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Information thermodynamics of a feedback control for cold damping

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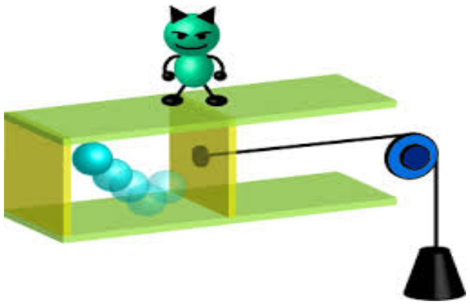
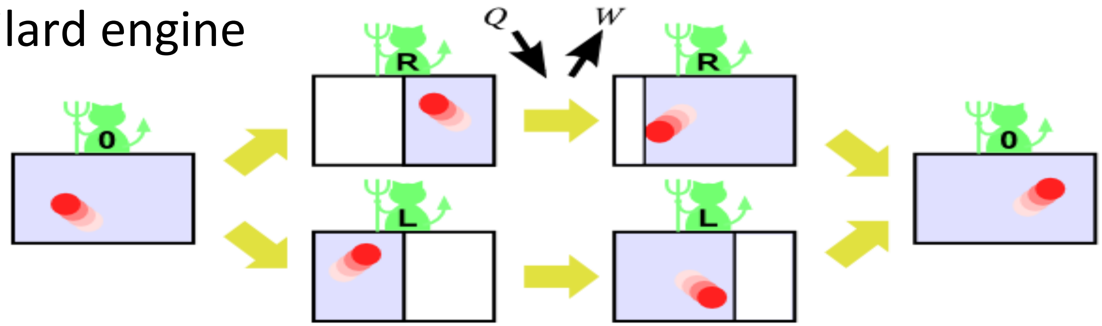
I. Feedback control: the exorcism of Maxwell's demon
II. Post-measurement process
III. Multi-step feedback control with time delay
IV. Cold damping
V. Summary

I. Feedback control: the exorcism of a Maxwell's demon

1. The paradox of Maxwell's demon: the violation of 2nd law
   -- Maxwell (1867), Szilard (1929), Landauer(1961), ….



Szilard engine



$$\Delta S = -\frac{Q}{T} = -k_B \ln 2 < 0$$

Incomplete exorcism

- Demonic entropy $\Delta S_{DM} = k_B \ln 2, \quad \Delta(S + S_{DM}) = 0$

  The state of demon?

- Landauer's principle

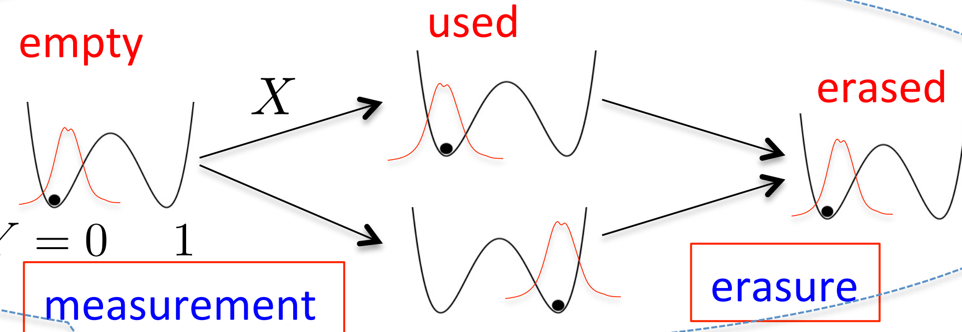
  $$W_{eraser} \geq k_B T \ln 2, \quad Q_{eraser}/T \geq k_B \ln 2$$

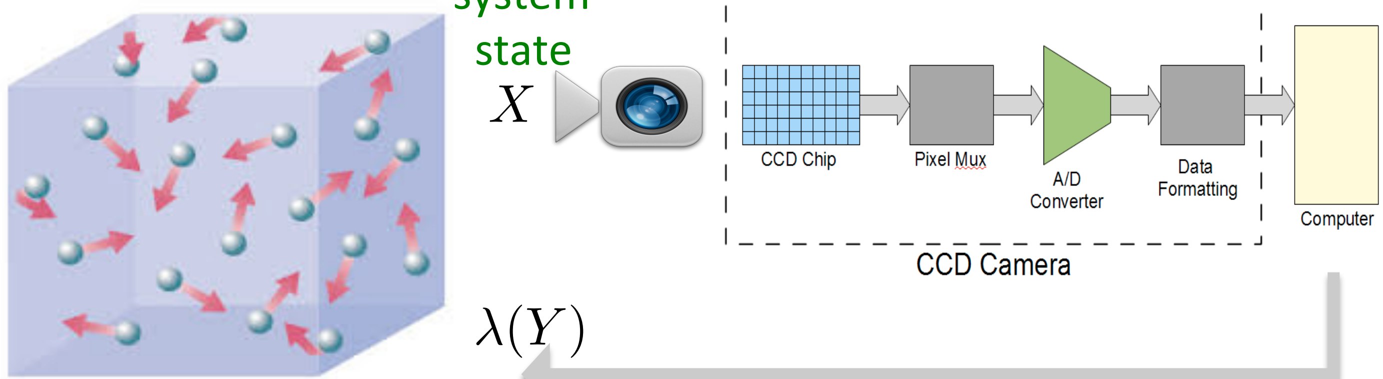
  Nothing to do with the system's time evolution

