First passage fluctuation relations ruled by cycles affinities

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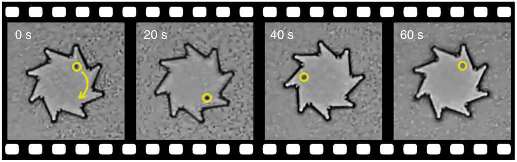
STOCHASTIC PROCESSES OF INTEREST

Semi-Markovian property

1.1 Example of processes of interest : a bacterial ratchet motor

*Di Leonardo & al. PNAS, **107** 9541 (2010)*

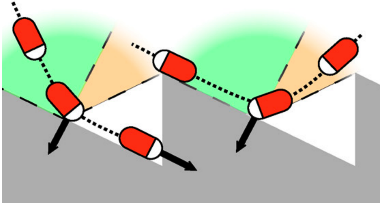
- **Experiment** : asymmetric gear (diameter : 48 μm, thickness 10 μm)
  in active bath of self-propelling bacteriae.



α_t: angle of black spot position at time t

$$\langle \frac{\alpha_t}{t} \rangle = 1 \textit{ revolution per minute}$$

- **Physical mechanism**



white "head" : self-propulsion direction

- perpendicular wall reaction reorients bacteria motion
- either bacteria slides to corner
  $\longrightarrow$ gets stuck $\longrightarrow$ torque

  or bacteria slides away from corner
  $\longrightarrow$ no torque

1.2 Modelization by a finite state semi-Markovian process

- **Finite number of configurations** $\mathcal{C}_m$:
  discretized values of angle α of black spot position : $\mathcal{C}_m \equiv \alpha_m = m2\pi/M$

- **Semi-Markovian process** (or **generalized renewal sequence**) :

$$\text{History :}\ \big((\mathcal{C}^0,\tau^0),(\mathcal{C},\tau^0+\tau),(\mathcal{C}',\tau 0+\tau+\tau'),\ldots\big)$$

After a waiting time τ distributed with probability $P_{\mathcal{C}}(\tau)$,
system jumps from $\mathcal{C}$ to $\mathcal{C}'$ with probability $(\mathcal{C}'|\mathbb{P}|\mathcal{C})$
($\mathbb{P}$ stochastic matrix with quantum mechanics convention for sense of evolution)

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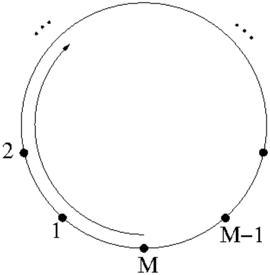
- **Graph representation** :



vertex • : $\begin{cases} \text{configuration } \mathcal{C} \\ \text{weight for waiting time at } \mathcal{C} : \end{cases}$
  - $P^0_{\mathcal{C}}(\tau)$ if $\mathcal{C}$ initial configuration of history
  - $P_{\mathcal{C}}(\tau)$ otherwise

bond —— : probability $(\mathcal{C}'|\mathbb{P}|\mathcal{C})$ to jump from $\mathcal{C}$ to $\mathcal{C}'$
when a jump is known to occur
and probability $(\mathcal{C}|\mathbb{P}|\mathcal{C}')$ of reverse jump

1.3 Questions

1) Probability that the cycle be performed at least once in positive (negative) sense in a infinite time interval ?

2) Fluctuation relation for first passage time at winding number $+1$ or -1 ?

winding number $=$ number of revolutions in the positive sense minus number of revolutions in the opposite sense

Answers use affinity concept

AFFINITY and ENTROPY PRODUCTION RATE

Known results for Markovian processes

2.1 Specific case : Markovian processes

- **Markov property** : specific form for probability of waiting time τ in configuration $\mathcal{C}$: exponential

$$P_{\mathcal{C}}(\tau) = r(\mathcal{C})e^{-r(\mathcal{C})\tau}$$

$r(\mathcal{C})$ escape rate from $\mathcal{C}$ = inverse mean waiting time at $\mathcal{C}$

- **From a Markov chain to a Markov process** :

$(\mathcal{C}'|\mathbb{P}|\mathcal{C})$ probability to jump from $\mathcal{C}$ to $\mathcal{C}'$ knowing that system jumps out of $\mathcal{C}$

$\longrightarrow(\mathcal{C}'|\mathbb{W}|\mathcal{C})dt$ probability to jump from $\mathcal{C}$ to $\mathcal{C}'$ during dt

- **Master equation** for evolution of probability $P(\mathcal{C};t)$ of configuration $\mathcal{C}$ at t

$$\frac{dP(\mathcal{C};t)}{dt} = \sum_{\mathcal{C}'\neq\mathcal{C}} [(\mathcal{C}|\mathbb{W}|\mathcal{C}')P(\mathcal{C}';t) - (\mathcal{C}'|\mathbb{W}|\mathcal{C})P(\mathcal{C};t)]$$

- **Microreversibility hypothesis** : $(\mathcal{C}'|\mathbb{W}|\mathcal{C}) \neq 0 \quad \Leftrightarrow \quad (\mathcal{C}|\mathbb{W}|\mathcal{C}') \neq 0$

