Dynamics of Ising spin systems

Reversibility, irreversibility, Gibbsianity

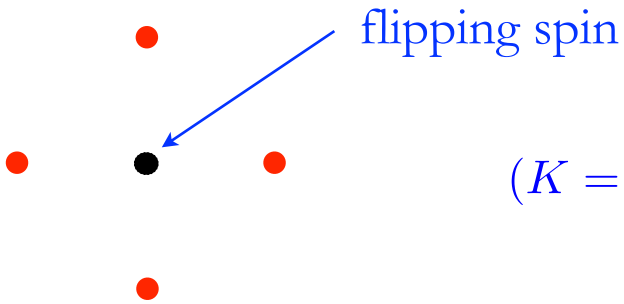
CG&AJ Bray 2009
CG 2011
CG 2013
CG&M Pleimling 2014
CG&JM Luck 2014

with thanks to J Lebowitz

GG Institute May 26, 2014

Take flipping rate for 2D square lattice

$$w(\sigma) = e^{-K\sigma(\sigma_N+\sigma_E+\sigma_W+\sigma_S)}$$



$(K = J/T)$

then stationary state is Boltzmann-Gibbs and detailed balance (w.r.t. Ising energy) satisfied

Künsch 1984:

if flipping rate $w(\sigma) = e^{-K\sigma(\sigma_N+\sigma_E)}$

then stationary state is Boltzmann-Gibbs but detailed balance violated

Take flipping rate for 2D square lattice (Glauber)

$$w(\sigma) = \frac{1}{2}(1 - \sigma \tanh K(\sigma_N + \sigma_E + \sigma_W + \sigma_S))$$

then stationary state is Boltzmann-Gibbs and detailed balance (w.r.t. Ising energy) satisfied

Lima&Stauffer 2006: simulations on 2D square lattice with flipping rate

$$w(\sigma) = \frac{1}{2}(1 - \sigma \tanh K(\sigma_N + \sigma_E))$$

then no phase transition (apparently the same up to 5D)

We are led to ask:

What is the stationary state for a given generic rate function (here in 2D)?

In particular which rate functions are compatible with Gibbs stat. measure?
(Without necessarily being reversible)

Idem in 1D, 3D, ...

Generic rate function (with spin symmetry)

$$w(\sigma) = c_0 + \sum_{i=1}^{2^z-1} c_i\, O_i, \qquad (z \text{ neighbours})$$

\+ positivity

2D:

i	O_i
1	$\sigma_n \sigma_{j_1} \sigma_{j_2} \sigma_{j_{\underline{1}}}$
2	$\sigma_n \sigma_{j_1} \sigma_{j_2} \sigma_{j_{\underline{2}}}$
3	$\sigma_n \sigma_{j_1} \sigma_{j_{\underline{1}}} \sigma_{j_{\underline{2}}}$
4	$\sigma_n \sigma_{j_2} \sigma_{j_{\underline{1}}} \sigma_{j_{\underline{2}}}$
5	$\sigma_{j_1} \sigma_{j_2} \sigma_{j_{\underline{1}}} \sigma_{j_{\underline{2}}}$
6	$\sigma_n \sigma_{j_1}$
7	$\sigma_n \sigma_{j_2}$
8	$\sigma_n \sigma_{j_{\underline{2}}}$
9	$\sigma_n \sigma_{j_{\underline{1}}}$
10	$\sigma_{j_1} \sigma_{j_2}$
11	$\sigma_{j_2} \sigma_{j_{\underline{1}}}$
12	$\sigma_{j_1} \sigma_{j_{\underline{1}}}$
13	$\sigma_{j_{\underline{1}}} \sigma_{j_{\underline{2}}}$
14	$\sigma_{j_1} \sigma_{j_{\underline{2}}}$
15	$\sigma_{j_2} \sigma_{j_{\underline{2}}}$

For example all rate functions on 2D square lattice with NEC spins:

$$w(\sigma) = c_0 + c_6\, \sigma\sigma_E + c_7\, \sigma\sigma_N + c_{10}\, \sigma_E\sigma_N \qquad + \text{positivity}$$

i	O_i
1	$\sigma_n\sigma_{j_1}\sigma_{j_2}\sigma_{j_{\underline{1}}}$
2	$\sigma_n\sigma_{j_1}\sigma_{j_2}\sigma_{j_{\underline{2}}}$
3	$\sigma_n\sigma_{j_1}\sigma_{j_{\underline{1}}}\sigma_{j_{\underline{2}}}$
4	$\sigma_n\sigma_{j_2}\sigma_{j_{\underline{1}}}\sigma_{j_{\underline{2}}}$
5	$\sigma_{j_1}\sigma_{j_2}\sigma_{j_{\underline{1}}}\sigma_{j_{\underline{2}}}$
6	$\sigma_n\sigma_{j_1}$
7	$\sigma_n\sigma_{j_2}$
8	$\sigma_n\sigma_{j_{\underline{2}}}$
9	$\sigma_n\sigma_{j_{\underline{1}}}$
10	$\sigma_{j_1}\sigma_{j_2}$
11	$\sigma_{j_2}\sigma_{j_{\underline{1}}}$
12	$\sigma_{j_1}\sigma_{j_{\underline{1}}}$
13	$\sigma_{j_{\underline{1}}}\sigma_{j_{\underline{2}}}$
14	$\sigma_{j_1}\sigma_{j_{\underline{2}}}$
15	$\sigma_{j_2}\sigma_{j_{\underline{2}}}$

Künsch:

$$w(\sigma) = c_0(1 - \gamma\,\sigma(\sigma_E + \sigma_N) + \gamma^2\sigma_E\sigma_N)$$

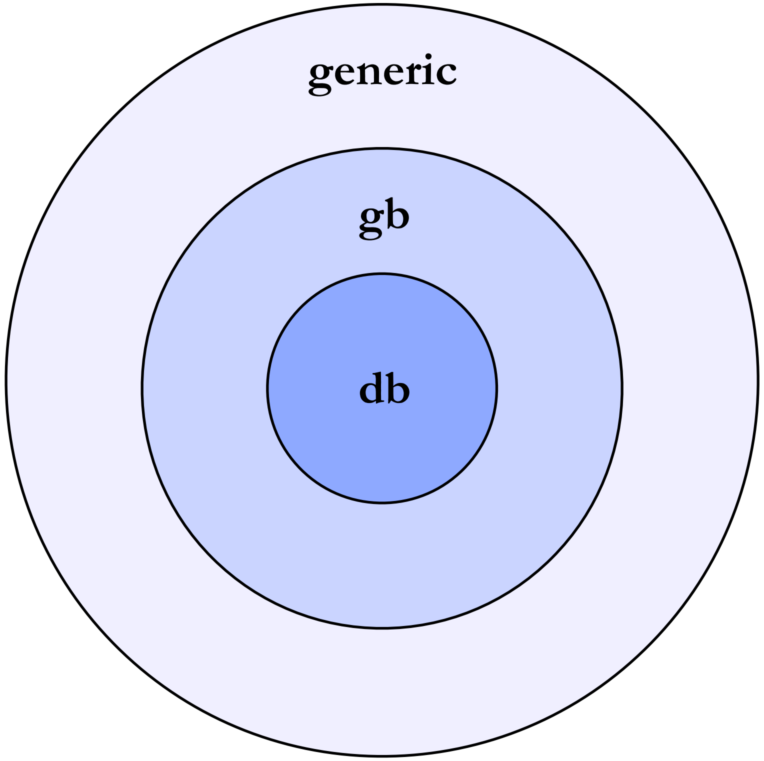
$$\gamma = \tanh 2K \qquad (K = J/T)$$

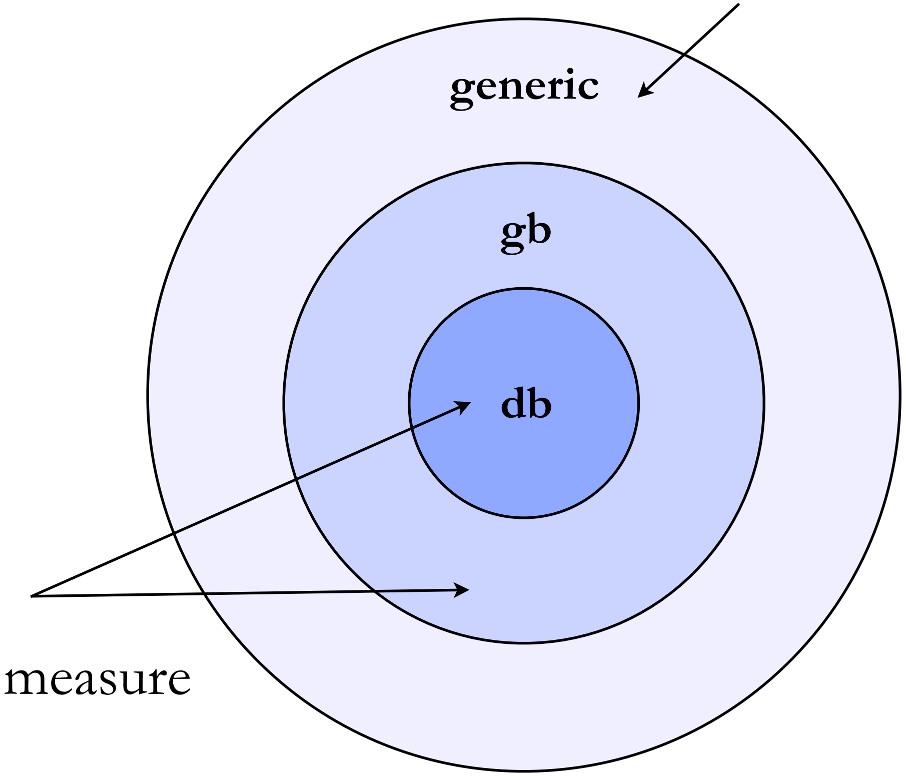
Lima&Stauffer:

$$w(\sigma) = c_0(1 - \tfrac{1}{2}\gamma\,\sigma(\sigma_E + \sigma_N))$$

A priori expected classification

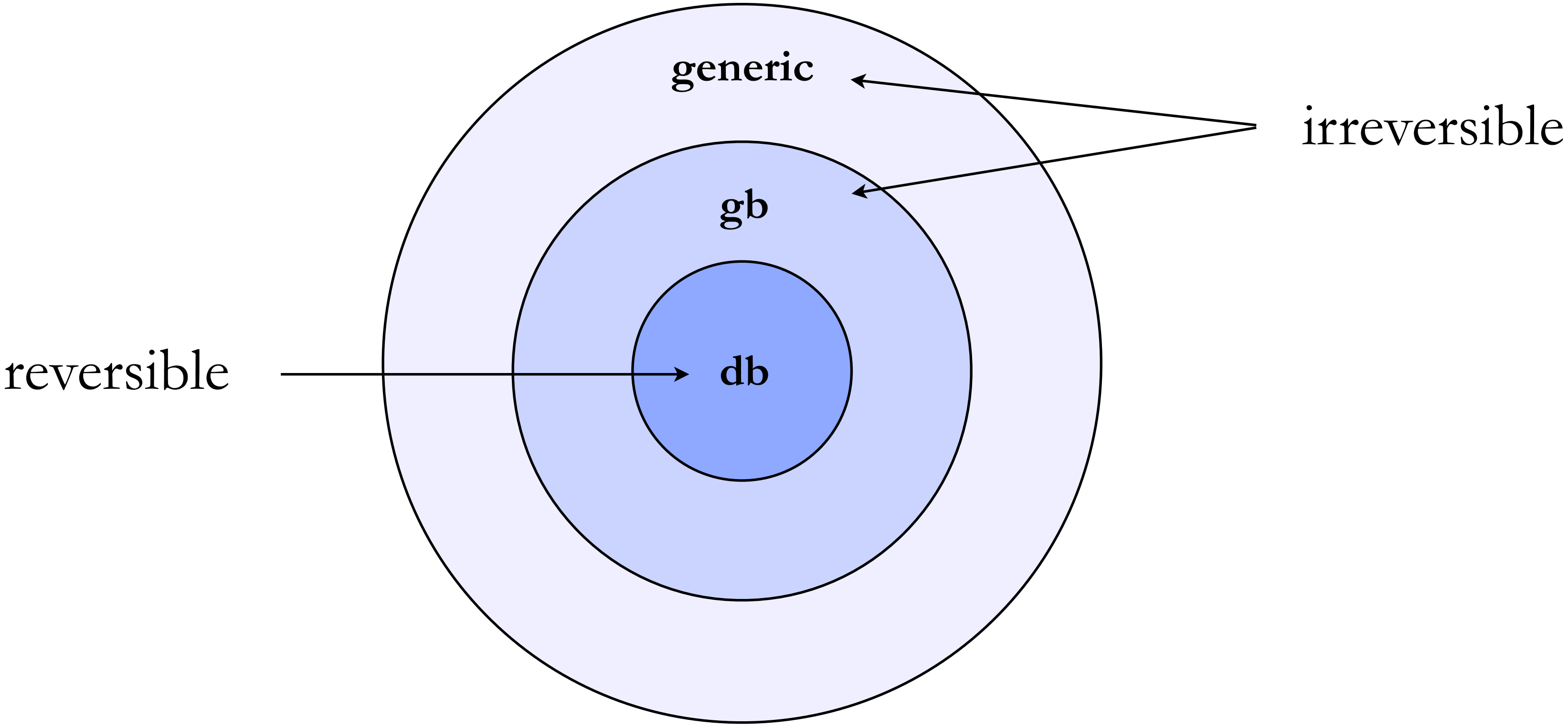
(in the space of coefficients defining the rate function):





