Universal current fluctuations in non-equilibrium systems

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Joint work with

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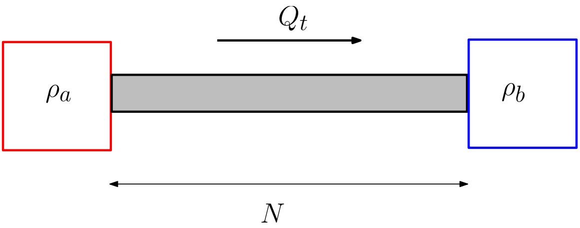
Outline

- Universality of the Fano factor
- Universality of the large deviations for the current

Open system with reservoirs



Q_t current of particles during the time interval $[0, t]$

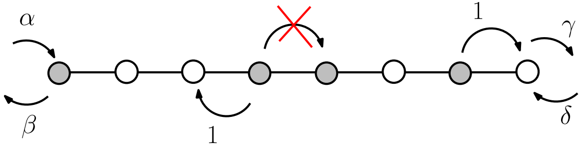
Diffusive systems : $\langle Q_t \rangle \simeq \frac{C(\rho_a, \rho_b)}{N} t$ (Fick's law)

Question : Distribution of the current ?

- Cumulants
- Large deviations

Symmetric Simple Exclusion Process

Particles : $\eta(t) = \{\eta_i(t)\}_{i \leq N} \in \{0,1\}^N$



Reservoir Domain with N sites Reservoir

Left reservoir acting at site 1

➪ particle creation at rate α

➪ particle annihilation at rate β

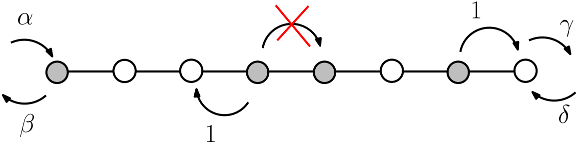
Right reservoir acting at site N

➪ particle creation at rate δ

➪ particle annihilation at rate γ

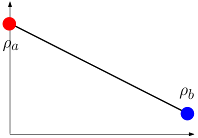
Symmetric Simple Exclusion Process

Particles : $\eta(t) = \{\eta_i(t)\}_{i \leq N} \in \{0,1\}^N$



Reservoir Domain with N sites Reservoir

Macroscopic scale



Linear density profile (Fick's law)

$$\rho_a = \frac{\alpha}{\alpha+\beta}, \quad \rho_b = \frac{\delta}{\delta+\gamma}$$

Current

Q_t = Number of jumps from i to $i+1$ up to time t
− Number of jumps from $i+1$ to i up to time t

Cumulants [Derrida, Douçot, Roche]

$$\lim_{t\to\infty} \frac{\langle Q_t \rangle}{t} \simeq \frac{1}{N}(\rho_a - \rho_b)$$

$$\lim_{t\to\infty} \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{t} \simeq \frac{1}{N}\left(\rho_a + \rho_b - \frac{2}{3}(\rho_a^2 + \rho_a\rho_b + \rho_b^2)\right)$$

Fano Factor

$$F_N = \lim_{t\to\infty} \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{\langle Q_t \rangle}$$

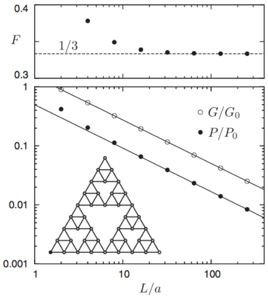
For $\rho_a = 1$ and $\rho_b = 0$ then $\lim_{N\to\infty} F_N = 1/3$

SSEP on a Sierpinski lattice

$(\rho_a = 1, \rho_b = 0)$

$$\lim_{N\to\infty} F_N = 1/3$$

[Groth, Tworzydlo, Beenakker]



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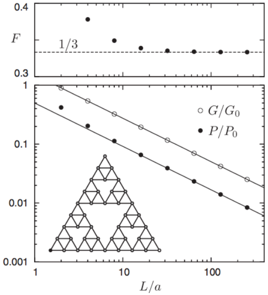
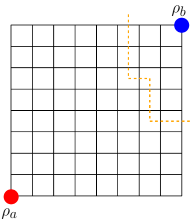
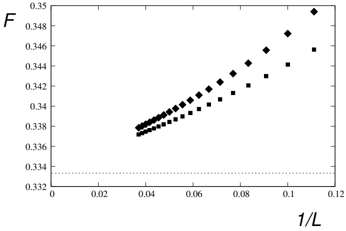
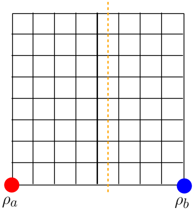