2.2 Shannon-Gibbs entropy evolution and irreversibility

- **Dimensionless Shannon-Gibbs entropy** $(k_B = 1)$

$$S^{SG}[P(t)] \equiv -\sum_{\mathcal{C}} P(\mathcal{C};t) \ln P(\mathcal{C};t)$$

$$\frac{dS^{SG}}{dt} = \sum_{\mathcal{C},\mathcal{C}'} (\mathcal{C}'|\mathbb{W}|\mathcal{C}) P(\mathcal{C};t) \ln \frac{P(\mathcal{C};t)}{P(\mathcal{C}';t)}$$

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- **Analogy with phenomenological thermodynamics of irreversible processes**

[Schnakenberg 1976]

$$\frac{dS^{SG}}{dt} = \frac{d_{\text{exch}} S^{SG}}{dt} + \frac{d_{\text{irr}} S^{SG}}{dt}$$

$$\frac{d_{\text{exch}} S^{SG}}{dt} \equiv -\sum_{\mathcal{C},\mathcal{C}'} (\mathcal{C}'|\mathbb{W}|\mathcal{C}) P(\mathcal{C};t) \ln \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')} \quad \text{with no definite sign}$$

$$\frac{d_{\text{irr}} S^{SG}}{dt} \equiv \frac{1}{2} \sum_{\mathcal{C},\mathcal{C}'} \left[(\mathcal{C}'|\mathbb{W}|\mathcal{C}) P(\mathcal{C};t) - (\mathcal{C}|\mathbb{W}|\mathcal{C}') P(\mathcal{C}';t)\right] \ln \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C}) P(\mathcal{C};t)}{(\mathcal{C}|\mathbb{W}|\mathcal{C}') P(\mathcal{C}';t)} \geq 0$$

$\frac{d_{\text{irr}} S^{SG}}{dt}$: irreversible entropy production rate

2.3 Comparison with kinetic theory : affinity of a chemical reaction (a)

- In a vessel with walls at inverse temperature β and exerting pressure P, one introduces species A and B prepared separately at (β, P)

$$\text{reversible reaction : } A \rightleftharpoons B$$

- **Phenomenological thermodynamics of irreversible processes**

$$\underbrace{\frac{d_{\text{irr}} S^{\text{ph}}}{dt}}_{\textbf{entropy production rate}} = \underbrace{\beta(\mu_A - \mu_B)}_{\textbf{affinity } A_{A \rightleftharpoons B}} \times \underbrace{\frac{dn_B^{A \rightleftharpoons B}}{dt}}_{\textbf{reaction extent rate } J_{A \rightleftharpoons B}}$$

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• **Kinetic theory** : $\dfrac{dn_B^{A\rightleftharpoons B}}{dt} = k_{B\leftarrow A}n_A - k_{A\leftarrow B}n_B$ with $k_{j\leftarrow i}$: kinetic constants

• **Thermodynamics of ideal solutions** : $n_i \propto e^{\beta\mu_i}$ and $\mu_A^{\rm eq} = \mu_B^{\rm eq} \rightarrow$

$$\beta(\mu_A - \mu_B) = \ln\frac{k_{B\leftarrow A}n_A}{k_{A\leftarrow B}n_B}$$

2.3 Comparison with kinetic theory : affinity of a chemical reaction (b)

- Correspondance:

$$\text{concentration } n_i(t) \longrightarrow P(\mathcal{C};t) \text{ configuration probability}$$

$$\text{kinetic constant } k_{j\leftarrow i} \longrightarrow (\mathcal{C}'|\mathbb{W}|\mathcal{C}) \text{ transition rate}$$

$\longrightarrow$ Rewriting

$$\frac{d_{\text{irr}}S^{SG}}{dt} = \frac{1}{2}\sum_{\mathcal{C},\mathcal{C}'} J_{\mathcal{C}\rightleftharpoons\mathcal{C}'} A_{\mathcal{C}\rightleftharpoons\mathcal{C}'}$$

bond current $\quad J_{\mathcal{C}\rightleftharpoons\mathcal{C}'} \equiv (\mathcal{C}'|\mathbb{W}|\mathcal{C})P(\mathcal{C};t) - (\mathcal{C}|\mathbb{W}|\mathcal{C}')P(\mathcal{C}';t)$

bond affinity $\quad A_{\mathcal{C}\rightleftharpoons\mathcal{C}'} \equiv \ln \dfrac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})P(\mathcal{C};t)}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')P(\mathcal{C}';t)}$

2.4 Affinity for a master equation corresponding to a graph made of a single cycle

- **Representation of a master equation by a graph**
  Graph **G** : vertex • : configuration $\mathcal{C}$
  bond —— : transtion rates $(\mathcal{C}'|\mathbb{W}|\mathcal{C})$ and $(\mathcal{C}|\mathbb{W}|\mathcal{C}')$

