rho,j] \;=\begin{cases} \frac{1}{4}\int(j-j_\rho)(\rho D)^{-1}(j-j_\rho)\,dt\,dx & \text{if}\quad \partial_t\rho+\nabla\cdot j=0 \\ \infty & \text{otherwise}\end{cases}
$$

- Large-deviations rate function(al)s for empirical densities or empirical currents only

$$\left\langle \delta[\varrho - \rho_N] \right\rangle \underset{N\to\infty}{\sim} \mathrm{e}^{-N\mathcal{I}[\rho]} \qquad \left\langle \delta[j - j_N] \right\rangle \underset{N\to\infty}{\sim} \mathrm{e}^{-N\mathcal{I}[j]}$$

are obtained by the **contraction principle**

$$\mathcal{I}[\rho] = \min_j \mathcal{I}[\rho, j] = \frac{1}{4}\int (\partial_t \rho + \nabla\cdot j_\rho)(-\nabla\cdot\rho D\nabla)^{-1}(\partial_t\rho + \nabla\cdot j_\rho)\, dt\, dx$$

$$\mathcal{I}[j] = \min_\rho \mathcal{I}[\rho, j] \quad \text{with appropriate boundary limiting conditions for } \rho$$

  - That empirical densities have dynamical large deviations with rate function given above was proven by **Dawson-Gartner** in 1987
  - To our knowledge, the large deviations of currents for mean field models were not studied in math literature
  - The formulae above have similar form as for the macroscopic density and current rate functions in stochastic lattice gases studied by the Rome group and **B. Derrida** with collaborators

- **Elements of the** (Roman) **Macroscopic Fluctuation Theory**

  - **Instantaneous fluctuations of empirical densities**

    - Time t distribution of the empirical density

$$\mathcal{P}_t[\varrho] \;=\; \Big\langle \delta[\varrho - \rho_{Nt}] \Big\rangle \;\sim\; \mathrm{e}^{-N\mathcal{F}_t[\varrho]} \qquad \leftarrow$$

leading WKB term

satisfies the functional equation $\partial_t \mathcal{P}_t \;=\; \mathcal{L}_{Nt}^{\dagger} \mathcal{P}_t$ which reduces for the large-deviations rate function $\mathcal{F}_t[\varrho]$ to the functional **Hamilton-Jacobi Equation** (**HJE**)

$$\partial_t \mathcal{F}_t[\varrho] + \int j_\rho \cdot \nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho} + \int \left(\nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho} \right) \cdot \rho D \left(\nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho} \right) \;=\; 0$$

    - In a **stationary state** the latter becomes the time-independent **HJE** for the rate function $\mathcal{F}[\varrho]$

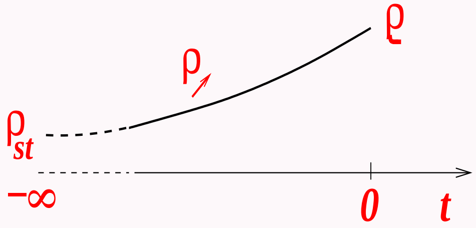
- **Relation between instantaneous and dynamical rate fcts**

  - By **contraction principle**

$$\mathcal{F}_t[\varrho] \;=\; \min_{\rho_t=\varrho}\Big(\mathcal{F}_{t_0}[\rho_{t_0}] + \mathcal{I}_{[t_0,t]}[\rho]\Big)$$

  - In the stationary state this reduces to

$$\mathcal{F}[\varrho] \;=\; \min_{\substack{\rho_{-\infty}=\rho_{st}\\ \rho_0=\varrho}} \mathcal{I}_{[-\infty,0]}[\rho]$$



where ρ_{st} is the stable stationary solution of **NFPE** minimizing $\mathcal{F}[\varrho]$

  - The minimum on the right is attained on the most probable (**Onsager**-**Machlup**) trajectory $\rho_{\nearrow}$ creating fluctuation ϱ from the "vacuum" ρ_{st}

- **Time reversal**

  - One defines the **time-reversed current** $j'_\rho(t,x)$ by

$$j'^{\,*}_{\rho^*} \;=\; j_\rho \,+\, 2\,\rho D\nabla\frac{\delta\mathcal{F}_t[\rho_t]}{\delta\varrho}$$

where $\rho^*(t,x)=\rho(-t,x)$ and $j^*(t,x)=-j(-t,x)$ and the **time-reversed process** in the density space by **Itô** eqn.