SSEP in 2D $(\rho_a = 1,\ \rho_b = 0)$ [Akkermans, B, Derrida, Shpielberg]

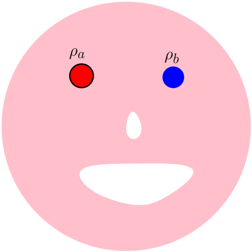






Question : Universality of the Fano factor ?

- Geometry of the domain ?
- Dimension ?
- $0 < \rho_b < \rho_a < 1$?



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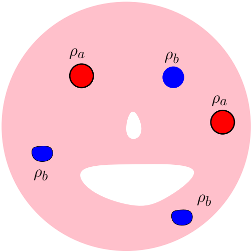

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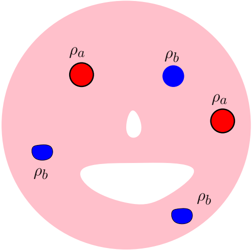

➪ Beyond the fano factor ? $\lim_{t\to\infty} \frac{\langle Q_t^3 \rangle_c}{\langle Q_t \rangle}$, $\lim_{t\to\infty} \frac{\langle Q_t^4 \rangle_c}{\langle Q_t \rangle} \ldots$

➪ Other microscopic dynamics ?

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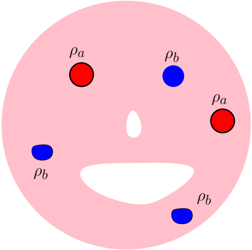

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➪ Other microscopic dynamics ?

Claim. Universality of the large deviations.

Large deviations

Probability of observing a current $\mathcal{J} \neq$ mean current ?

N large and $t \to \infty$ $$\left\langle \frac{Q_t}{t} \sim \frac{\mathcal{J}}{N} \right\rangle \approx \exp\left(-\frac{t}{N} G(\mathcal{J})\right)$$

Equivalently

$$\left\langle \exp\left(\lambda Q_t\right) \right\rangle \approx \exp\left(\frac{t}{N}\mu(\lambda)\right)$$

For the 1D SSEP :

$$\lim_{N\to\infty} \lim_{t\to\infty} \frac{N}{t} \log \left\langle \exp(\lambda Q_t) \right\rangle = \left(\mathrm{Arcsinh}(\sqrt{\omega})\right)^2$$

with

$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a(e^{\lambda} - 1)\rho_b(e^{-\lambda} - 1)$$

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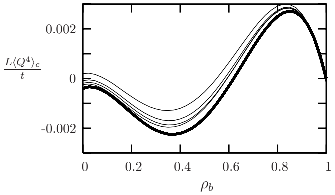
Formal derivation of the cumulants

$$\lim_t \frac{\langle Q_t^k \rangle_c}{t} = \frac{1}{N} \partial^k \mu \Big|_{\lambda=0}$$



$N = 5, 9, 13, 17 \quad (\rho_a = 1)$

Large deviations 1D SSEP

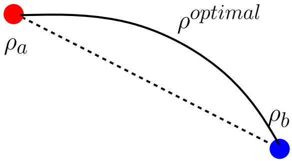
$$\mu_1(\lambda) = \lim_{T\to\infty} \lim_{N\to\infty} -\frac{N}{TN^2} \log \left\langle \exp\left(\lambda Q_{TN^2}\right) \right\rangle$$

with

$$\mu_1(\lambda) = \inf_{\mathcal{J},\rho} \left\{ \lambda \mathcal{J} - \int_0^1 dx \, \frac{(\mathcal{J} + \rho'_x)^2}{4\rho_x(1-\rho_x)} \right\} \quad \text{with} \quad \begin{array}{l} \rho(0) = \rho_a \\ \rho(1) = \rho_b \end{array}$$

[B, Derrida], [Bertini,De Sole,Gabrielli,Jona Lasinio,Landim]

Variational problem
$\Rightarrow$ coupling density/current



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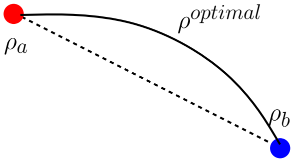