- Case where graph **G** is a **cycle C of M vertices**.
  Fixed orientation along **C** with $\mathcal{C}_{M+1} \equiv \mathcal{C}_1$

cycle affinity $\quad A_{\mathbf{C}} \equiv \sum_{m=1}^{M} A_{\mathcal{C}_m \rightleftharpoons \mathcal{C}_{m+1}}$ with $A_{\mathcal{C}_m \rightleftharpoons \mathcal{C}_{m+1}} \equiv \ln \frac{(\mathcal{C}_{m+1}|\mathbb{W}|\mathcal{C}_m) P(\mathcal{C}_m;t)}{(\mathcal{C}_m|\mathbb{W}|\mathcal{C}_{m+1}) P(\mathcal{C}_{m+1};t)}$

$$A_{\mathbf{C}} = \ln \prod_{m=1}^{M} \frac{(\mathcal{C}_{m+1}|\mathbb{W}|\mathcal{C}_m)}{(\mathcal{C}_m|\mathbb{W}|\mathcal{C}_{m+1})} \quad \textit{independent from } P(\mathcal{C}, t)$$

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- **Property of stationary state** $P_{\mathrm{st}}(\mathcal{C})$

  Cycle current : $J_{\mathbf{C}}[P_{\mathrm{st}}] \equiv J_{\mathcal{C}_1 \rightleftharpoons \mathcal{C}_2}[P_{\mathrm{st}}] = J_{\mathcal{C}_2 \rightleftharpoons \mathcal{C}_3}[P_{\mathrm{st}}] = \cdots$

  Entropy production rate: $\left.\frac{d_{\mathrm{irr}} S^{SG}}{dt}\right|_{P_{\mathrm{st}}} = J_{\mathbf{C}}[P_{\mathrm{st}}]\, A_{\mathbf{C}}$

2.5 Affinity class in graph theory

- **Exchange processes in configuration jumps** $\leftrightarrow$ antisymmetric matrices
  - $\mathbb{S}$ for the exchange entropy variation
  - $\mathbb{A}$ for the affinity variation

$$(\mathcal{C}'|\mathbb{S}|\mathcal{C}) \equiv \ln \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')} \quad \textit{and} \quad (\mathcal{C}'|\mathbb{A}^{[P]}|\mathcal{C}) \equiv \ln \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})P(\mathcal{C};t)}{(\mathcal{C}|\mathbb{W}|\mathcal{C}')P(\mathcal{C}';t)} \equiv A_{\mathcal{C}\rightleftharpoons\mathcal{C}'}$$

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- For any $P(\mathcal{C};t)$ $\quad (\mathcal{C}'|\mathbb{A}^{[P]}|\mathcal{C}) - (\mathcal{C}'|\mathbb{S}|\mathcal{C}) = -\ln P(\mathcal{C}') + \ln P(\mathcal{C})$

$\longrightarrow$ For any $P(\mathcal{C};t)$, $\mathbb{A}^{[P]}$ in **cohomology class of** $\mathbb{S}$:
set of antisymmetric $\mathbb{Q}$ such that "integration" along any cycle subgraph **C** gives the same result as for $\mathbb{S}$

$$\forall \mathbf{C} \quad \sum_{m=1}^{M} (\mathcal{C}_{m+1}|\mathbb{Q}|\mathcal{C}_m) = \sum_{m=1}^{M} (\mathcal{C}_{m+1}|\mathbb{S}|\mathcal{C}_m) = \sum_{m=1}^{M} \ln \frac{(\mathcal{C}_{m+1}|\mathbb{W}|\mathcal{C}_m)}{(\mathcal{C}_m|\mathbb{W}|\mathcal{C}_{m+1})} \equiv A_{\mathbf{C}}$$

$\longrightarrow$ cohomology class of $\mathbb{S}$ called "affinity class"

AFFINITY CLASS INVARIANCE

under probabilistic constructions

3.1 From a Markov process to a Markov chain

- **Hypothesis** : **G** connected :
  $\rightarrow$ no absorption configuration : $r(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} (\mathcal{C}'|\mathbb{W}|\mathcal{C}) \neq 0$ for all $\mathcal{C}$

  $(\mathcal{C}'|\mathbb{W}|\mathcal{C})dt$ probability to jump from $\mathcal{C}$ to $\mathcal{C}'$ during dt

$\longrightarrow$ $(\mathcal{C}'|\mathbb{P}|\mathcal{C})$ probability to jump from $\mathcal{C}$ to $\mathcal{C}'$ knowing that system jumps out of $\mathcal{C}$

$$\textit{for} \quad \mathcal{C}' \neq \mathcal{C} \quad (\mathcal{C}'|\mathbb{P}|\mathcal{C}) = \frac{(\mathcal{C}'|\mathbb{W}|\mathcal{C})}{r(\mathcal{C})}$$

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- **Comparison of cycle affinities**

$$\text{cycle affinity for process } \mathbb{W} \quad A_{\mathbf{C}}[\mathbb{W}] \equiv \ln \prod_{m=1}^{M} \frac{(\mathcal{C}_{m+1}|\mathbb{W}|\mathcal{C}_m)}{(\mathcal{C}_m|\mathbb{W}|\mathcal{C}_{m+1})}$$

