# Supersonic propagation in long-range lattice models

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NITheP National Institute for Theoretical Physics

Universiteit Stellenbosch University

GGI Florence, 29 May 2014

based on:

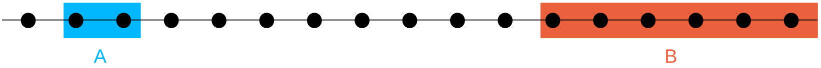
D. Métivier, R. Bachelard, and M. K., PRL (in press)

J. Eisert, M. van den Worm, S. R. Manmana, and M. K., PRL **111**, 260401 (2013)

R. Bachelard, M. K., PRL **110**, 170603 (2013)

# Stellenbosch


# Propagation in spatially extended systems



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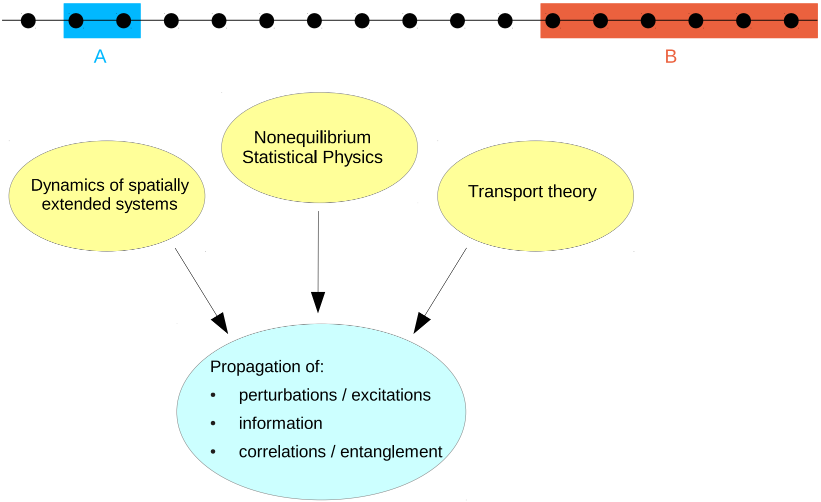

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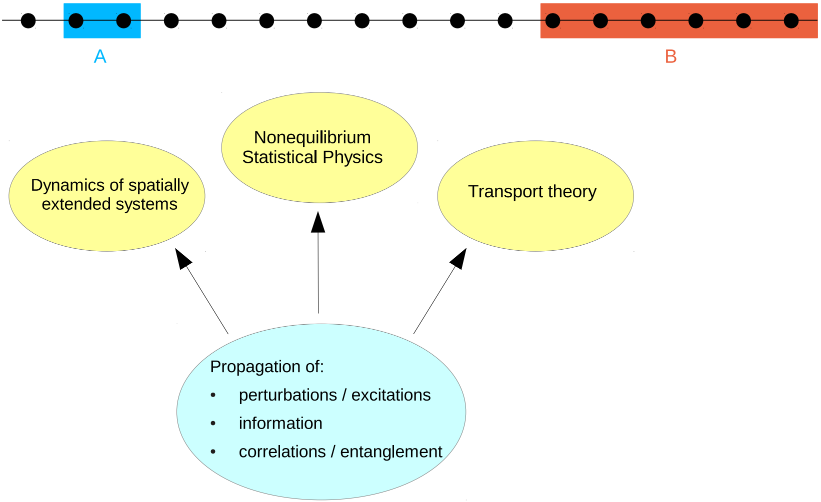