unknown stationary measure

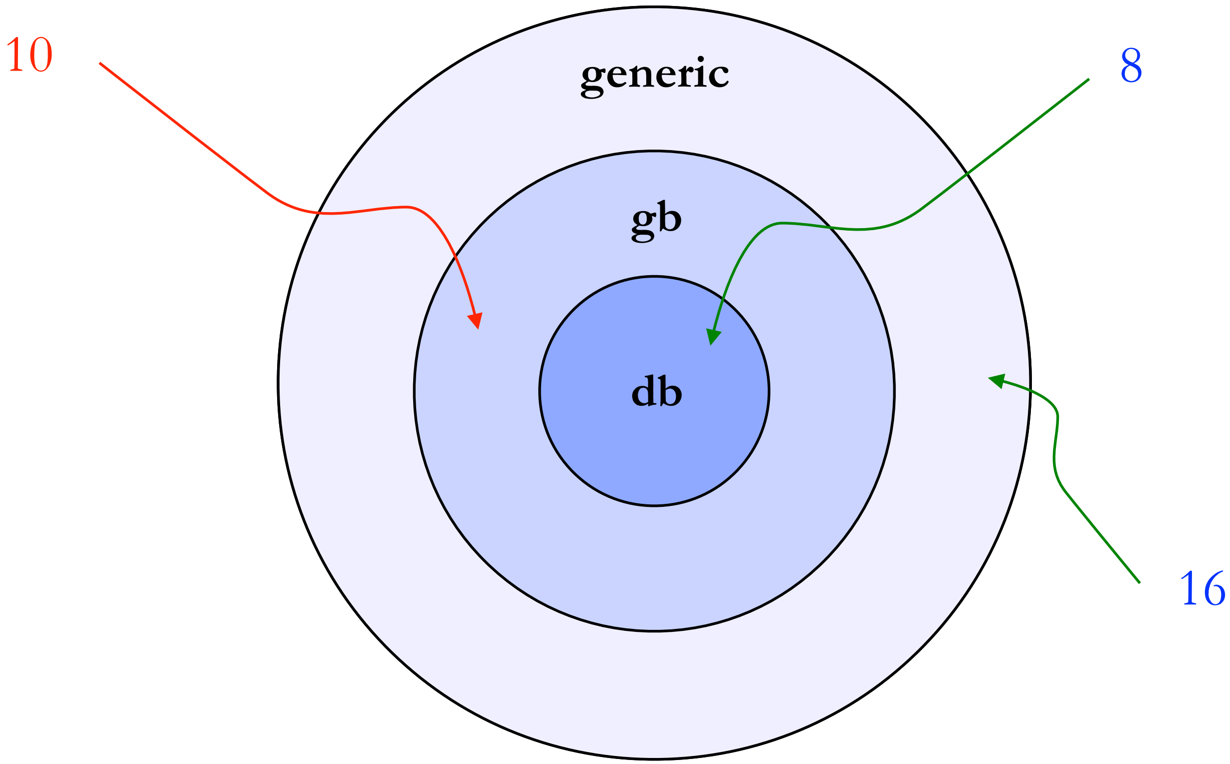
Gibbsian stationary measure



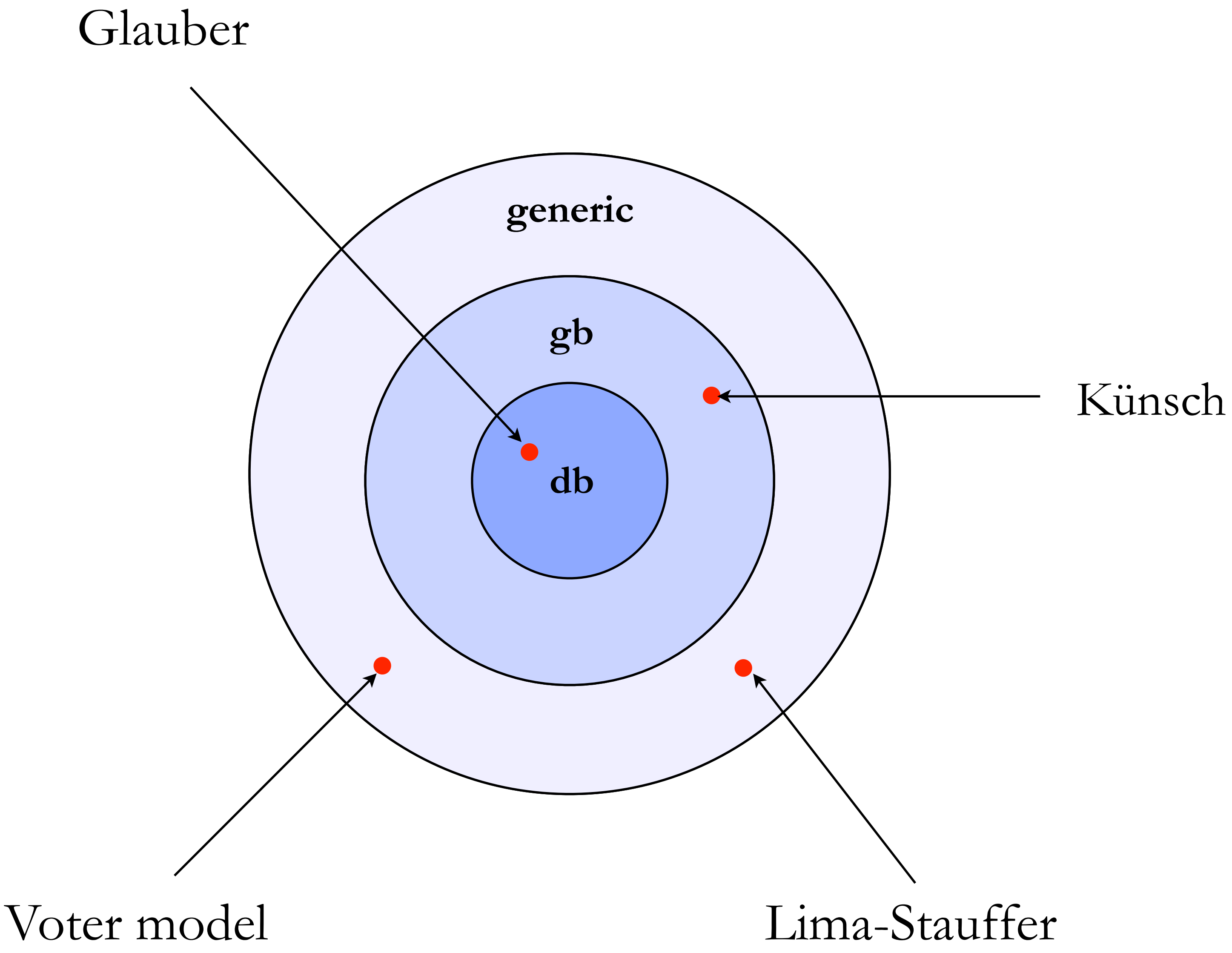
irreversible

reversible

2D square lattice: results



CG&AJ Bray 2009
CG 2013


Glauber
generic
gb
Künsch
db
Voter model
Lima-Stauffer

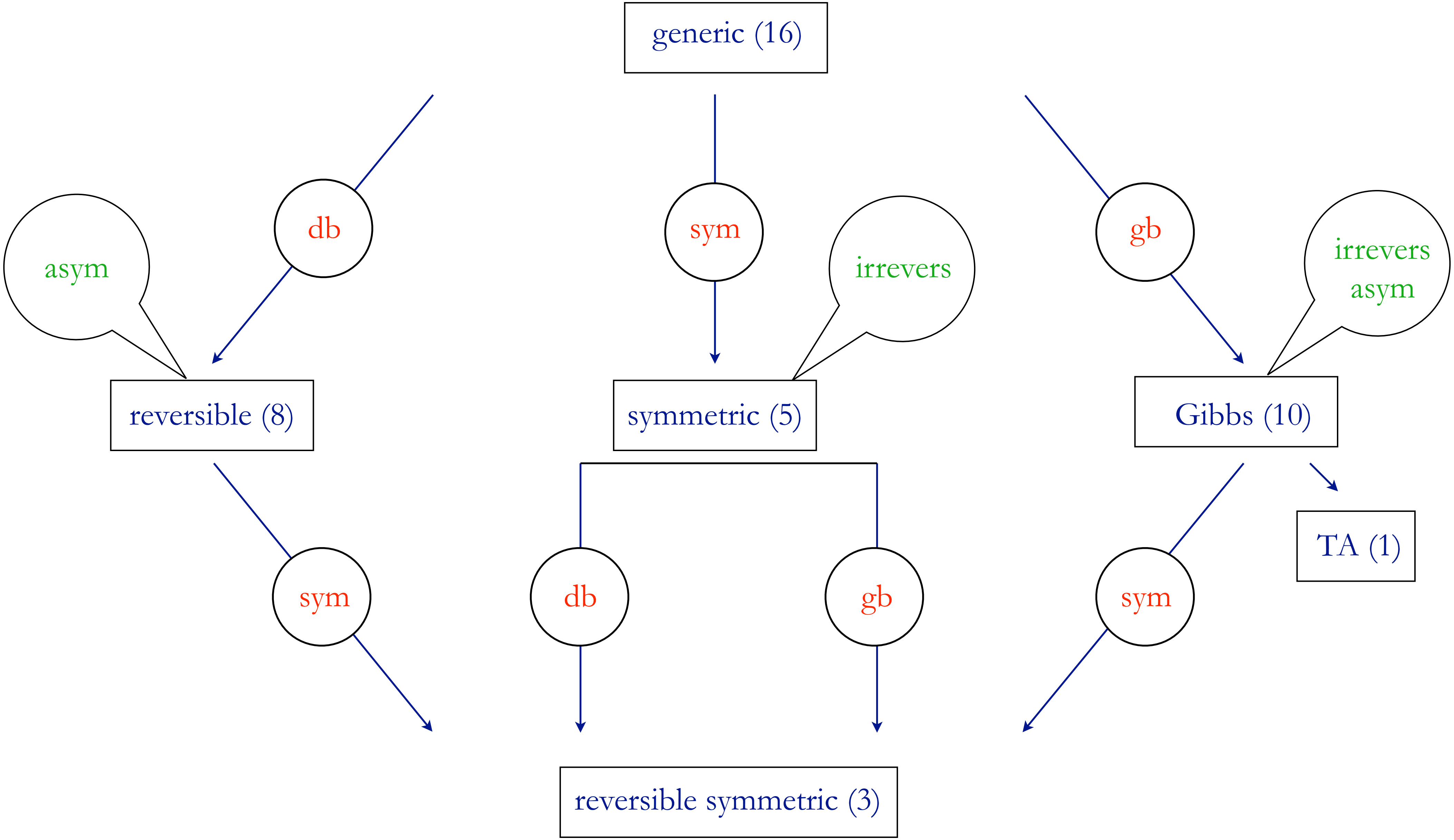
2D, totally asymmetric dynamics (Künsch): unicity

CG&AJ Bray 2009
CG 2013

$$w(\sigma) = c_0(1 - \gamma\,\sigma(\sigma_E + \sigma_N) + \gamma^2\sigma_E\sigma_N)$$

Lima&Stauffer: truncated Voter model

$$w(\sigma) = c_0(1 - \tfrac{1}{2}\gamma\,\sigma(\sigma_E + \sigma_N))$$


generic (16)
db
sym
gb
asym
irrevers
irrevers
asym
reversible (8)
symmetric (5)
Gibbs (10)
sym
db
gb
sym
TA (1)
reversible symmetric (3)

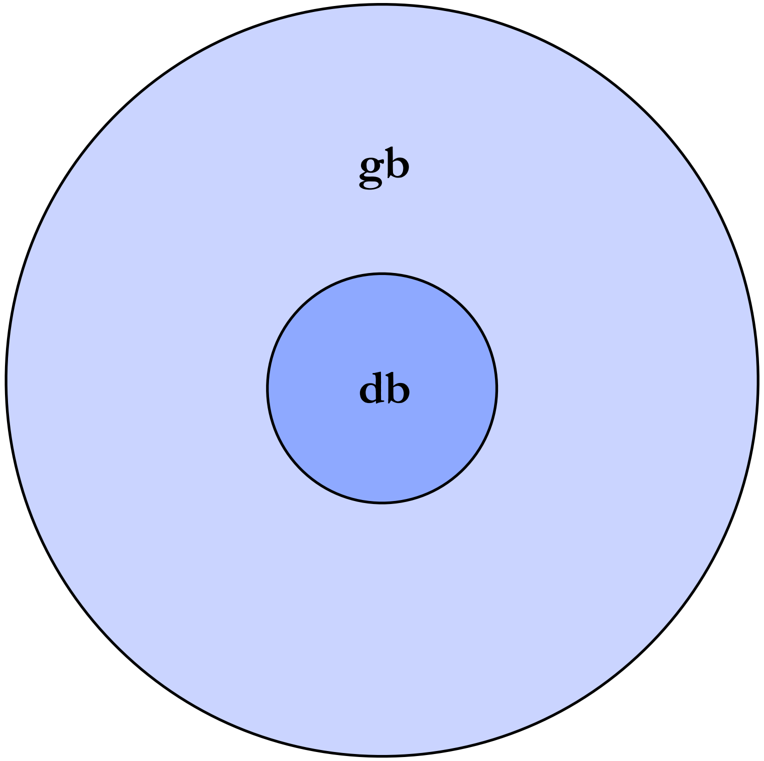
1D

Generic rate function (with spin symmetry)

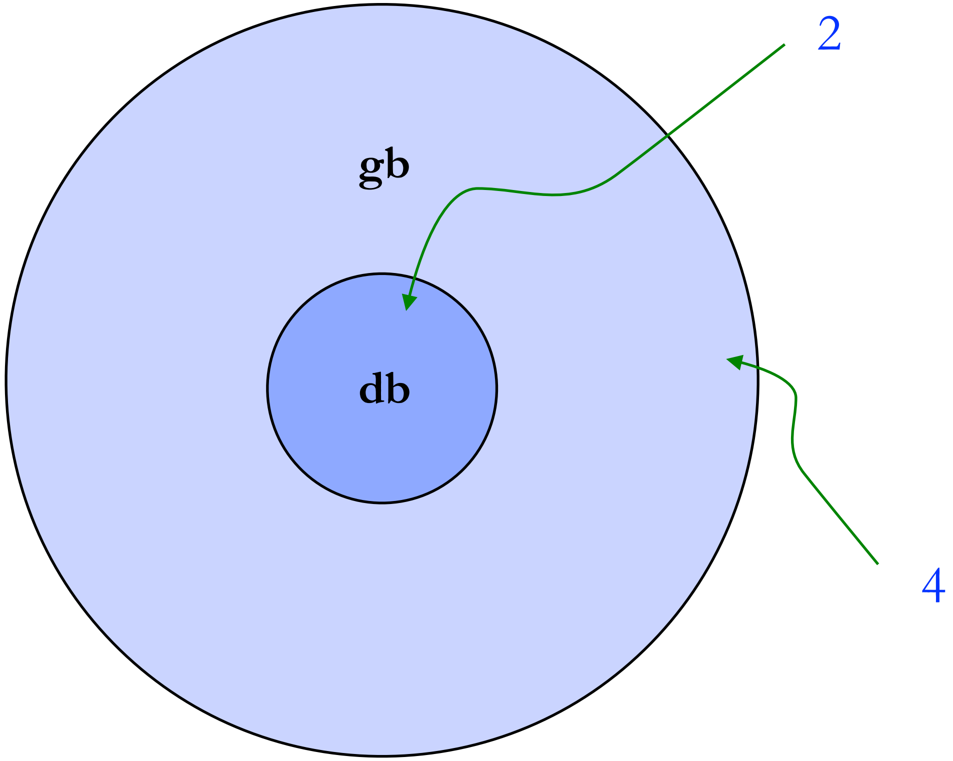
$$w(\sigma_n) = c_0 + c_1\,\sigma_n\sigma_{n+1} + c_2\,\sigma_{n-1}\sigma_n + c_3\,\sigma_{n-1}\sigma_{n+1}$$

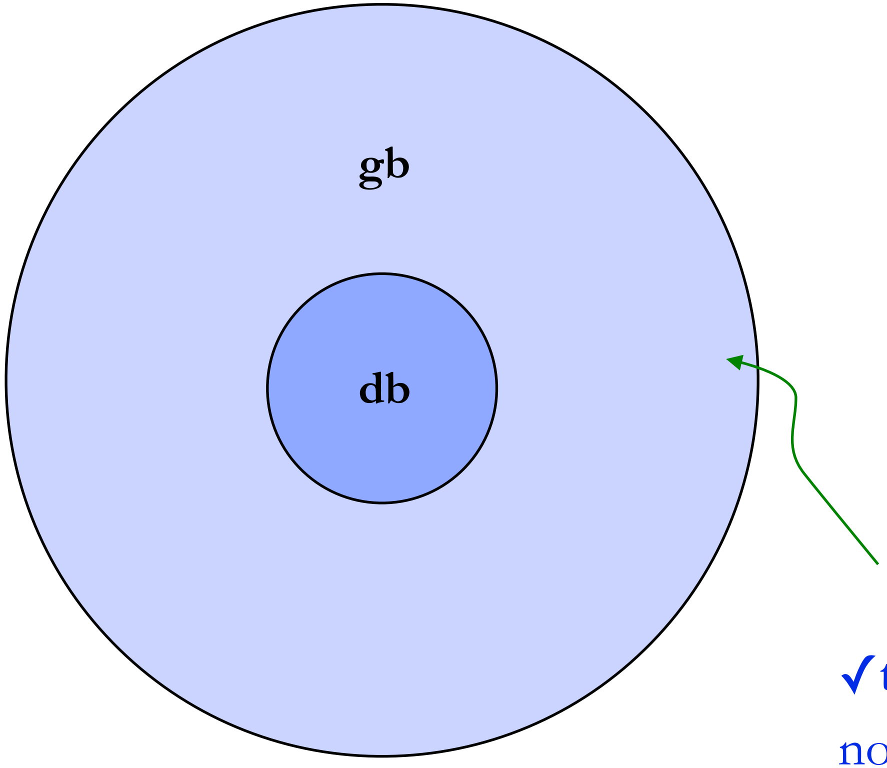
4 parameters, with constraints of positivity to fulfill