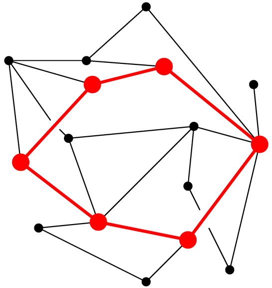
$$\text{cycle affinity for chain } \mathbb{P} \quad A_{\mathbf{C}}[\mathbb{P}] \equiv \ln \prod_{m=1}^{M} \frac{(\mathcal{C}_{m+1}|\mathbb{P}|\mathcal{C}_m)}{(\mathcal{C}_m|\mathbb{P}|\mathcal{C}_{m+1})}$$

$$A_{\mathbf{C}}[\mathbb{W}] = A_{\mathbf{C}}[\mathbb{P}]$$

Invariance under description change from Markov process to Markov chain

3.2 From a Markov process to processes defined on a subgraph (a)

- Generic connected **graph G**. Consider red subgraph **H** (a cycle here)



- Initial process with $\begin{cases} \text{transition rate } (\mathcal{C}'|\mathbb{W}|\mathcal{C}) \\ \text{waiting time probability } P_{\mathcal{C}}(\tau) \end{cases}$

  Markov property $P_{\mathcal{C}}(\tau) = r(\mathcal{C})e^{-r(\mathcal{C})\tau}$

- Derived process only between configurations of **H**

  with $\begin{cases} \text{transition rate } (\mathcal{C}'|\widetilde{\mathbb{W}}|\mathcal{C}) \\ \text{waiting time probability } \widetilde{P_{\mathcal{C}}}(\tau) \end{cases}$

- Examples of derived processes such that, if **H** is a cycle **C**, then $A_{\mathbf{C}}[\widetilde{\mathbb{W}}] = A_{\mathbf{C}}[\mathbb{W}]$

3.2 From a Markov process to processes defined on a subgraph (b)

- **1) restriction to a subgraph H** :
  Markov process for different histories where
  system jumps only along red bonds with same transition rates

  - $(\mathcal{C}'|\mathbb{W}^{\mathsf{rest}}|\mathcal{C}) = (\mathcal{C}'|\mathbb{W}|\mathcal{C}) \longrightarrow$ different escape rate $r^{\mathsf{rest}}(\mathcal{C}) = \sum_{\mathcal{C}' \in \mathbf{H}}(\mathcal{C}'|\mathbb{W}|\mathcal{C})$

  - If **H** is a cycle **C** $\qquad A_{\mathbf{C}}[\mathbb{W}^{\mathsf{rest}}] = A_{\mathbf{C}}[\mathbb{W}]$

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  - If **H** is a cycle **C** $\qquad A_{\mathbf{C}}[\mathbb{W}^{\text{rest}}] = A_{\mathbf{C}}[\mathbb{W}]$

- **2) Conditionning**
  Only histories where system jumps along red bonds are retained

  - $\longrightarrow$ Markov process with $\qquad (\mathcal{C}'|\mathbb{W}^{\text{cond}}|\mathcal{C}) = g(\mathcal{C}')(\mathcal{C}'|\mathbb{W}|\mathcal{C})\,[g(\mathcal{C})]^{-1}$

  - If **H** is a cycle **C** $\qquad A_{\mathbf{C}}[\mathbb{W}^{\text{cond}}] = A_{\mathbf{C}}[\mathbb{W}]$

3.2 Processes defined on a subgraph (c)

- **3) Drag and drop**

  A box is bound to move on the subgraph.

  All histories are considered but only the following events are retained :

  a walker meets the box on a red site

  and then jumps through a red bond while carrying the box along

  The box moves according to a semi-Markovian process with

  - probability to jump from $\mathcal{C}$ to $\mathcal{C}'$: $(\mathcal{C}'|\mathbb{P}^{\text{dd}}|\mathcal{C}) = (\mathcal{C}'|\mathbb{P}^{\text{rest}}|\mathcal{C})$

  - waiting time probability $\widetilde{P}_{\mathcal{C}}(\tau)$ not exponential

  - If **H** is a cycle **C** $\quad A_{\mathbf{C}}[\mathbb{P}^{\text{dd}}] = A_{\mathbf{C}}[\mathbb{P}]$

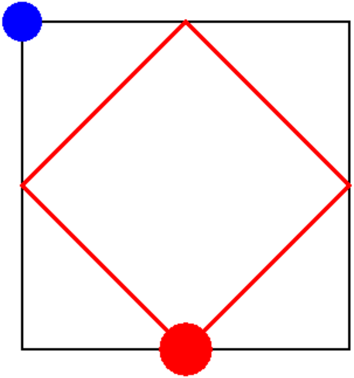
  - *Example* :

    ⋆ graph **G** : positions of a complex inside a cell

    ⋆ subgraph **H** : heteropolymer

    ⋆ box : a ligand bound to move along the heteropolymer when carried by the complex

Drag and Drop



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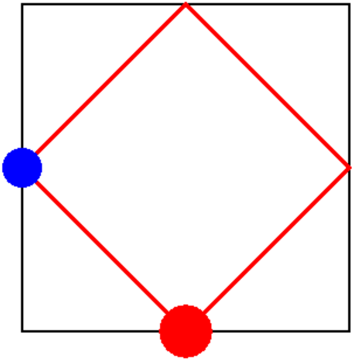