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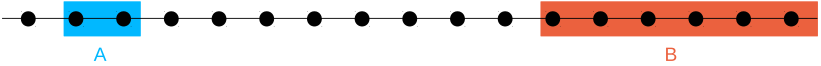
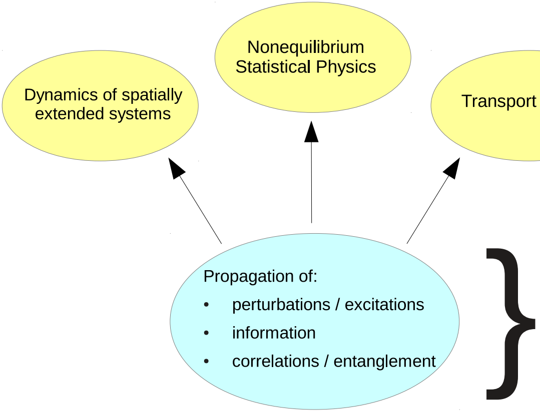
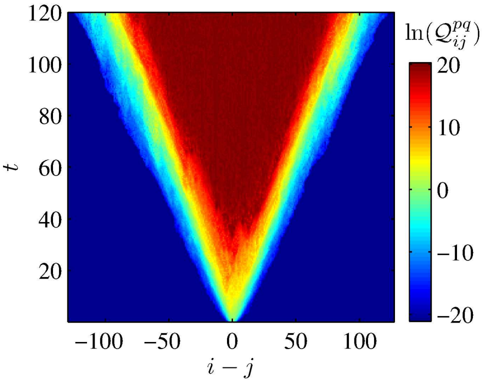



- basic “building block”
- key quantity: **group velocity**

# Velocity of propagation




In (some) condensed matter systems: propagation velocity is group velocity $\frac{\partial\omega(k)}{\partial k}$ obtained from quasi-particle dispersion

General behaviour??? $\Longrightarrow$ Lieb-Robinson bound

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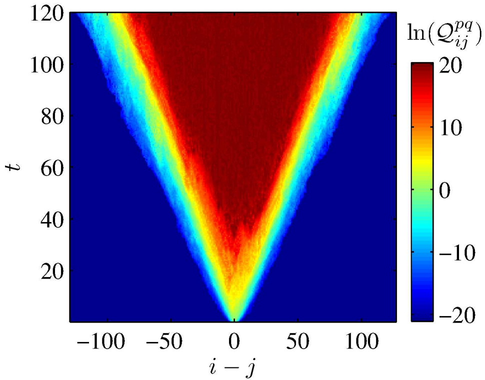


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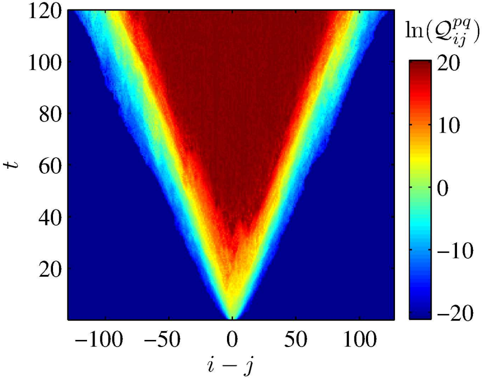


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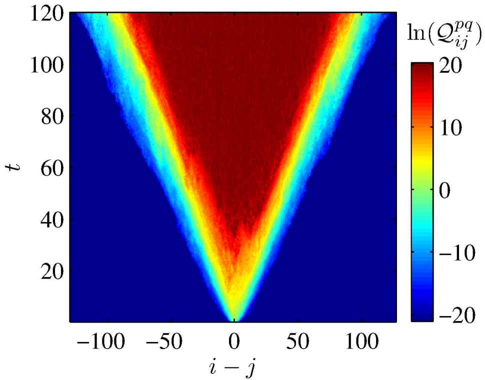


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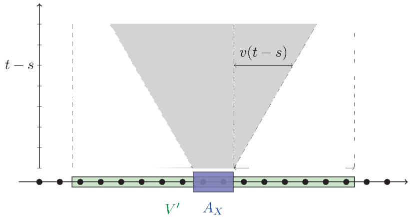
# Spatio-temporal evolution

Relativistic theory: ∃ finite maximum propagation speed




Nonrelativistic quantum lattice systems, finite local dimension, finite-range interactions:
∃ finite group velocity, with exponentially small effects outside an effective light cone

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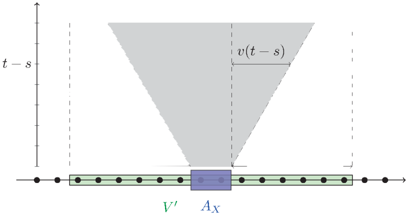


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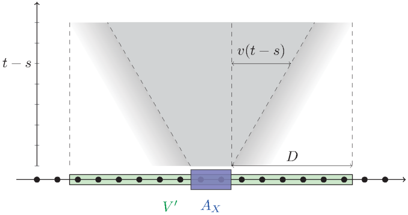