For SSEP **time independent** variational problem

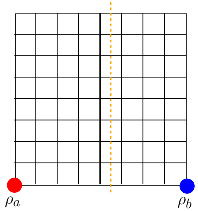
SSEP in higher dimension

Microscopic description

- $\eta_i(t)$ for $i \in \mathbb{Z}^d$
- Edge current $Q_t(i,j)$
- Sources at A, B

Macroscopic description for $x \in \mathbb{R}^d$

- Density profile $\rho(x,t)$
- Current $q(x,t) = \big(q_k(x,t)\big)_{k \le d}$
- Conservation law $\partial_t \rho = -\nabla_x \cdot q$



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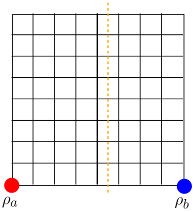
Total current

$$Q_t = \sum_{i,j} \frac{V_i - V_j}{2} Q_t(i,j)$$

$$\forall i, \quad \sum_{j \sim i} V_j - V_i = 0$$

At the sources : $V_A = 1,\ V_B = 0$



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Total current

$$Q_t = \sum_{i,j} \frac{V_i - V_j}{2} Q_t(i,j)$$

$$q_t = \frac{1}{2} \int_{\mathbb{D}} dx \; \nabla_x v(x) \cdot q(x,t)$$

$$\forall i, \quad \sum_{j \sim i} V_j - V_i = 0$$

At the sources : $V_A = 1,\ V_B = 0$

$$\Delta v(x) = 0 \quad \text{and } v(A) = 1, v(B) = 0$$

A, B : **macroscopic contacts**

SSEP in higher dimensions

Macroscopic domain $\mathbb{D} \subset \mathbb{R}^d$ with macroscopic contacts at A, B

$$\mu(\lambda) = \lim_{T\to\infty} \lim_{N\to\infty} -\frac{1}{TN^2 N^{d-2}} \log \left\langle \exp\left(\lambda Q_{TN^2}\right) \right\rangle$$

with

$$\mu(\lambda) = \inf_{q,\rho} \left\{ \int_{\mathbb{D}} dx \left[\frac{\lambda}{2} \nabla_x v(x) \cdot q(x) - \sum_{k \le d} \frac{\left(q_k(x) + \partial_k \rho(x)\right)^2}{4\rho_x(1-\rho_x)} \right] \right\}$$

with boundary conditions $\rho(A) = \rho_a$ and $\rho(B) = \rho_b$

[Bertini,De Sole,Gabrielli,Jona Lasinio,Landim]
[B., Derrida, Lebowitz]
[Hurtado, Espigares, Del Pozi, Garrido]

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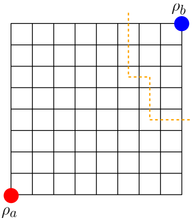
Claim [Akkermans, B, Derrida, Shpielberg]

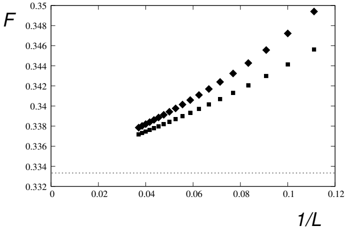
$$\mu(\lambda) = C(\mathbb{D})\mu_1(\lambda) \qquad \textit{optimal density}: \ \rho(x) = \rho_1(v(x))$$

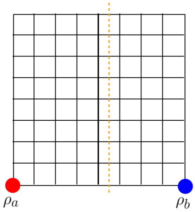
where $C(\mathbb{D}) = \int_{\mathbb{D}} (\nabla_x v)^2$ with $\Delta v = 0, \quad v(A) = 1, v(B) = 0.$

Structure of the contacts

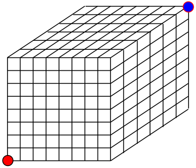
SSEP in 2D ($\rho_a = 1$, $\rho_b = 0$)

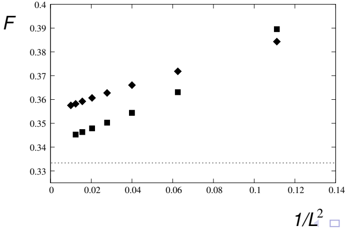


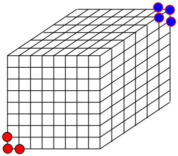




SSEP in 3D ($\rho_a = 1$, $\rho_b = 0$)