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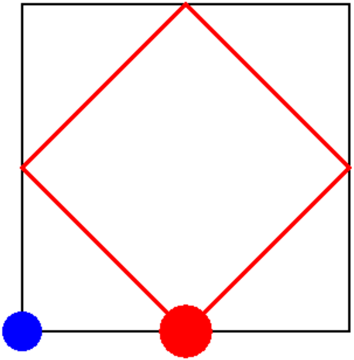

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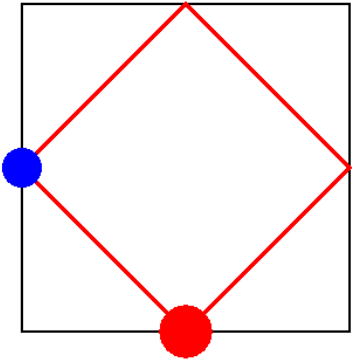

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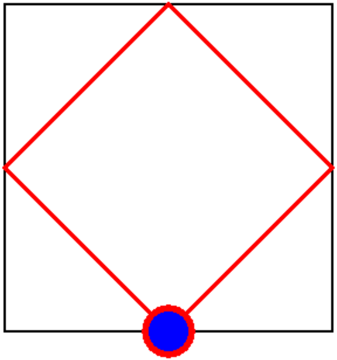

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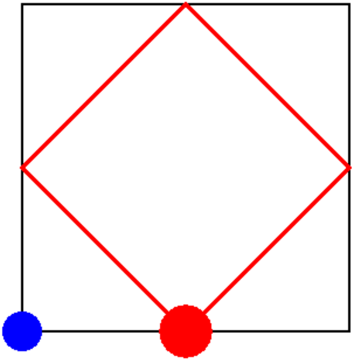

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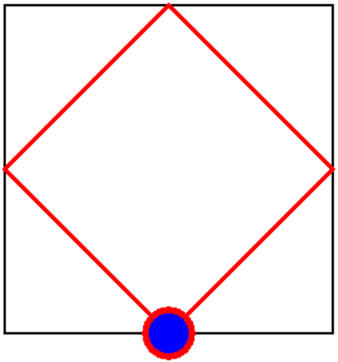

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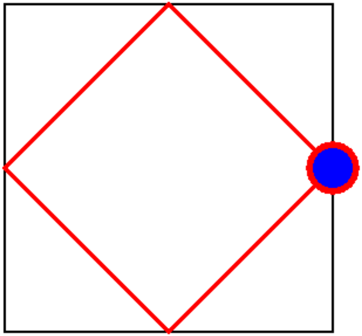

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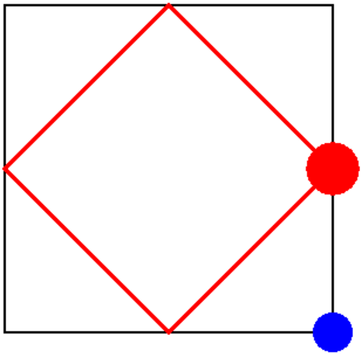

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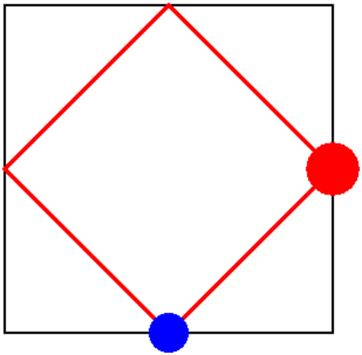

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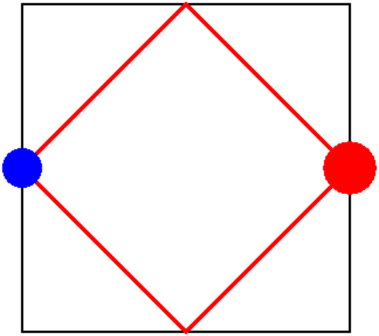

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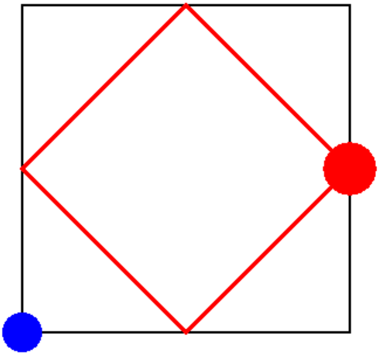

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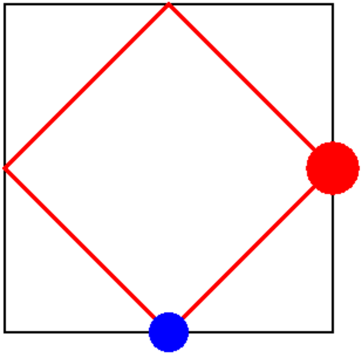

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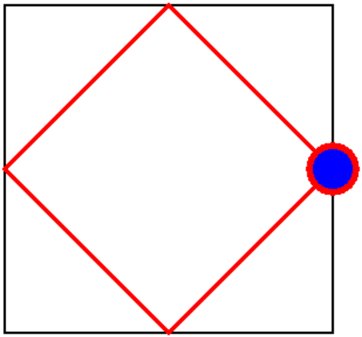