# Lieb-Robinson bound

Commun. Math. Phys. **28**, 251 (1972)

$$\| [O_A(t), O_B(0)] \| \leqslant C \, \|O_A\| \, \|O_B\| \min(|A|, |B|) \mathrm{e}^{(v|t|-d(A,B))/\xi}$$

$\exists$ finite group velocity, with exponentially small effects outside an effective light cone

- physical relevance: transmission of information, growth of entanglement, clustering of correlations, Lieb-Schultz-Mattis in $D > 1$, finite-size errors of simulations…
- very general result
- restrictions:
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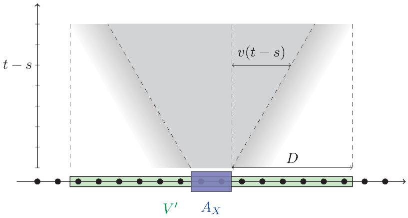


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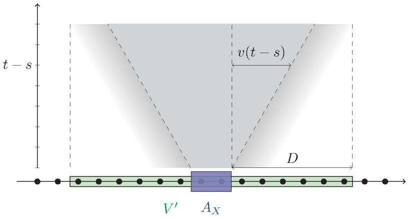


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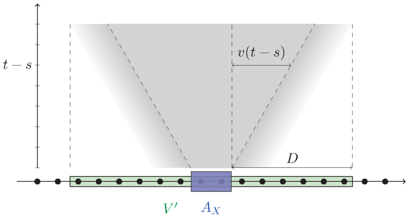


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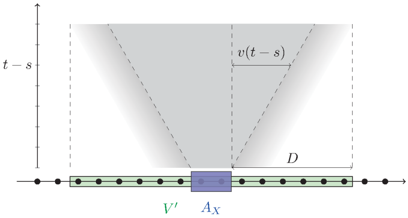


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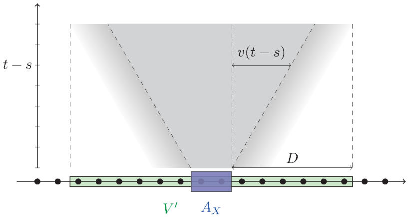


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**Short-range:** finite-range (e.g. nearest-neighbour)
or exponentially decaying ($\propto e^{-cr}$ with $c > 0$)

**Long-range:** power law decaying, $\propto 1/r^{\alpha}$ with $\alpha \geqslant 0$

Realisations of long-range many-body systems:

- Dipolar materials
- Free Electron Laser
- Rydberg atoms
- Cavity QED
- Crystals of trapped ions: $1/r^{\alpha}$

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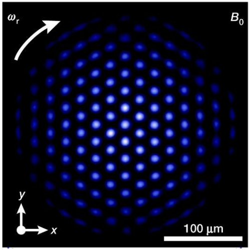


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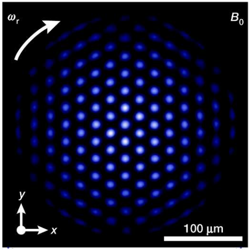


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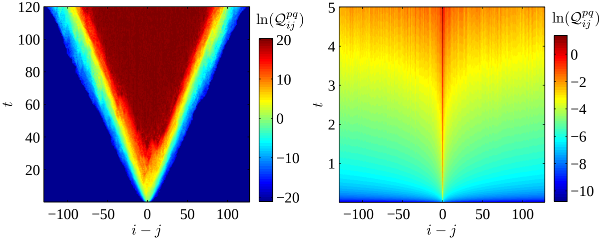


short-range long-range

Absence of a finite propagation velocity!

General predictions? Long-range Lieb-Robinson bounds?