$$\partial_t\rho' \;+\; \nabla\cdot\big(j'_{\rho'} \,+\, \sqrt{2N^{-1}D'\rho'}\,\xi\big) \;=\; 0$$

with $D'(t,x)=D(-t,x)$

  - (**Gallavotti**-**Cohen**-type) **Fluctuation Relation**

$$\mathcal{I}_{[t_0,t_1]}[\rho,j]+\mathcal{F}_{t_0}[\rho_{t_0}]-\mathcal{F}_{t_1}[\rho_{t_1}] \;=\; \mathcal{I}'_{[-t_1,-t_0]}[\rho^*,j^*]$$

follows from the comparison of the direct and reversed rate functions and the **HJE** for $\mathcal{F}_t$

- **Generalized Onsager-Machlup Relation**

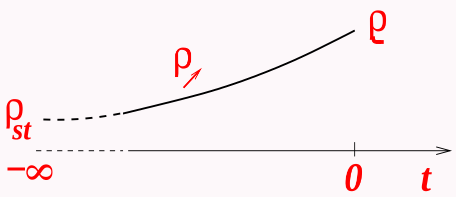
  - Upon minimizing over currents in a stationary state, Fluctuation Relation reduces to

$$\mathcal{I}_{[t_0,t_1]}[\rho] + \mathcal{F}[\rho_{t_0}] - \mathcal{F}[\rho_{t_1}] = \mathcal{I}'_{[-t_1,-t_0]}[\rho^*]$$

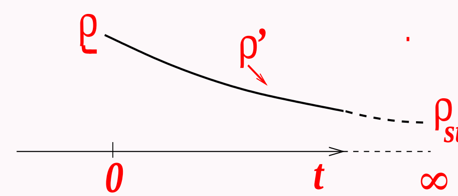
  - For $t_0 = -\infty$, $\rho_{t_0} = \rho_{st}$ and $t_1 = 0$, $\rho_{t_1} = \varrho$ the minimum of the **LHS** is attained on trajectory $\rho_{\nearrow}$ and is zero

  - It must be equal to the minimum of the **RHS** that is realized on trajectory $\rho'_{\searrow}$ that describes the decay of fluctuation ϱ to vacuum ρ_{st} and satisfies time-reversed **NFPE** $\partial_t \rho'_{\searrow} + \nabla \cdot j'_{\rho'_{\searrow}} = 0$

  - Hence the generalized **Onsager-Machlup** relation:



$$\rho_{\nearrow}(t,x) = \rho'_{\searrow}(-t,x)$$



- **Solutions for $\mathcal{F}_t$ in special cases**

  - For decoupled systems with $Y = 0$ and independent $x_n(0)$ all distributed with initial **PDF** ρ_0

    $$\mathcal{F}_t[\varrho] \;=\; \int \varrho(x)\,\ln\frac{\varrho(x)}{\rho_t(x)}\,dx \;\equiv\; k_B^{-1} S[\varrho\|\rho_t] \quad \leftarrow \textbf{relative entropy}$$

    where ρ_t solves the linear **FP** equation with initial condition ρ_0 (**Sanov Theorem**)

  - For stationary **equilibrium** evolutions with $\widehat{X}(x) = -M(x)\nabla U(x)$, $Y(x,y) = -M(x)(\nabla V)(x-y)$ and **diffusivity** and **mobility** matrices related by the **Einstein** relation $D(x) = k_B T\, M(x)$

    $$\mathcal{F}[\varrho] = \int \varrho(x)\Big(\frac{1}{k_B T}\Big(U(x) + \frac{1}{2}\int V(x,y)\,\rho(y)\,dy\Big) + \ln \varrho(x)\Big)\,dx + const.$$

    i.e. $k_B T\,\mathcal{F} = E - TS$ is the equilibrium mean-field **free energy** ($\Rightarrow$ **a well known large deviations interpretation of the latter**)

- **Perturbative calculation of the non-equilibrium free energy** $\mathcal{F}[\varrho]$