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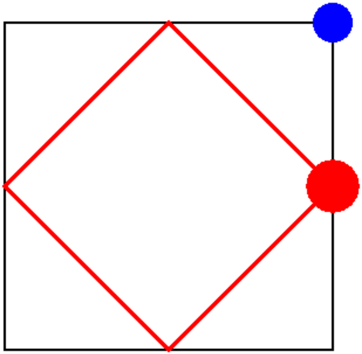
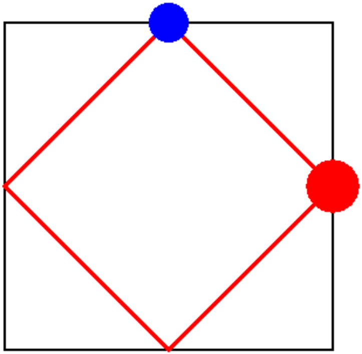


Drag and Drop



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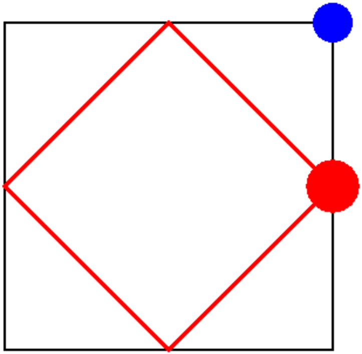
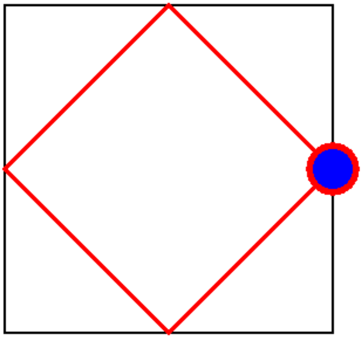
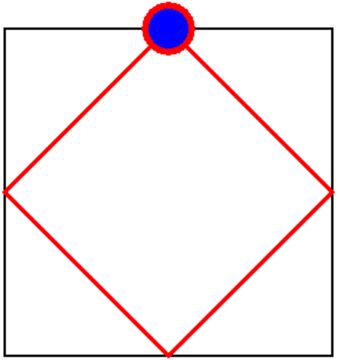


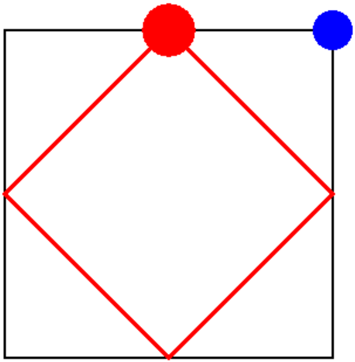
Drag and Drop



Drag and Drop



Drag and Drop



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AFFINITY AND FLUCTUATION RELATIONS at fixed time

Exchange Markovian processes

Known results

4.1 Exchange processes : cumulative currents

- **Exchange observable** $\mathbb{Q}$: (antisymmetric) $(\mathcal{C}'|\mathbb{Q}|\mathcal{C}) = -(\mathcal{C}|\mathbb{Q}|\mathcal{C}')$
- Process $\mathcal{C}_t \longrightarrow$ **Exchange cumulative process**

$$X_t^{\mathbb{Q}} \equiv \sum_{s\in]0,t]} (\mathcal{C}_s|\mathbb{Q}|\mathcal{C}_{s^-})$$

- Example : (microreversibility hyp.: $(\mathcal{C}'|\mathbb{W}|\mathcal{C}) \neq 0 \quad \Leftrightarrow \quad (\mathcal{C}|\mathbb{W}|\mathcal{C}') \neq 0$)
  Stochastic exchange entropy variation along a history :

  Lebowitz-Spohn action functional (1999) : $X_t^{\mathbb{S}} = \sum_{s\in]0,t]} (\mathcal{C}_s|\mathbb{S}|\mathcal{C}_{s^-})$

  For a history from $\mathcal{C}_0$ to $\mathcal{C}_N$ in time interval $[0,t]$

$$X_t^{\mathbb{S}} = \ln \frac{(\mathcal{C}_N|\mathbb{W}|\mathcal{C}_{N-1})(\mathcal{C}_{N-1}|\mathbb{W}|\mathcal{C}_{N-2})\cdots(\mathcal{C}_1|\mathbb{W}|\mathcal{C}_0)}{(\mathcal{C}_0|\mathbb{W}|\mathcal{C}_1)\cdots(\mathcal{C}_{N-2}|\mathbb{W}|\mathcal{C}_{N-1})\cdots(\mathcal{C}_{N-1}|\mathbb{W}|\mathcal{C}_N)}$$

4.2 Fluctuation relation for $X^{\mathbb{S}}$ at fixed time

- Extra hypothesis : **graph G connected** $\Rightarrow$ unique stationary $P_{\text{st}}(\mathcal{C})$
- Large deviation function $f_{X^{\mathbb{S}}}(\mathcal{J})$ for cumulative current $\mathcal{J}_t \equiv X_t^{\mathbb{S}}/t$

$$\lim_{t\to+\infty} \frac{1}{t} \ln P\left(\frac{X_t^{\mathbb{S}}}{t} \in [\mathcal{J}, \mathcal{J}+d\mathcal{J}]\right) = f_{X^{\mathbb{S}}}(\mathcal{J})$$