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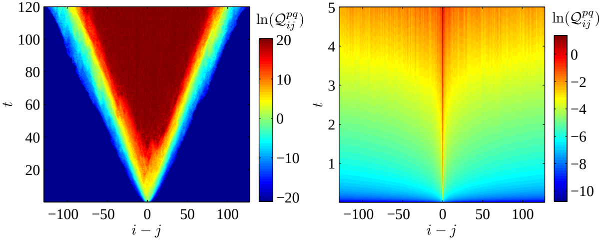


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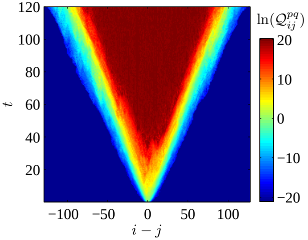


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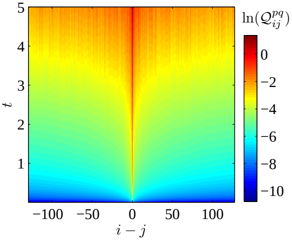


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# Long-range Lieb-Robinson bounds

Here: classical-mechanical, think of

$$H = \sum_{i\in\Lambda} \frac{p_i^2}{2} - \frac{J_\Lambda}{2} \sum_{\substack{i,j\in\Lambda \\ i\neq j}} \frac{\cos(q_i - q_j)}{|i-j|^\alpha}$$

$$|\{f_i(0), g_j(t)\}| \leq c \max\left\{ \left|\frac{\partial p_j(t)}{\partial p_i(0)}\right|, \left|\frac{\partial q_j(t)}{\partial p_i(0)}\right|, \left|\frac{\partial p_j(t)}{\partial q_i(0)}\right|, \left|\frac{\partial q_j(t)}{\partial q_i(0)}\right| \right\}$$

"Spreading of a perturbation"

$$\left|\frac{\partial q_j(t)}{\partial q_i(0)}\right| \leq \frac{\sum_{n=1}^{\infty} U_n^{ij} t^{2n}}{|i-j|^\alpha} \leq \text{const.} \times \frac{\cosh(v_\alpha t) - 1}{|i-j|^\alpha}$$

D. Métivier, R. Bachelard, M. K., PRL (in press);

M. B. Hastings and T. Koma, CMP **265**, 781 (2006); B. Nachtergaele, Y. Ogata, and R. Sims, JSP **124**, 1 (2006)

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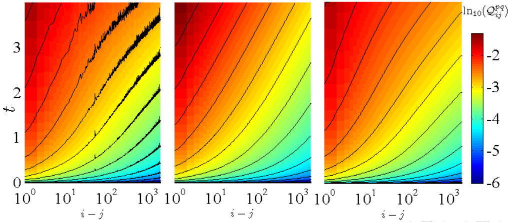

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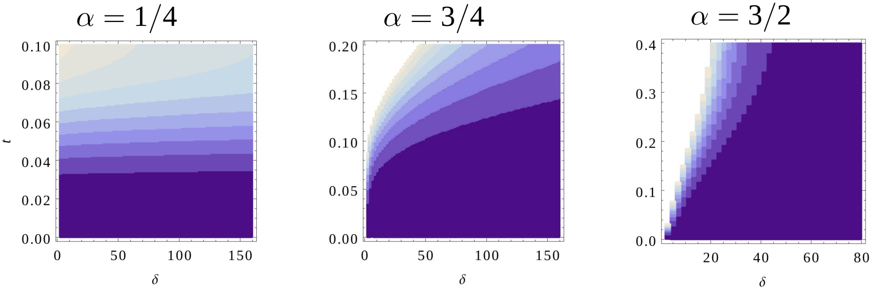


J. Eisert, M. van den Worm, S. R. Manmana, M. K., PRL **111**, 260401 (2013)

Propagation is qualitatively different in the regimes

$0 \leq \alpha < D/2$ $D/2 < \alpha < D$ $D < \alpha$

Two threshold values: $\alpha = D/2$ and $\alpha = D$

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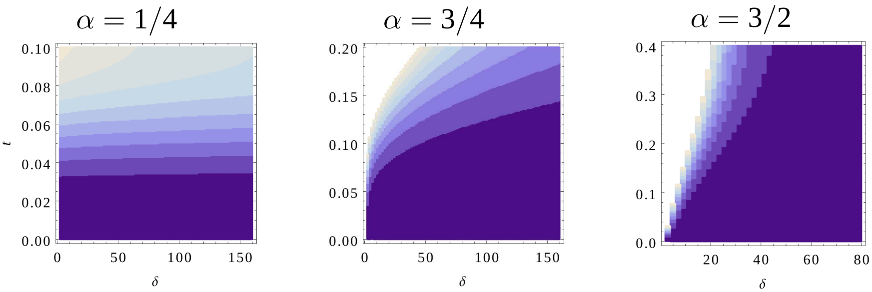