1D lattice: results



1D lattice: results





~~4~~ → 2

✓ time scale

non linearity

asymmetry

✓ temperature

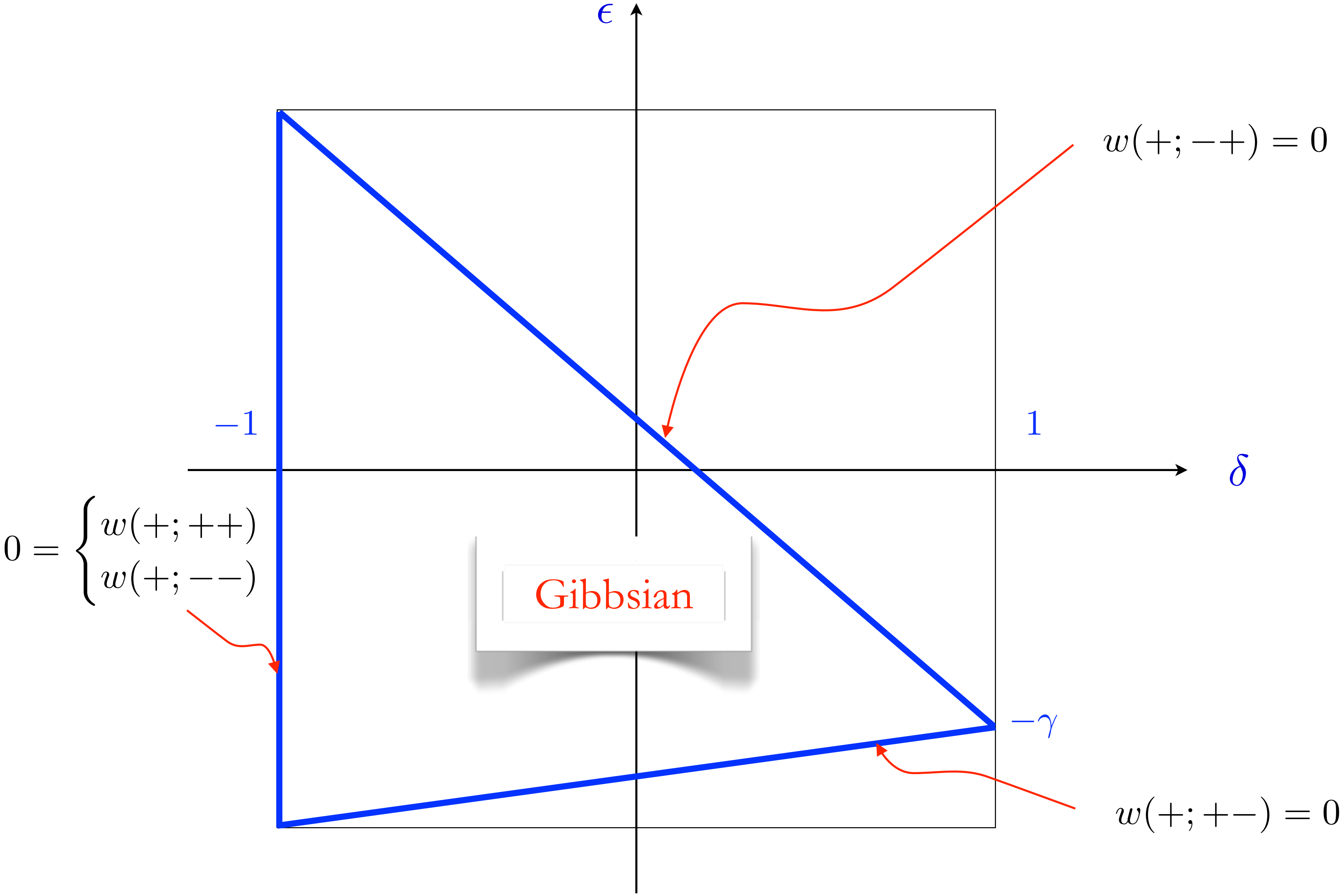
Most general rate function satisfying global balance (1D)

$$w(\sigma_n) = \underset{\underset{\text{time scale}}{\uparrow}}{\frac{\alpha}{2}}\Big[1 - (\underset{\underset{\text{temperature}}{\uparrow}}{\gamma}(1+\delta) + \epsilon)\sigma_n\sigma_{n+1} + \underset{\underset{\text{(asymmetry)}}{\uparrow}}{\epsilon}\,\sigma_{n-1}\sigma_n + \underset{\underset{\text{non linearity}}{\uparrow}}{\delta}\,\sigma_{n-1}\sigma_{n+1}\Big]$$

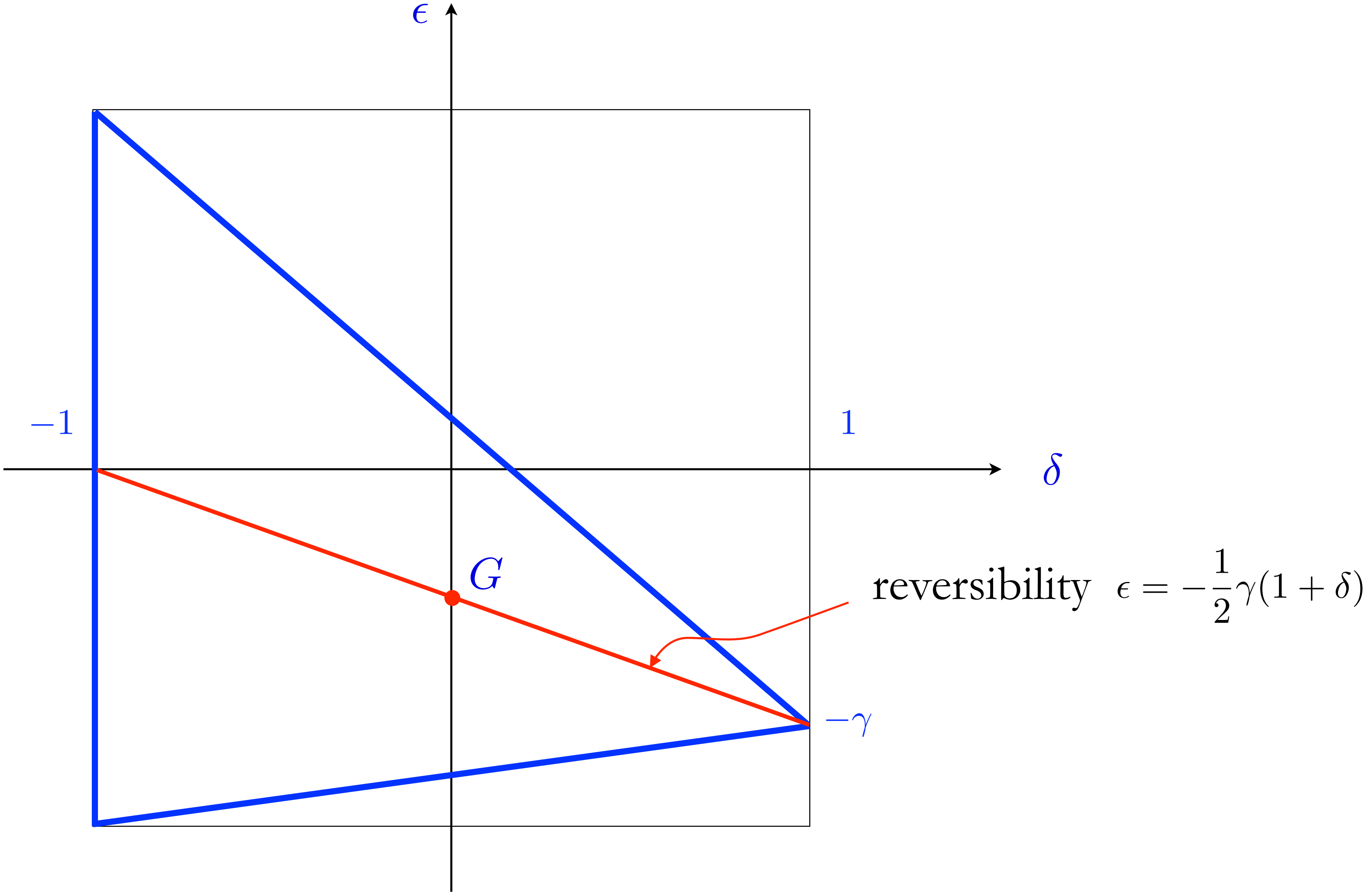
Most general rate function satisfying detailed balance (1D)

$$w(\sigma_n) = \frac{1}{2}\Big[1 - \frac{1}{2}\gamma(1+\delta)\sigma_n(\sigma_{n-1} + \sigma_{n+1}) + \delta\,\sigma_{n-1}\sigma_{n+1}\Big]$$

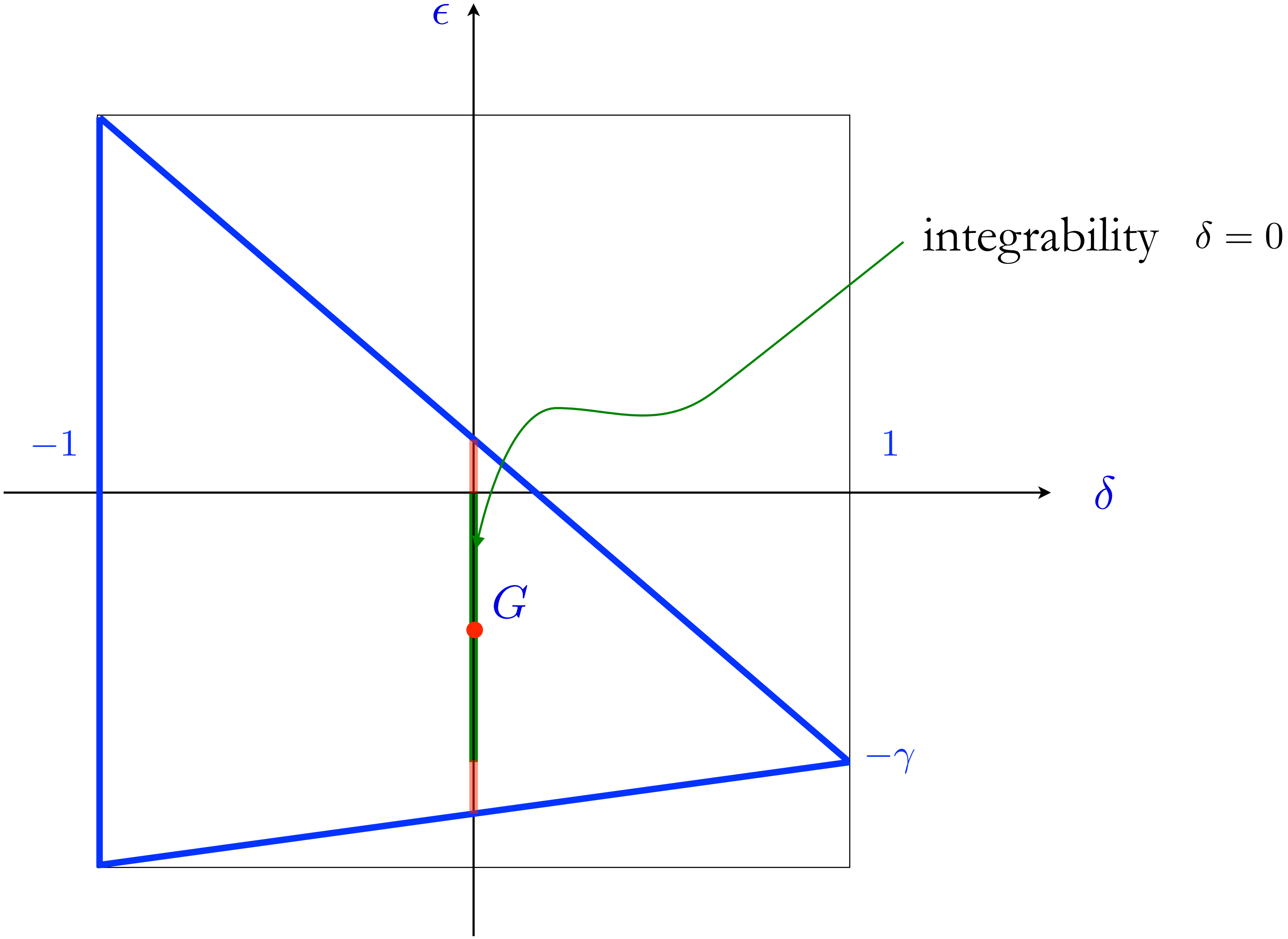
obtained from above by $\epsilon = -\frac{1}{2}\gamma(1+\delta)$