  - $\mathcal{F}[\varrho]$ may be expanded around its minimum ρ_{st} that is a stable stationary solution of **NFPE**

    $$\mathcal{F}[\varrho] \;=\; \sum_{k=1}^{\infty} \mathcal{F}^k[\widetilde{\varrho}]$$

    where $\widetilde{\varrho} = \varrho - \rho_{st}$ and

    $$\mathcal{F}^k[\widetilde{\varrho}] \;=\; \tfrac{1}{(k+1)!} \int \phi^k(x_0, \ldots, x_n)\, \widetilde{\varrho}(x_0) \cdots \widetilde{\varrho}(x_k)\, dx_0 \cdots dx_k$$

    with ϕ^k symmetric in the arguments and fixed by demanding that $\int \phi^k(x_0, x_1, \ldots, x_k)\, dx_0 = 0$

  - Kernels ϕ^k of $\mathcal{F}^k[\widetilde{\varrho}]$ may be represented in terms of a sum over tree diagrams that solves the recursion obtain by substituting the expansion into the stationary **HJE**

- The recursion has for $k > 1$ the form:

$$\int \widetilde{\varrho}\,\Phi R \Phi^{-1} \frac{\delta \mathcal{F}^k[\widetilde{\varrho}]}{\delta \widetilde{\varrho}} = \int \widetilde{\varrho}\Big[\big(Y * \widetilde{\varrho}\big)\cdot \nabla \frac{\delta \mathcal{F}^{k-1}[\widetilde{\varrho}]}{\delta \widetilde{\varrho}} + \sum_{l=1}^{k-1} \Big(\nabla \frac{\delta \mathcal{F}^{l}[\varrho]}{\delta \varrho}\Big) \cdot D \Big(\nabla \frac{\delta \mathcal{F}^{k-l}[\varrho]}{\delta \varrho}\Big)\Big] + \sum_{l=2}^{k-1} \int \Big(\nabla \frac{\delta \mathcal{F}^{l}[\widetilde{\varrho}]}{\delta \widetilde{\varrho}}\Big) \cdot \rho_{st} D \Big(\nabla \frac{\delta \mathcal{F}^{k+1-l}[\widetilde{\varrho}]}{\delta \widetilde{\varrho}}\Big)$$

where R is the linearization of the nonlinear **Fokker**-**Planck** operator around ρ_{st} and

$$\big(\Phi \widetilde{\varrho}\big)(x) = \int \phi^1(x,y)\, \widetilde{\varrho}(y)\, dy$$

solves the operator equation

$$R\Phi^{-1} + \Phi^{-1}R^\dagger = 2\nabla \cdot \rho D \nabla$$

(coming from the stochastic **Lyapunov** eqn.) and determines $\mathcal{F}^1[\widetilde{\varrho}]$

• Large deviations for currents

- Following the Romans, one defines for time-independent current $J(x)$

$$I_0[J] \quad = \quad \lim_{\tau \to \infty} \frac{1}{\tau} \min_{\substack{\rho(t,x),\, j(t,x) \\ J(x) = \frac{1}{\tau} \int_0^\tau j(t,x)\, dt}} \mathcal{I}_{[0,\tau]}[\rho, j]$$

- This is the rate function of large deviations for the temporal means J of current fluctuations

- In the stationary phase, for J close to $j_{st} = j_{\rho_{st}}$ the minimum is attained on time independent (ρ, j) so that

$$I_0[J] = \begin{cases} \min\limits_{\rho(x)} \frac{1}{4} \int (J - j_\rho)(\rho D)^{-1}(J - j_\rho)\, dx & \text{if} \quad \nabla \cdot J = 0 \\ \infty & \text{otherwise} \end{cases}$$

- This does not necessarily hold for all J

- In the periodic phase, it is more natural to fix the periodic means:

$$I_{\omega,\varphi}[J] \quad = \quad \lim_{\tau\to\infty} \frac{1}{\tau} \min_{\substack{\rho(t,x),\, j(t,x) \\ J(x)=\frac{1}{\tau}\int_0^\tau \sin(\omega t+\varphi)\, j(t,x)\, dt}} \mathcal{I}_{[0,\tau]}[\rho, j]$$

where ω is a multiple of the basic frequency

- **New phenomenon** that does not occur in equilibrium:

> At the 2^{nd} order non-equilibrium phase transitions the covariance of temporal averages of current fluctuations around j_{st} on the scale $\frac{1}{N\tau}$ diverges in special directions

$\Rightarrow$ amplification of current fluctuations around such transitions

- In other words, at such transition, the variance of the random variable

$$\frac{\sum_{n=1}^{N} \int_0^\tau A(t, x_n(t)) \circ dx_n(t) \; - \; \Big\langle \cdots \Big\rangle}{\sqrt{N\tau}}$$

(note the central-limit-like rescaling) diverges when $N, \tau \to \infty$ for some time-independent or periodic functions $A(t, x)$

- A somewhat related enhancement of fluctuations at the saddle-node transition of the **rotator model** was observed numerically and analyzed in **Ohta-Sasa**, Phys. Rev. E **78**, 065101(R) (2008), see also **Iwata-Sasa**, Phys. Rev. E. **82**, 011127 (2010)

- The simplest way to access the above variance is via the calculation of its inverse by expanding $I_0(J)$ or $I_{\omega,\varphi}(J)$ to the 2$^{\text{nd}}$ order around their minima

- To the 2$^{\text{nd}}$ order around (j_{st}, ρ_{st}) the rate functional $\mathcal{I}[\rho, j] = \frac{1}{4}\int (j - j_\rho)(\rho D)^{-1}(j - j_\rho)$ is

$$\mathcal{I}[\rho_{st} + \delta\rho, j_{st} + \delta j] = \frac{1}{4}\int (\delta j - S\delta\rho)(\rho_{st} D)^{-1}(\delta j - S\delta\rho)$$

for $\partial_t \delta\rho + \nabla \cdot \delta j = 0$ where $S(x.y) = \frac{\delta j_\rho(x)}{\delta\rho(y)}\Big|_{\rho=\rho_{st}}$

- The linearized **Fokker- Planck** operator is $R = -\nabla \cdot S$

- At critical points corresponding to a **saddle-node** or a **pitchfork bifurcations**, R has a zero mode $\delta\rho_0(x)$ and then for $(\delta\rho, \delta j) = (\delta\rho_0, S\delta\rho_0)$

$$\partial_t \delta\rho + \nabla \cdot \delta j = \nabla \cdot S\, \delta\rho_0 = -R\, \delta\rho_0 = 0 \qquad \text{and}$$

$$\delta j - S\delta\rho = 0$$

so that $\mathcal{I}[\rho_{st} + \delta\rho, j_{st} + \delta j]$, and consequently $I_0[j_{st} + \delta j]$, vanish to the 2$^{\text{nd}}$ order on such a perturbation

- At critical points corresponding to a **Hopf bifurcation**, R has complex conjugate modes $\delta\rho_0(x)$, $\overline{\delta\rho_0(x)}$ with eigenvalues $\pm i\omega$ and then for $\big(\delta\rho, \delta j\big) = \mathrm{Re}\,\big(\mathrm{e}^{i\omega(t+t_0)}\delta\rho_0,\, \mathrm{e}^{i\omega(t+t_0)}S\delta\rho_0\big)$ again

$$\partial_t\delta\rho + \nabla\cdot\delta j \;=\; 0 \qquad \text{and} \qquad \delta j - S\delta\rho \;=\; 0$$

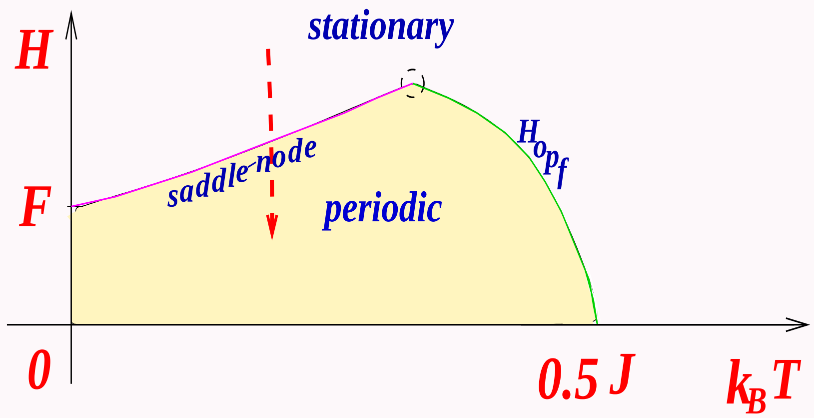
  and again $\mathcal{I}[\rho_{st}+\delta\rho, j_{st}+\delta j]$, and consequently $I_{\omega,\varphi}[\mathrm{Re}\,\mathrm{e}^{i\psi}S\delta\rho_0]$ for any phase ψ vanish to the 2^{nd} order