- **Fluctuation relation obeyed by** $f_{X^{\mathbb{S}}}(\mathcal{J})$ *[Lebowitz and Spohn (1999)]*

$$f_{X^{\mathbb{S}}}(\mathcal{J}) - f_{X^{\mathbb{S}}}(-\mathcal{J}) = \mathcal{J}$$

Other "sloppy" formulation

$$\frac{P\left(X_t^{\mathbb{S}} = t\mathcal{J}\right)}{P\left(X_t^{\mathbb{S}} = -t\mathcal{J}\right)} \underset{t\to+\infty}{\asymp} e^{t\mathcal{J}}$$

4.3 Case of a graph made of a single cycle : fluctuation relation for the cycle current at fixed time

• $X_t^{\mathbb{N}_M}$:number of passages through the bond $(\mathcal{C}_M, \mathcal{C}_1)$ of cycle **C** during $[0, t]$ in the positive sense minus the number of passages in the negative sense

with $\mathbb{N}_M$ defined by

- $(\mathcal{C}_M|\mathbb{N}_M|\mathcal{C}_1) = +1$
- $(\mathcal{C}_1|\mathbb{N}_M|\mathcal{C}_M) = -1$
- $(\mathcal{C}'|\mathbb{N}_M|\mathcal{C}) = 0$ if $\{\mathcal{C}, \mathcal{C}'\} \neq \{1, M\}$

• **Fluctuation relation for the cycle current at fixed time**

special case of more general results in Gaspard & Andrieux (2007)

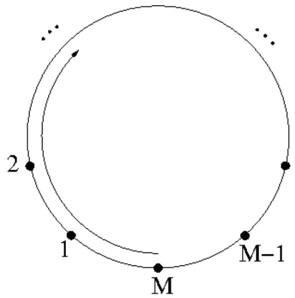
$$\frac{P\left(X_t^{\mathbb{N}_M} = t\mathcal{V}\right)}{P\left(X_t^{\mathbb{N}_M} = -t\mathcal{V}\right)} \underset{t\to+\infty}{\asymp} e^{t\,\mathcal{V}A_{\mathsf{C}}}$$

FLUCTUATION RELATIONS FOR FIRST PASSAGE TIMES

AT INTEGER WINDING NUMBERS

Semi-Markovian processes

5.1 Cycle graph and winding number



• Only jumps between successive configurations on the cycle
with probability knowing that a jump occurs :
$(\mathcal{C}_{m\pm1}|\mathbb{P}|\mathcal{C}_m)$

• Probability for waiting time τ at site m :
$P_m(\tau)$

• W_t : **winding number around the cycle C during $[0,t]$: number of clockwise jumps minus number of anticlockwise jumps divided by M**

$$W_t = X_t^{\mathbb{N}_w} \text{ with } \quad \forall m = 1 = \cdots = M$$

$$(\mathcal{C}_{m+1}|\mathbb{N}_w|\mathcal{C}_m) = +\frac{1}{M} \qquad \textit{and} \qquad (\mathcal{C}_m|\mathbb{N}_w|\mathcal{C}_{m+1}) = -\frac{1}{M}$$

5.2 Probability for winding number ± 1 to be reached

- **Cycle affinity in the clockwise sense**

$$A_{\mathbf{C}} \equiv \ln \prod_{m=1}^{M} \frac{(\mathcal{C}_{m+1}|\mathbb{P}|\mathcal{C}_m)}{(\mathcal{C}_{m-1}|\mathbb{P}|\mathcal{C}_m)}$$

- Method : generating function. Probabilistic arguments and strong Markov property $\to$ recursive relations

$$\frac{P\left(\exists t \in [0, +\infty[\quad \textit{such that} \quad W_t = -1\right)}{P\left(\exists t \in [0, +\infty[\quad \textit{such that} \quad W_t = +1\right)} = e^{-A_{\mathbf{C}}}$$

More precisely, if $A_{\mathbf{C}} > 0$

- winding number $+1$ is reached with probability 1
- winding number -1 is never reached with finite probability $1 - e^{-A_{\mathbf{C}}}$

5.3 Fluctuation relation for first passage time at winding number 1

- $T_\pm$: first passage time at winding number ± 1

Method : Laplace transform $\langle e^{-\lambda T_+} \rangle \equiv \int_{t\in[0,\infty[} e^{-\lambda t} P\left(T_+ \in [t, t+dt[\right)$

Result :
$$\frac{\langle e^{-\lambda T_+} \rangle}{\langle e^{-\lambda T_-} \rangle} = e^{A_\mathcal{C}}$$

$\rightarrow$ Radon-Nikodym derivative

$$\frac{P\left(T_+ \in [t, t+dt[\right)}{P\left(T_- \in [t, t+dt[\right)} = e^{A_\mathcal{C}}$$