$$w(\sigma_n) = \frac{\alpha}{2}\Big[1 - (\gamma(1+\delta) + \epsilon)\sigma_n\sigma_{n+1} + \epsilon\,\sigma_{n-1}\sigma_n + \delta\,\sigma_{n-1}\sigma_{n+1}\Big]$$

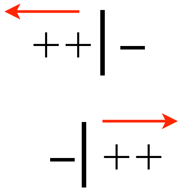


$$w(\sigma_n) = \frac{\alpha}{2}\Big[1 - (\gamma(1+\delta)+\epsilon)\sigma_n\sigma_{n+1} + \epsilon\,\sigma_{n-1}\sigma_n + \delta\,\sigma_{n-1}\sigma_{n+1}\Big]$$



$$w(\sigma_n) = \frac{\alpha}{2}\left[1 - (\gamma(1+\delta)+\epsilon)\sigma_n\sigma_{n+1} + \epsilon\,\sigma_{n-1}\sigma_n + \delta\,\sigma_{n-1}\sigma_{n+1}\right]$$

Moves



$$
\begin{aligned}
w(+;++) &= \frac{1}{2}(1-\gamma)(1+\delta) \\
w(+;--) &= \frac{1}{2}(1+\gamma)(1+\delta) \\
w(+;+-) &= \frac{1}{2}(1-\delta+\gamma(1+\delta))+\epsilon \\
w(+;-+) &= \frac{1}{2}(1-\delta-\gamma(1+\delta))-\epsilon
\end{aligned}
$$

elementary excitations

motion of domain walls
(asymmetric)

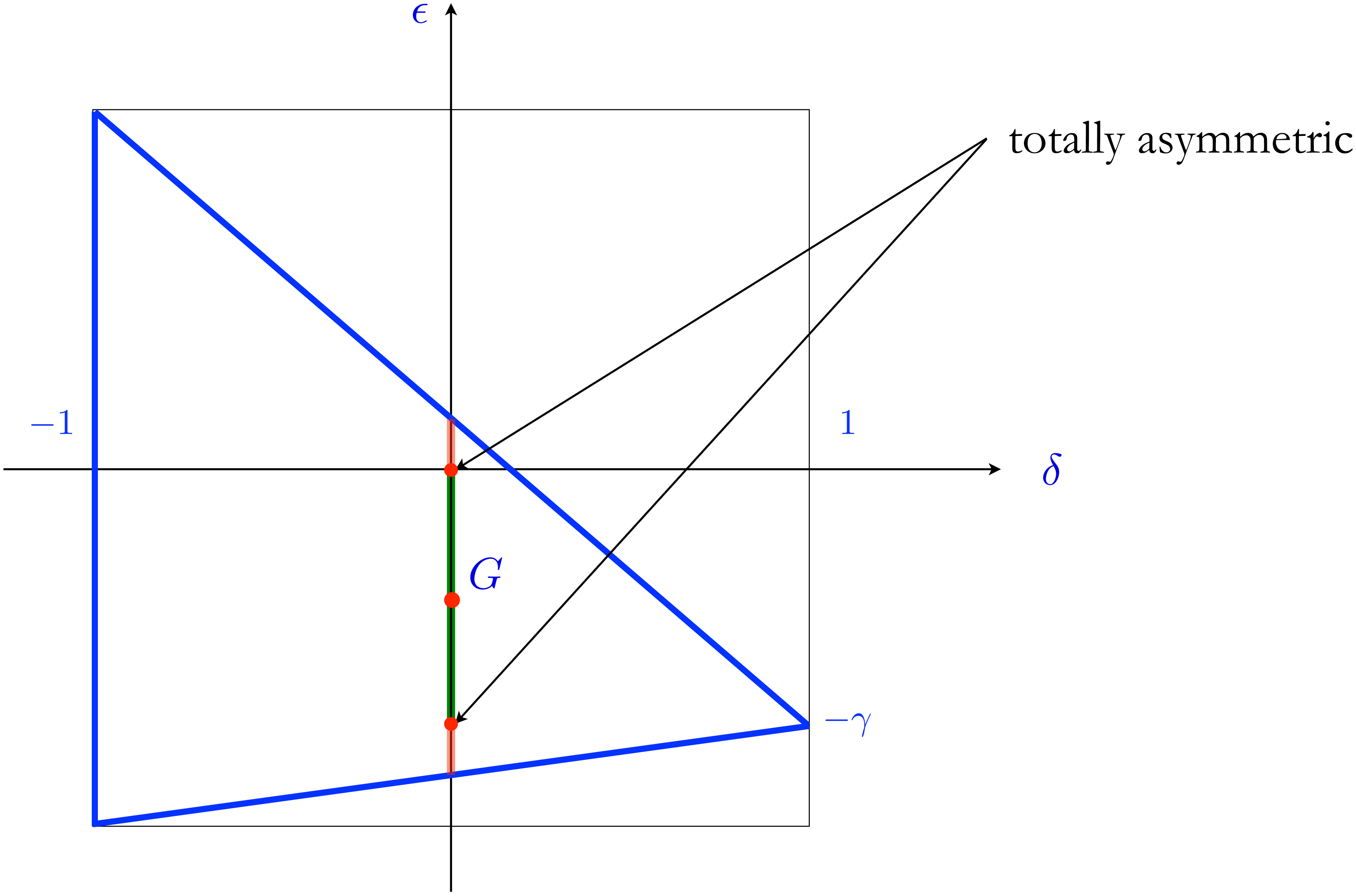
$$w(\sigma_n;\sigma_{n-1},\sigma_{n+1}) = \frac{\alpha}{2}\Big[1-(\gamma(1+\delta)+\epsilon)\sigma_n\sigma_{n+1}+\epsilon\,\sigma_{n-1}\sigma_n+\delta\,\sigma_{n-1}\sigma_{n+1}\Big]$$

Totally asymmetric rates

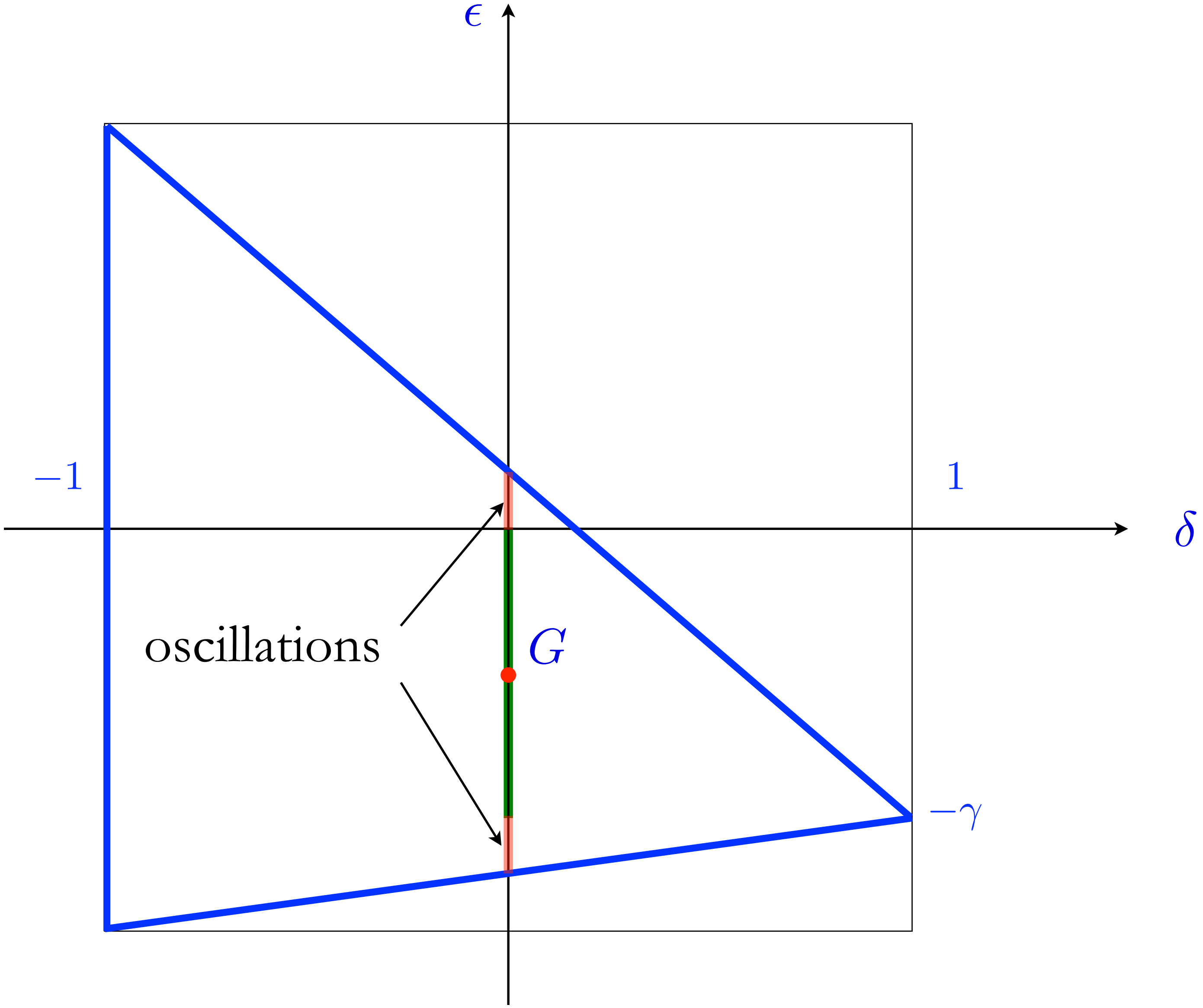
$$w(\sigma_n) = \frac{\alpha}{2}\Big[1 - (\gamma(1+\delta)+\epsilon)\cancel{\sigma_n\sigma_{n+1}} + \epsilon\,\sigma_{n-1}\sigma_n + \delta\,\cancel{\sigma_{n-1}\sigma_{n+1}}\Big]$$