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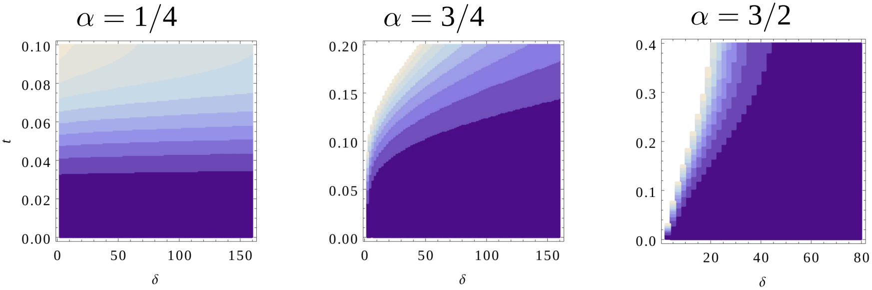


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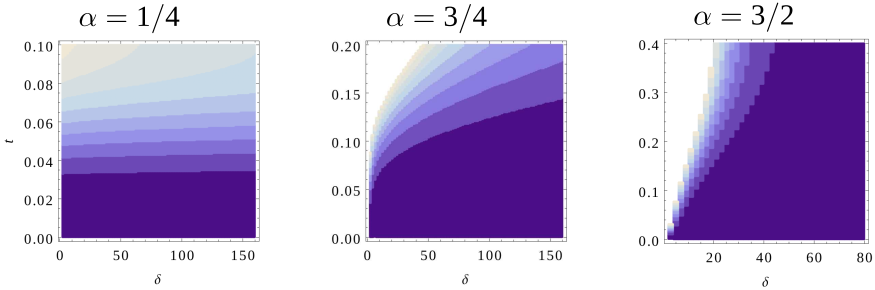


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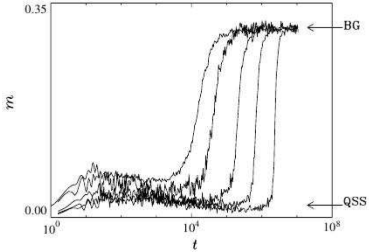


A. Campa *et al.*, Phys. Rep. **480**, 57 (2009)

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HMF model: $\tau \propto N^q$
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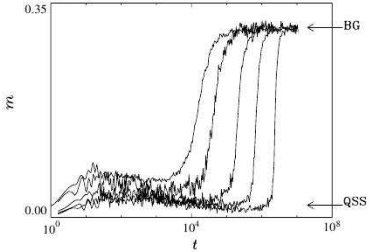


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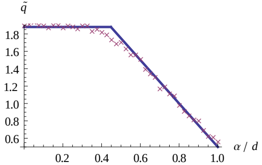


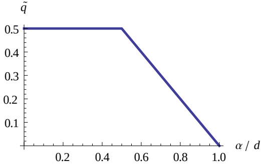


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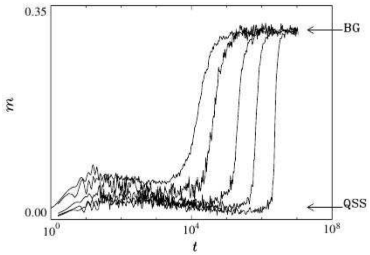


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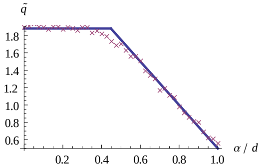


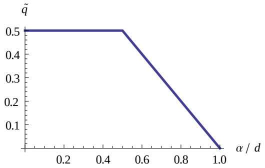


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# Experimental realisation of long-range interactions

## Beryllium ions in a Penning trap

J. W. Britton *et al.*, Engineered two-dimensional Ising interactions in a trapped-ion quantum simulator with hundreds of spins,
Nature **484**, 489 (2012).