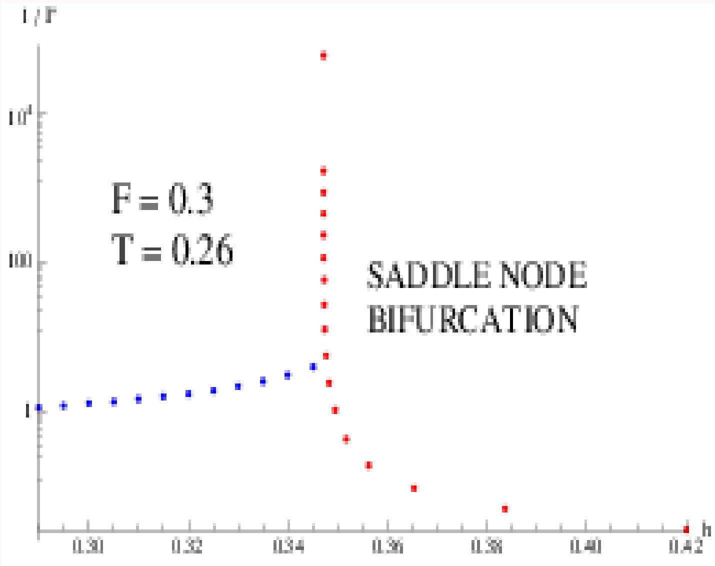
- Vanishing of I_0 or $I_{\omega,\varphi}$ to the 2^{nd} order around j_{st} means that the variance of current fluctuations in the corresponding direction diverges on the central-limit scale $\frac{1}{N\tau}$

- The reason is that such fluctuations are realized in $N=\infty$ dynamics

- In equilibrium, R cannot have non-zero imaginary eigenvalues and for its zero modes $\delta\rho_0$, one also has $S\delta\rho_0 = 0$, unlike in nonequilibrium where $\delta j = S\delta\rho_0$ represents a non-trivial current fluctuation

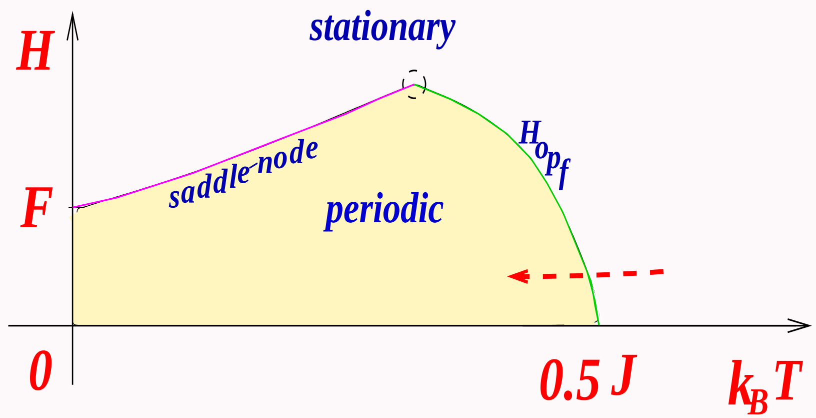
Example of the **rotator model** for $J = 1, \quad F = 0.5$