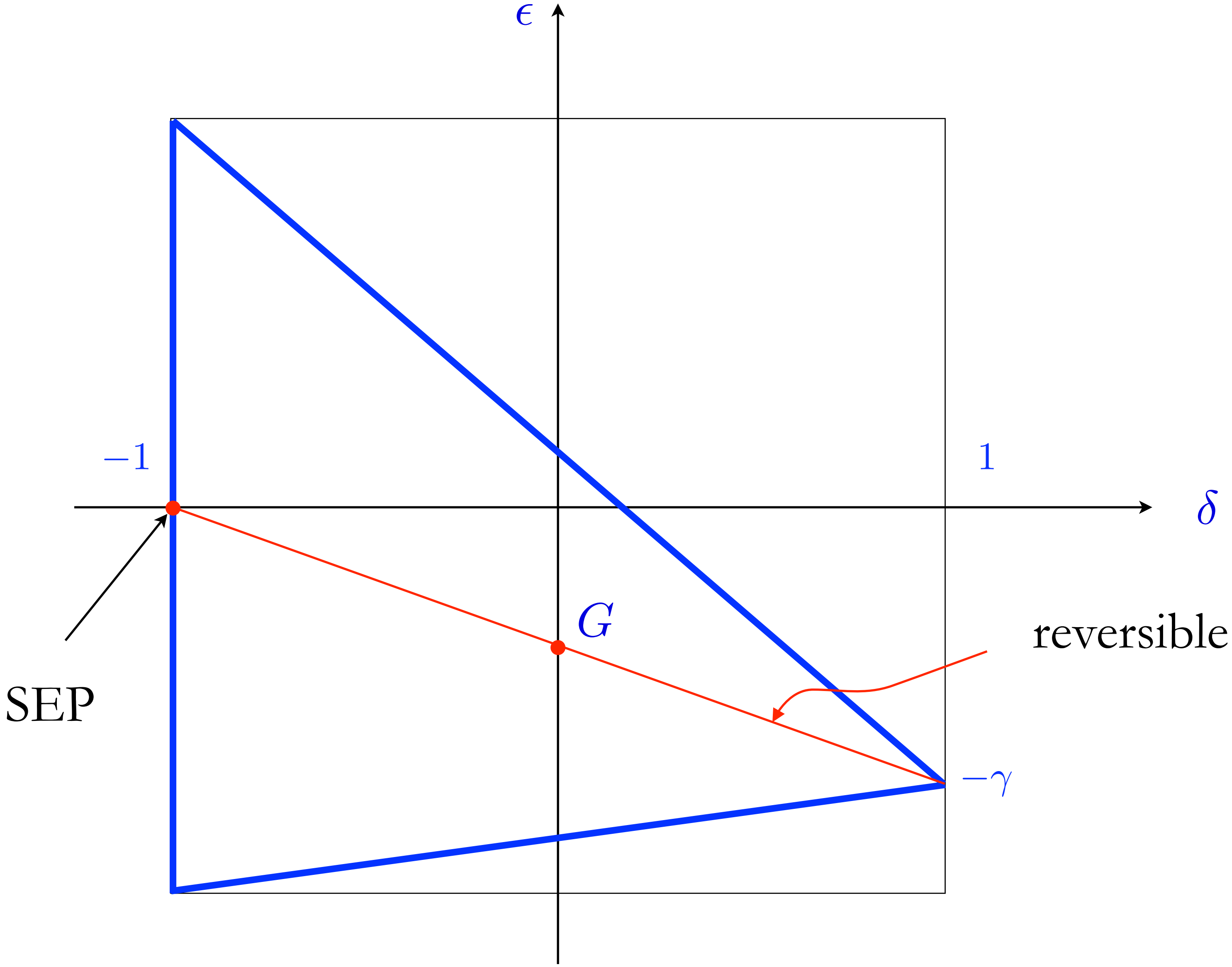
unicity (up to the time scale)

$$w(\sigma_n) = \frac{\alpha}{2}\Big[1 - \gamma\sigma_{n-1}\sigma_n\Big]$$

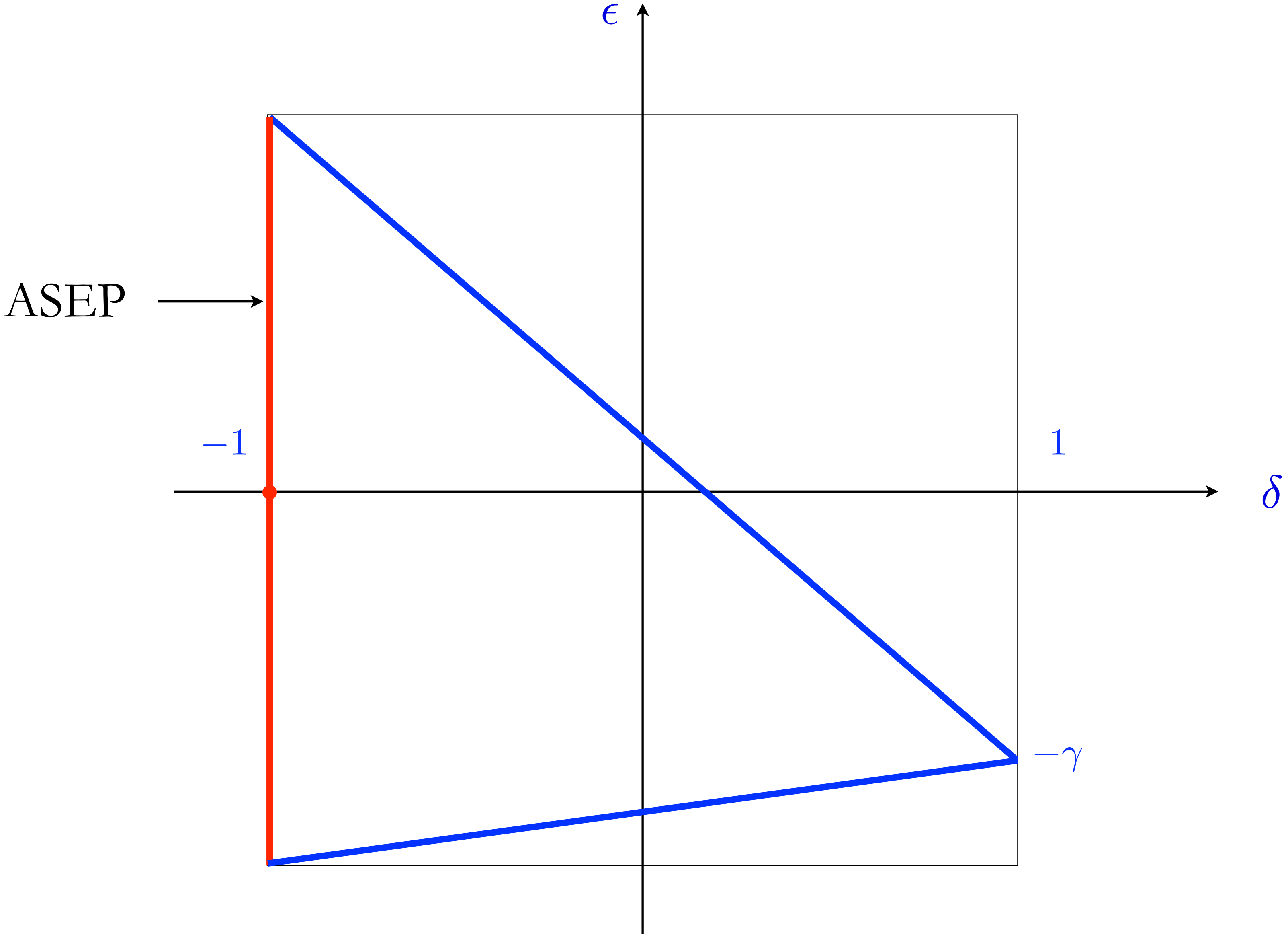


$$w = \frac{1}{2}\Big[1 - \gamma\sigma_{n-1}\sigma_n\Big], \qquad w = \frac{1}{2}\Big[1 - \gamma\sigma_n\sigma_{n+1}\Big]$$