The ratio is independent from the various distributions of waiting times $P_{\mathcal{C}_m}(\tau)$ along the cycle

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$$\frac{P\left(T_+ \in [t, t+dt[\right)}{P\left(T_- \in [t, t+dt[\right)} = e^{A_{\mathrm{c}}}$$

The ratio is independent from the various distributions of waiting times $P_{\mathcal{C}_m}(\tau)$ along the cycle

- Comparison with dual relation for a history corresponding to winding number $+1$ (without restriction of first passage)

$$\frac{P\left(\text{history}_{\text{with } W=+1}\right)}{P\left(\text{time-reversed history}_{\text{with } W=-1}\right)} = e^{X^{\mathbb{S}}[\text{history}_{\text{with } W=+1}]} = e^{A_{\mathrm{c}}}$$

5.4 Fluctuation relation for large winding numbers (a)

$T_{\pm w}$ first passage time at winding number $\pm w$ with w integer

- If the first waiting time plays no role,
  semi-Markov (or renewal) property $\rightarrow \langle e^{-\lambda T_{-w}} \rangle = \langle e^{-\lambda T_{-}} \rangle^{w}$

$$\frac{\langle e^{-\lambda T_{w}} \rangle}{\langle e^{-\lambda T_{-w}} \rangle} = e^{w\, A_{\mathrm{c}}}$$

Remarks :

1) valid for any finite winding number w

2) valid for any cycle in a more general graph of transitions as long as the procedure to define the process of the cycle preserves the affinity class

5.4 Fluctuation relation for large winding numbers (b)

- If the first passage time plays a special role (case of drag-and-drop construction)

  law of large numbers $\rightarrow \left\langle e^{-\lambda T_{-w}} \right\rangle^{1/w} \underset{|w|\to+\infty}{\sim} \left\langle e^{-\lambda T_{-}} \right\rangle$

$$\lim_{w\to\pm\infty} \frac{\left[\left\langle e^{-\lambda T_{+w}} \right\rangle\right]^{1/w}}{\left[\left\langle e^{-\lambda T_{-w}} \right\rangle\right]^{1/w}} = e^{A_{\mathrm{c}}}$$

5.4 Fluctuation relation for large winding numbers (b)

- If the first passage time plays a special role (case of drag-and-drop construction)

  law of large numbers $\rightarrow \left\langle e^{-\lambda T_{-w}} \right\rangle^{1/w} \underset{|w|\to+\infty}{\sim} \left\langle e^{-\lambda T_{-}} \right\rangle$

$$\lim_{w\to\pm\infty} \frac{\left[\left\langle e^{-\lambda T_{+w}} \right\rangle\right]^{1/w}}{\left[\left\langle e^{-\lambda T_{-w}} \right\rangle\right]^{1/w}} = e^{A_{\mathbf{C}}}$$

- **Comparison with fluctuation relations at fixed time**

  W_t : winding number : number of clockwise jumps minus number of anticlockwise jumps divided by M

$\underset{|W_t|\to+\infty}{\sim}$

$X_t^{\mathbb{N}_M}$: number of passages through the bond $(\mathcal{C}_M, \mathcal{C}_1)$ of cycle $\mathbf{C}$ during $[0, t]$ in the positive sense minus the number of passages in the negative sense

$$\frac{P\left(W_t = t\mathcal{V}\right)}{P\left(W_t = -t\mathcal{V}\right)} \underset{t\to+\infty}{\asymp} e^{\,t\,\mathcal{V}A_{\mathbf{C}}}$$

5.5 Mean first passage time at winding number 1

- T_{+w} is a sum of w independent random variables with mean $\langle T_+ \rangle$

strong law of large numbers $\rightarrow$

$$\lim_{w \to +\infty} \frac{T_{+w}}{w} = \langle T_+ \rangle \quad \textit{with probability } 1$$

$$\lim_{t \to +\infty} \frac{W_t}{t} = \frac{1}{\langle T^+ \rangle} \quad \textit{with probability } 1$$

In the long time limit fluctuations are suppressed and cycle is performed at velocity $1/\langle T^+ \rangle$

$$\langle T^+ \rangle = \frac{\sum_{m=1}^{M} \sum_{k=1}^{M} \left(\prod_{1 \le i < k} p_{m+i}^+ \right) \tau_{m+k} \left(\prod_{k < j \le M} p_{m+j}^- \right)}{\left(\prod_{m=1}^{M} p_m^+ - \prod_{m=1}^{M} p_m^- \right)}$$

with $p_m^+ \equiv (\mathcal{C}_{m+1} | \mathbb{P} | \mathcal{C}_m)$, $p_m^- \equiv (\mathcal{C}_{m-1} | \mathbb{P} | \mathcal{C}_m)$, τ_m mean waiting time in $\mathcal{C}_m$.

Conclusion

• Robustness of cycle affinities when edges are discarded by conditioning or drag and drop

→ properties for a single cycle are also valid for a cycle embedded in a more generic pattern of transitions

• In out-of-equilibrium state a current associated to winding number flows through cycle

Fluctuation relations for first-passage time at winding number $\pm w$ are ruled by cycle affinity

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