For general 1D diffusive dynamics :

- Diffusion constant $D(\rho)$: $\rho_a = \rho + \varepsilon, \rho_b = \rho$

$$\lim_{t\to\infty} \left\langle \frac{Q_t}{\tau} \right\rangle \approx D(\rho)\frac{\varepsilon}{N} \qquad \text{for SSEP :} \quad D(\rho) = 1$$

- Conductivity $\sigma(\rho)$: $\rho_a = \rho_b = \rho$

$$\lim_{t\to\infty} \left\langle \frac{(Q_t)^2}{t} \right\rangle \approx \sigma(\rho)\frac{1}{N} \qquad \text{for SSEP :} \quad \sigma(\rho) = 2\rho(1-\rho)$$

For isotropic dynamics in higher dimensions

$$\mu(\lambda) = \lim_{T} \inf_{q,\rho} \left\{ \frac{1}{T} \int_0^T dt \int_{\mathbb{D}} dx \left[\frac{\lambda}{2} \nabla_x v(x,t) \cdot q(x,t) - \frac{\left(q(x,t) + D(\rho(x,t))\nabla\rho(x,t)\right)^2}{4\sigma\left(\rho(x,t)\right)} \right] \right\}$$

If the large deviations are time independent

$$\mu(\lambda) = \inf_{q,\rho} \left\{ \int_{\mathbb{D}} dx \left[\frac{\lambda}{2} \nabla_x v(x) \cdot q(x) - \frac{\left(q(x) + D(\rho(x)) \nabla \rho(x)\right)^2}{4\sigma\left(\rho(x)\right)} \right] \right\}$$

Then there is a **universal** structure

$$\mu(\lambda) = C(\mathbb{D}) \mu_1(\lambda) \qquad \textit{optimal density}: \ \rho(x) = \rho_1(v(x))$$

where $C(\mathbb{D}) = \int_{\mathbb{D}} (\nabla_x v)^2$ with $\Delta v = 0, \quad v(A) = 1, v(B) = 0.$

[Sukhorukov, Loss]

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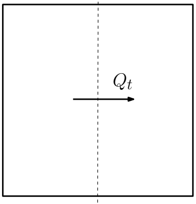
[Sukhorukov, Loss]

Universality of the cumulants [B., Derrida 2004]

$$I_n = \int_{\rho_b}^{\rho_a} D(u)\sigma(u)^{n-1} du \qquad \frac{\langle Q_t \rangle}{t} \simeq C(\mathbb{D}) N^{d-2} I_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} \simeq C(\mathbb{D}) N^{d-2} \frac{I_2}{I_1} \qquad \frac{\langle Q_t^3 \rangle_c}{t} \simeq C(\mathbb{D}) N^{d-2} \frac{3(I_3 I_1 - I_2^2)}{I_3} \dots$$

Higher dimensions



Two-dimensional SSEP on the periodic domain $\Lambda_N = \{1, \cdots, N\}^2$

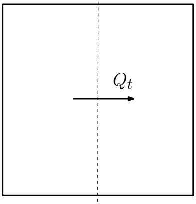
$G(\mathcal{J})$ large deviation function of the total current in the direction $\vec{e}_1$

$$G(\mathcal{J}) = \lim_{T\to\infty} \lim_{N\to\infty} -\frac{1}{TN^2} \log \mathbb{P}_{[0,TN^2]}\left(\frac{Q_{TN^2}}{TN^2} \sim \mathcal{J} \right)$$

Macroscopic current : $q(x,t) = \binom{q^1_{(x,t)}}{q^2_{(x,t)}}$

Current constraint : $\mathcal{J} = \frac{1}{T}\int_0^T dt \int_0^1 dx_2\, q^1_{\left((1/2,x_2),t\right)}$

Higher dimensions



Two-dimensional SSEP on the periodic domain $\Lambda_N = \{1, \cdots, N\}^2$

$G(\mathcal{J}) = \lim_{T\to\infty} \inf_\rho \left\{ \frac{1}{T} \mathcal{I}_{[0,T]}(\rho) \right\}$