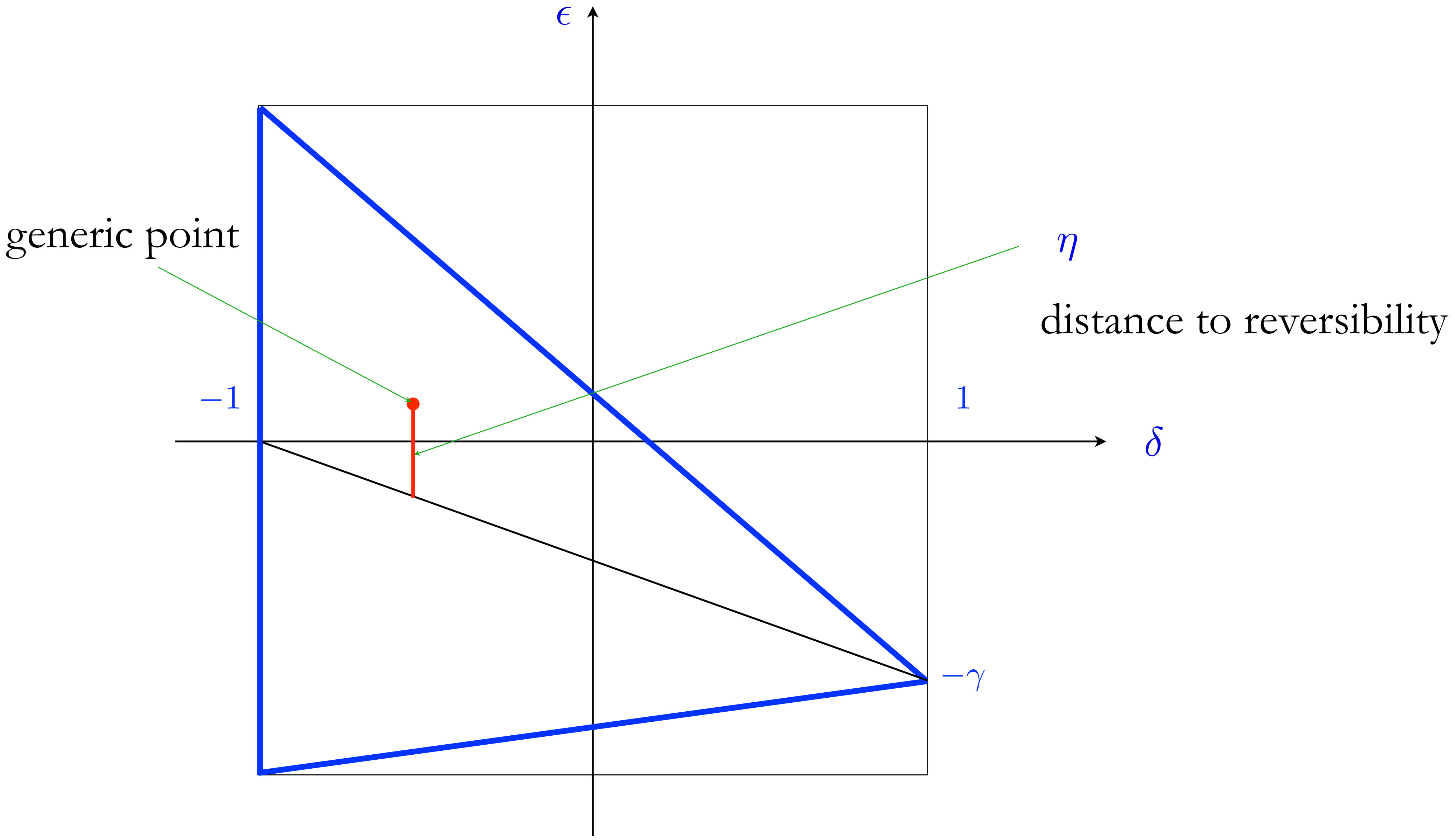

ϵ
-1
1
δ
oscillations
G
$-\gamma$



ϵ
-1
1
δ
G
reversible
SEP
$-\gamma$



ϵ
ASEP
-1
1
δ
$-\gamma$


ϵ
generic point
η
distance to reversibility
-1
1
δ
$-\gamma$

What can be said on the stationary state?

What can be said on the fluctuations of the system in the stat. state?

What can be said on the transient?

In 1D

In 2D, for those special rates leading to Gibbsian stat. state

Observables (1D)

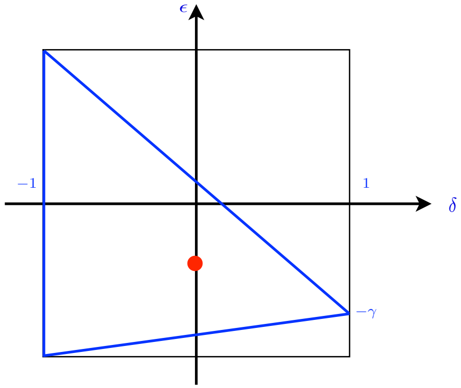
$$C_{\mathrm{transient}}(t) = \langle \sigma_0(0)\sigma_0(t) \rangle \qquad \text{(disordered initial state)}$$

relaxation of system to stationary state

$$C_{\mathrm{stat}}(t) = \langle \sigma_0(0)\sigma_0(t) \rangle \qquad \text{(thermalized initial state)}$$

fluctuations of system in stationary state

Glauber



$$C_{\text{transient}}(t) = e^{-t} I_0(\gamma t) \sim e^{-\alpha t}$$

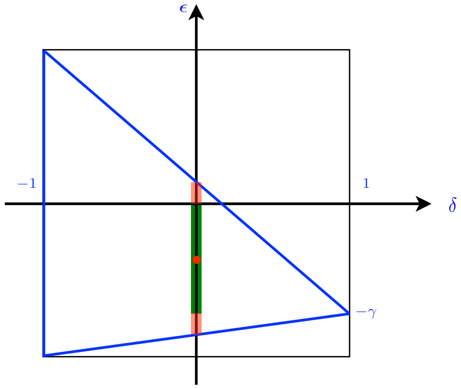
$$\alpha = 1 - \gamma$$

$$C_{\text{stat}}(t) = \sqrt{1-\gamma^2} \int_t^{\infty} du\, C_{\text{transient}}(u) \sim e^{-\alpha t}$$

cf. fluctuation-dissipation theorem at equilibrium

Glauber 1963

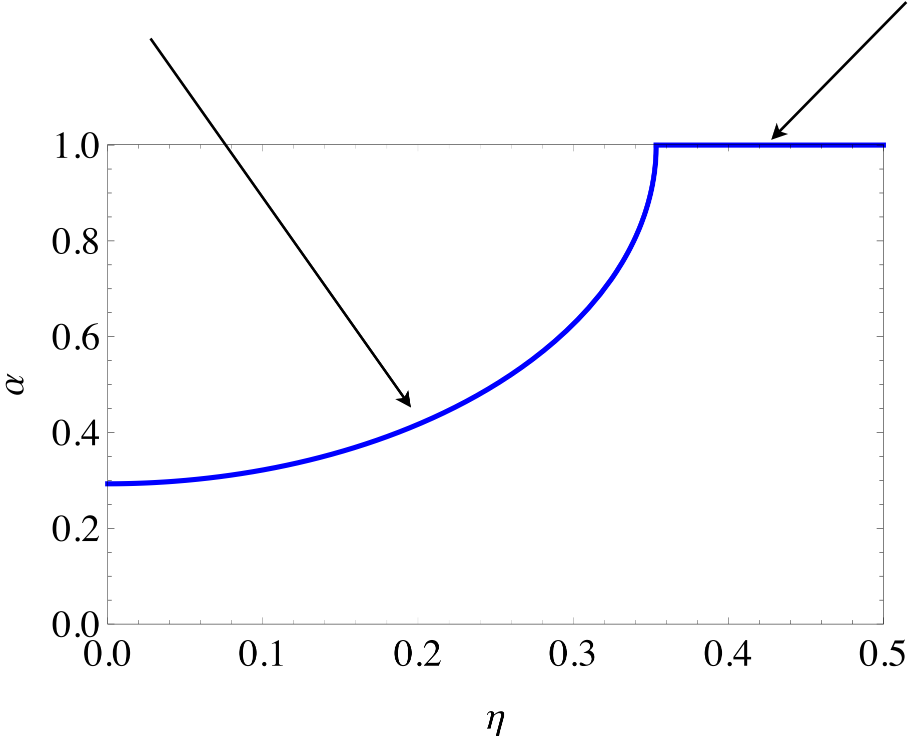
Integrable line



$$C_{\text{transient}}(t) = \langle \sigma_0(0)\sigma_0(t) \rangle = e^{-t} I_0(t(\dots)) \sim e^{-\alpha t}$$

$$\alpha = 1 - \sqrt{\gamma^2 - 4\eta^2}$$

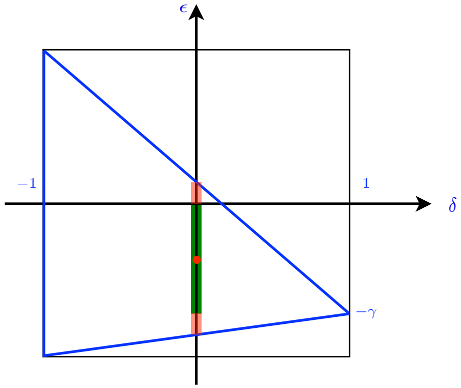
oscillations



acceleration of dynamics by irreversibility

CG 2011

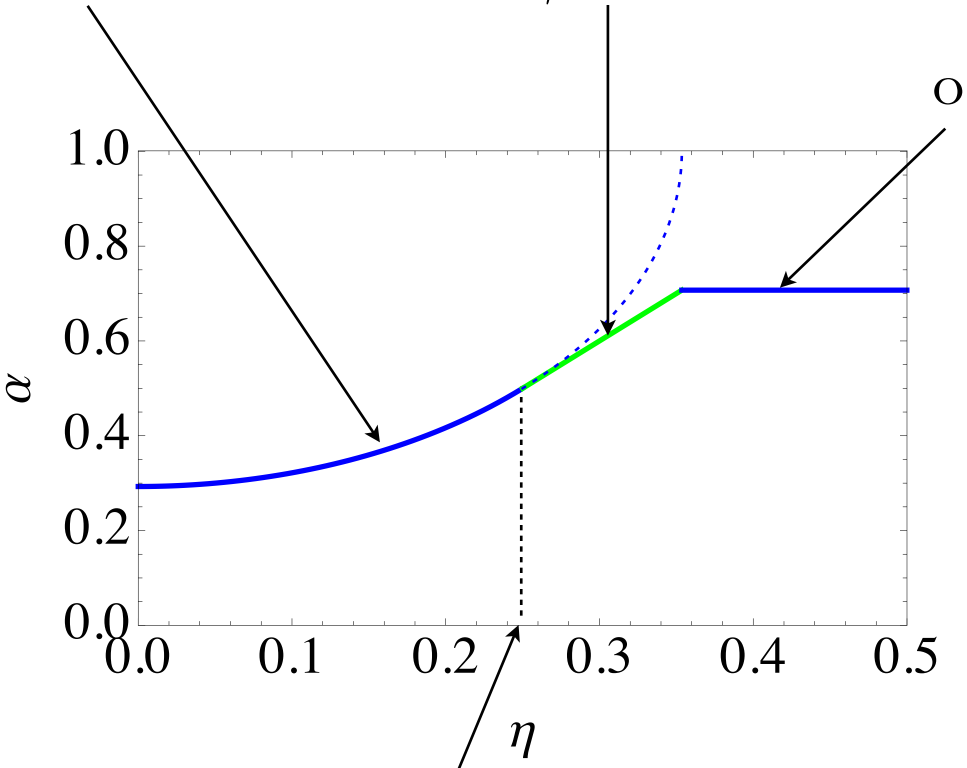
Integrable line



$$C_{\mathrm{stat}}(t) = \langle \sigma_0 \sigma_n \rangle_{\mathrm{eq}} \star C_{\mathrm{transient},n}(t) \sim e^{-\alpha_{1,2} t}$$

$$\alpha_1 = 1 - \sqrt{\gamma^2 - 4\eta^2}, \qquad \alpha_2 = \frac{2|\eta|}{\gamma}\sqrt{1-\gamma^2}$$