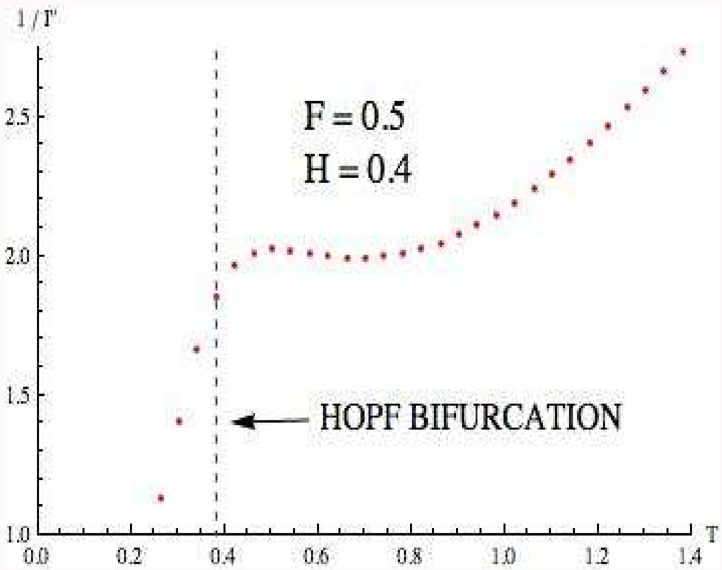




Right figure: the variance $1/I_0''[j_{st}]$ of current fluctuations as a function of magnetic field h in log-lin plot for $k_B T = 0.2$

- $1/I_0''[j_{st}]$ diverges at the saddle-node bifurcation for $h = h_{cr} \approx 0.56$ (the points for $h < h_{cr}$ correspond to an unstable stationary branch within the periodic phase)

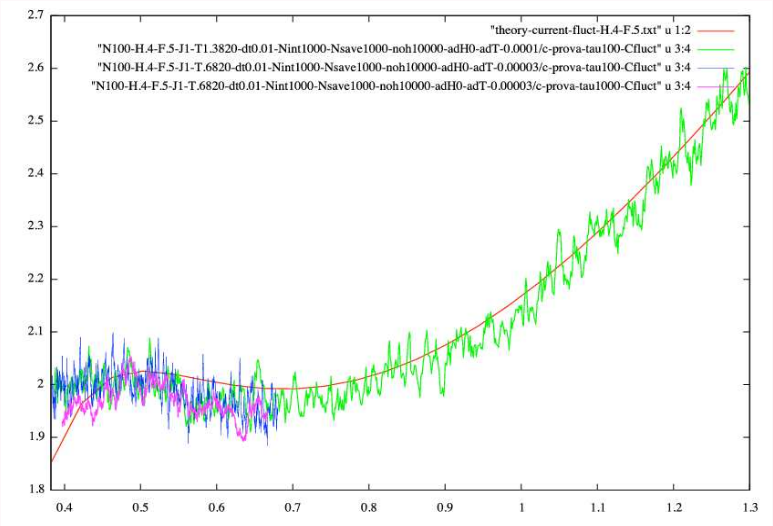




Right figure: the variance $1/I_0''[j_{st}]$ of the current fluctuations as a function of temperature $k_B T$ in lin-lin plot for $h = 0.2$

- $1/I_0''[j_{st}]$ is regular near the Hopf bifurcation at $T = T_{cr} \approx 0.5$ (again, the $T < T_c$ curve corresponds to an unstable stationary branch within the periodic phase)

- **Comparison to finite N simulations**
  - Divergence of the variance of current fluctuations around the saddle-node bifurcation is difficult to see in DNS as it occurs in a narrow window of h
  - Its theoretical behavior around the Hopf bifurcation is easier to reproduce for finite N



Variance of current fluctuations over times $\tau = 100$ and $\tau = 1000$ for 10^4 histories of $N = 100$ rotators compared to the theoretical $N = \infty, \tau = \infty$ curve

Conclusions and open problems

- Diffusions with mean-field coupling are a good laboratory for non-equilibrium statistical mechanics
- At $N = \infty$ they are described by the non-linear **Fokker-Planck** equation and may exhibit interesting phase diagrams with dynamical phase transitions.
- For large but finite N the large deviations of their empirical densities and currents are described by rate functionals similar to those for stochastic lattice gases, governed by **Macroscopic Fluctuation Theory**
- In particular, the non-equilibrium free energy solves a functional **Hamilton-Jacobi** equation and may be studied in perturbation theory
- Unlike in equilibrium, the covariance of current fluctuations diverges in specific directions at the 2^{nd} order transition points of such systems
- Similar methods should apply to underdamped diffusions with mean-field coupling leading at $N = \infty$ to **Vlasov-Fokker-Planck** equation. We hope also to apply them to randomly forced 2D **Navier-Stokes** eqns.