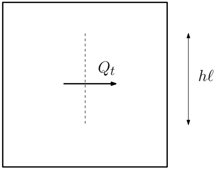
$$\mathcal{I}_{[0,T]}(\rho) = \int_0^T dt \int_\Lambda dx \, \frac{(q^1_{(x,t)} + \partial_{x_1}\rho_{(x,t)})^2}{2\sigma(\rho_{(x,t)})} + \frac{(q^2_{(x,t)} + \partial_{x_2}\rho_{(x,t)})^2}{2\sigma(\rho_{(x,t)})}$$

with $\partial_t \rho = -\mathrm{div}(q)$ and $q(x,t) = \begin{pmatrix} q^1_{(x,t)} \\ q^2_{(x,t)} \end{pmatrix}$

Minimization under the constraint : $\mathcal{J} = \frac{1}{T} \int_0^T dt \int_0^1 dx_2 \, q^1_{((\frac{1}{2}, x_2), t)}$

$$G(\mathcal{J}) > 0 \quad \text{for} \quad \mathcal{J} \neq 0$$

Higher dimensions : partial current



Slit $\ell = hN$ with $0 < h < 1$

Large deviations of the partial current Q_t^ℓ

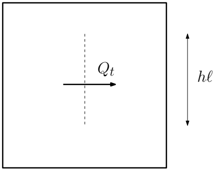
$G^h(\mathcal{J})$ large deviation function of the total current in the direction $\vec{e}_1$

$$G^h(\mathcal{J}) = \lim_{T\to\infty} \lim_{N\to\infty} -\frac{1}{TN^2} \log \mathbb{P}_{[0,TN^2]} \left(\frac{Q^\ell_{TN^2}}{TN^2} \sim \mathcal{J} \right)$$

Macroscopic current : $q(x,t) = \binom{q^1(x,t)}{q^2(x,t)}$

Current constraint : $\mathcal{J} = \frac{1}{T} \int_0^T dt \int_0^h dx_2 \, q^1_{((1/2,x_2)),t}$

Higher dimensions : partial current



Slit $\ell = hN$ with $0 < h < 1$

$G^h(\mathcal{J}) = \lim_{T\to\infty} \inf_\rho \left\{ \frac{1}{T}\mathcal{I}_{[0,T]}(\rho) \right\}$

$$\mathcal{I}_{[0,T]}(\rho) = \int_0^T dt \int_\Lambda dx \, \frac{(q^1_{(x,t)} + \partial_{x_1}\rho_{(x,t)})^2}{2\sigma(\rho_{(x,t)})} + \frac{(q^2_{(x,t)} + \partial_{x_2}\rho_{(x,t)})^2}{2\sigma(\rho_{(x,t)})}$$

with $\partial_t \rho = -\mathrm{div}(q)$ and $\mathcal{J} = \frac{1}{T}\int_0^T dt \int_0^h dx_2 \, q^1_{\left((1/2,x_2),t\right)}$

Claim. [B, Derrida, Lebowitz]

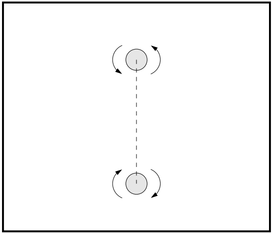
Fix $h < 1$. For any $\mathcal{J} \neq 0$ then $G^h(\mathcal{J}) = 0$

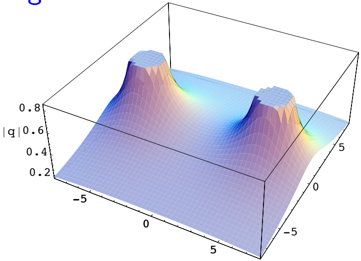
Current Vortices

$$G^h(\mathcal{J}) = \lim_{T\to\infty} \inf_{\rho} \left\{ \frac{1}{T} \mathcal{I}_{[0,T]}(\rho) \right\} \leq \inf_{q(x)} \frac{1}{2\sigma(\bar{\rho})} \int_\Lambda dx \, (q^1_{(x)})^2 + (q^2_{(x)})^2$$

with the constraint $\mathcal{J} = \int_0^h dx_2 \, q^1_{((1/2, x_2))}$

$\Rightarrow$ Current vortices localized at the edges of the slit





Conclusion

Summary

- Universal current large deviations (in the macroscopic limit)
- SSEP and other diffusive dynamics
- Predictions for all the cumulants

Open problems

- Time dependence ; Non uniqueness of the minimizers
- Structure of the contacts