- 2d Coulomb crystal on a triangular lattice
- Valence-electron spin states as qubits (Ising spins)
- Spin–spin interactions mediated by crystal's transverse motional degrees of freedom

- Effective (anti-)ferromagnetic Ising Hamiltonian $H = \sum_{i \neq j} J_{ij} \sigma_i^z \sigma_j^z - \sum_i \boldsymbol{B} \cdot \boldsymbol{\sigma}_i$

  D. Porras and J. I. Cirac, Phys. Rev. Lett. **96**, 250501 (2006).

- $J_{ij} \approx -\frac{J}{|i-j|^\alpha}$ with $0.05 \lesssim \alpha \lesssim 1.4$

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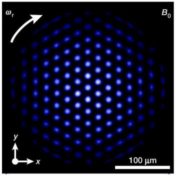


- 2d Coulomb crystal on a triangular lattice
- Valence-electron spin states as qubits (Ising spins)
- Spin–spin interactions mediated by crystal's transverse motional degrees of freedom

- Effective (anti-)ferromagnetic Ising Hamiltonian $H = \sum_{i \neq j} J_{ij} \sigma_i^z \sigma_j^z - \sum_i \boldsymbol{B} \cdot \boldsymbol{\sigma}_i$

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# Experimental realisation of long-range interactions

## Beryllium ions in a Penning trap

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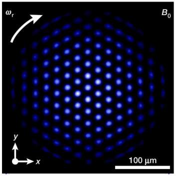


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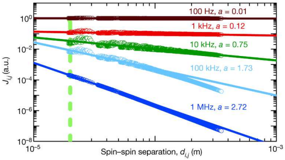
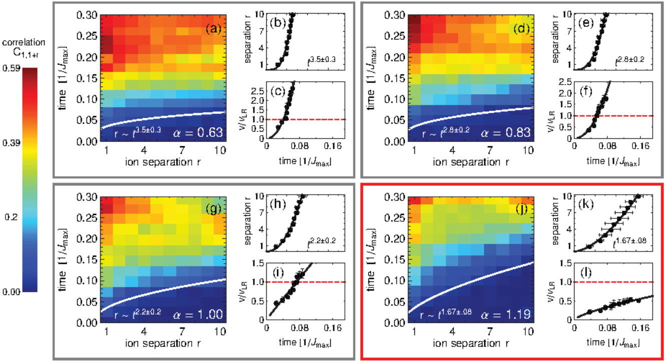


# Experimental results

long-range *XY* model $$H = -J \sum_{i,j} \frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y}{|i-j|^\alpha},$$

realised in a linear Paul ion trap




Richerme *et al.*, arXiv1401.5088

# Conclusions

- Nonequilibrium dynamics: spreading of whatsoever
- Long-range Lieb-Robinson bounds

$$\|\cdot\| \leqslant C\frac{e^{v|t|}-1}{|i-j|^{\alpha}}$$

- $\alpha$-dependence of the propagation front
- $\Longrightarrow$ $\alpha$-dependence of thermalisation
- Ion-trap emulation of long-range spin systems

D. Métivier, R. Bachelard, and M. K., PRL (in press)
J. Eisert, M. van den Worm, S. R. Manmana, and M. K., PRL **111**, 260401 (2013)
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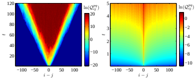

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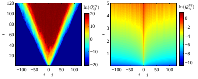

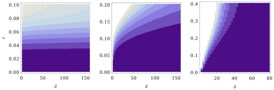

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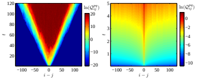

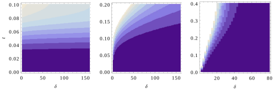

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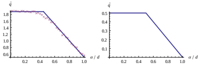

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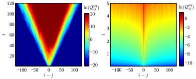

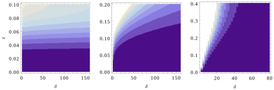

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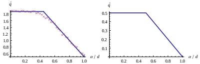

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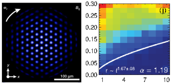

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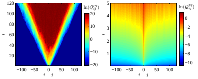

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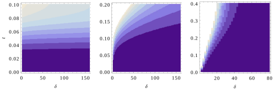

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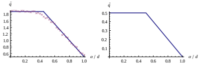

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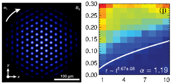


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