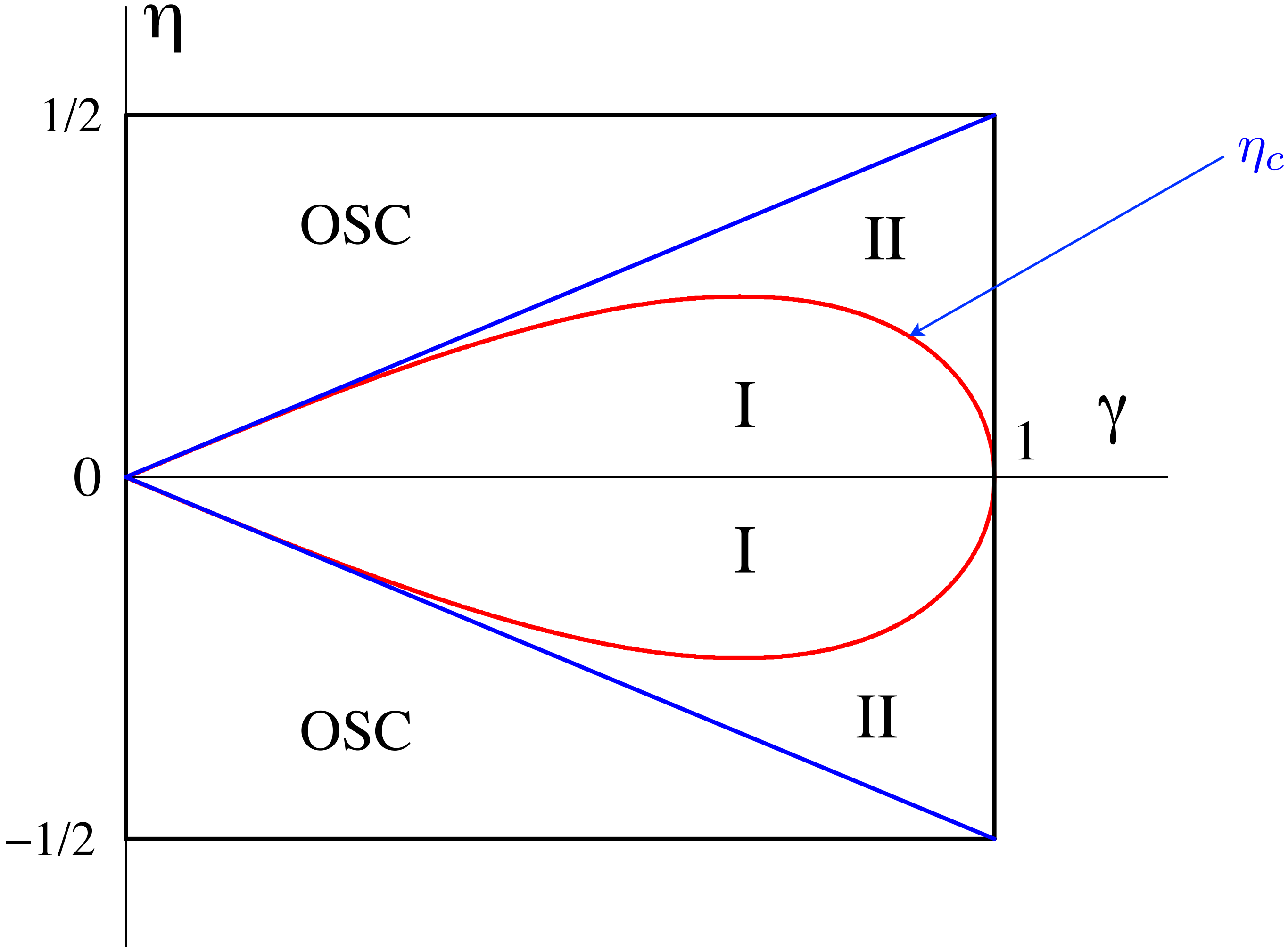
oscillations

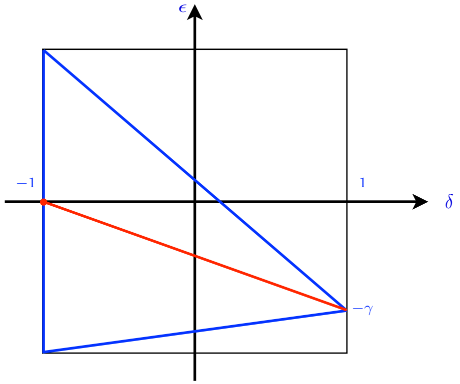


η_c bifurcation

CG 2011

Integrable line





Reversible line, infinite temperature

$$C_{\mathrm{stat}}(t) \equiv C_{\mathrm{transient}}(t) \sim e^{-\alpha t}$$

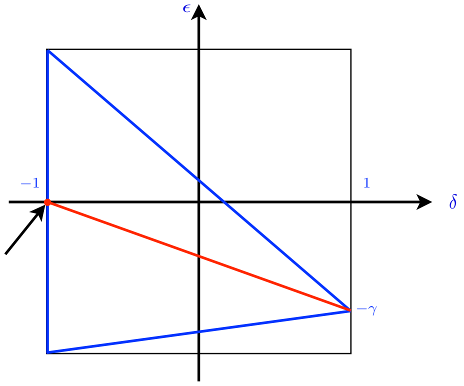
$$\alpha = \sqrt{1 - \delta^2}$$

Reversible line, finite temperature

α only known numerically

CG&JM Luck 2014

SEP point

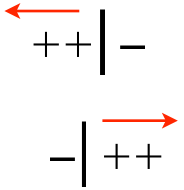


$$\delta = -1, \quad \epsilon = 0$$

$$w_n = \frac{1}{2}\Big[1 - \sigma_{n-1}\sigma_{n+1}\Big]$$

$$\begin{array}{rcl} w(+;++) & = & 0 \\ w(+;--) & = & 0 \end{array}$$

← elementary excitations



$$\begin{array}{rcl} w(+;+-) & = & 1 \\ w(+;-+) & = & 1 \end{array}$$

← motion of domain walls

infinite temperature point (no dependence in temperature)

SEP point

$$\alpha (= \sqrt{1-\delta^2}) = 0$$

$$C_{\text{stat}}(t) \equiv C_{\text{transient}}(t) = \langle (-1)^{\underset{\uparrow}{Q_t}} \rangle \sim e^{-A(\rho)\sqrt{t}}$$

number of particles (domain walls) crossing the origin during time t

cf. Spohn 1989: $A(\rho)$?

using Derrida-Gerschenfeld 2009, find

$$A(\rho) = \frac{1}{\sqrt{\pi}} \sum_{k\geq 1} \frac{(4\rho(1-\rho))^k}{k^{3/2}} \qquad (\rho = 1/2)$$

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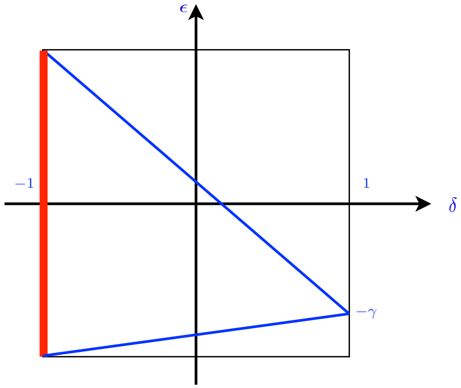
ASEP line

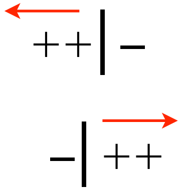
$\delta = -1$



$$w_n = \frac{1}{2}\Big[1 + \epsilon\,\sigma_n(\sigma_{n-1} - \sigma_{n+1}) - \sigma_{n-1}\sigma_{n+1}\Big]$$

$$\begin{aligned} w(+;++) &= 0 \\ w(+;--) &= 0 \end{aligned}$$

← elementary excitations



$$\begin{aligned} w(+;+-) &= 1+\epsilon \\ w(+;-+) &= 1-\epsilon \end{aligned}$$

← motion of domain walls

conjecture:

$$C_{\text{stat}}(t) \equiv C_{\text{transient}}(t) \sim e^{-B(\rho)\sqrt{t}}, \qquad B(\rho)?$$

use Tracy-Widom 2010

CG&JM Luck 2014

2D

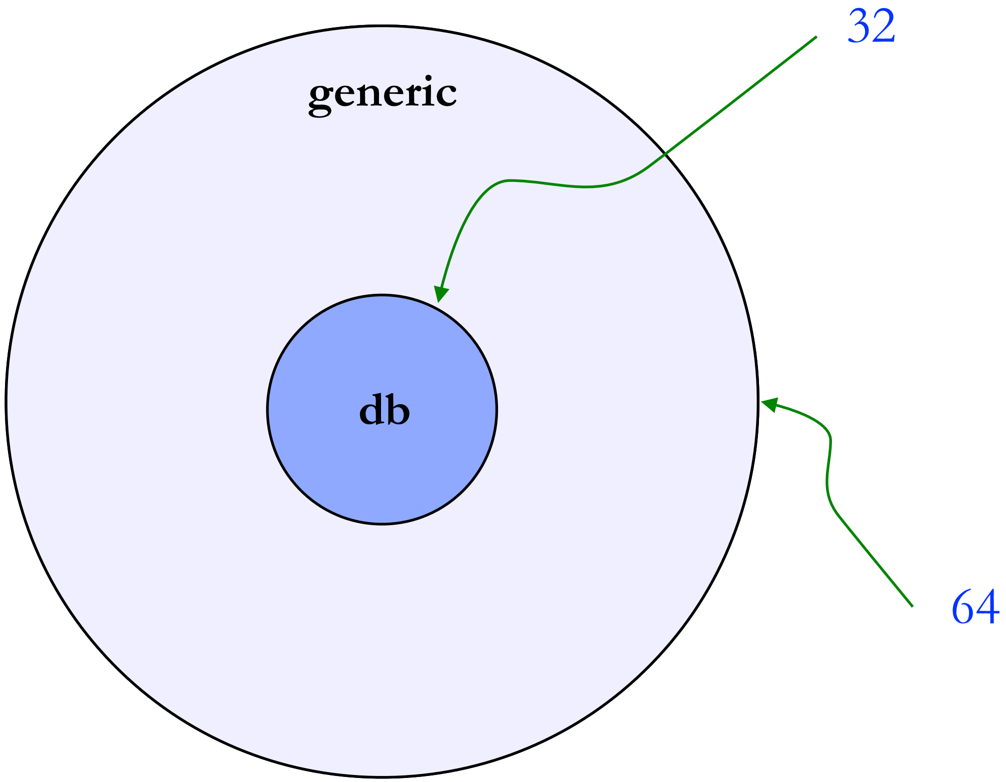
2D paramagnetic phase shares a number of properties of 1D

2D ferromagnetic phase, qualitatively different from 1D: ballistic coarsening,

power law persistence (while exponential in 1D), metastable/blocked configurations

CG&M Pleimling 2014

3D cubic lattice



Detailed balance $\Longleftarrow$ Global balance

CG 2013

More

- Relation to 2D Toom model
- Entropy production
- ...