2. Feedback control

Bit information process

Hidden process





Post-measurement process fixed Y $\quad X \longrightarrow X'$ **observable**

Fluctuation theorem $\langle e^{-\Delta S_{total}} \rangle = 1$ $\quad \Delta S_{total} = \Delta S_{sys} + \Delta S_{env} - \Delta I$

Generalized 2nd law $\langle \Delta S_{total} \rangle \geq 0$

T. Sagawa and M. Ueda, PRL (2012); NJP (2013)
S. Ito and T. Sagawa, PRL (2013)

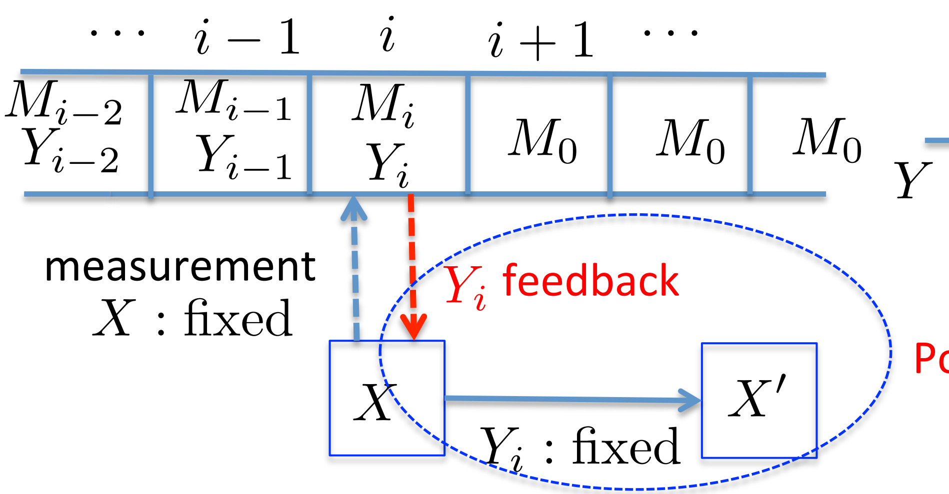
II. Fluctuation theorem for post-measurement process

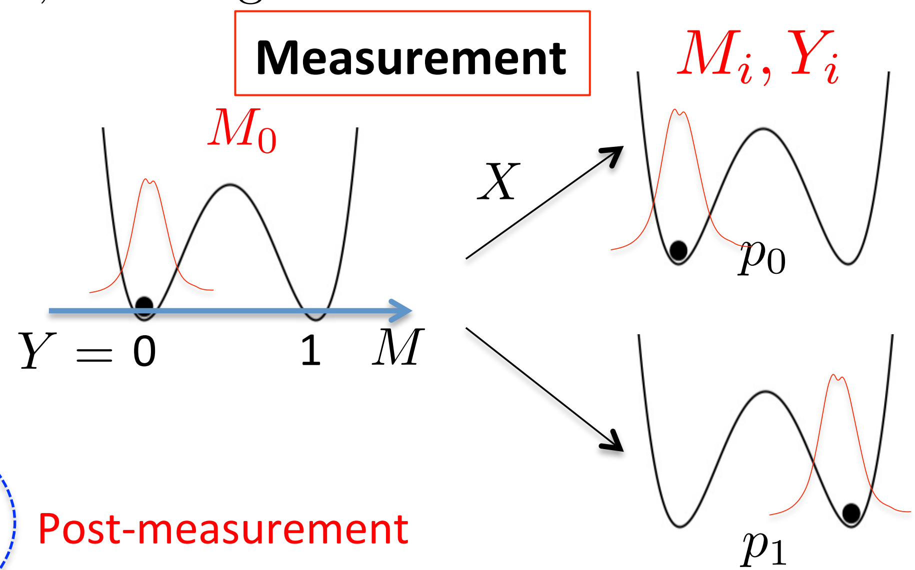
Schematic view

X : state of the system (gas)

M : microstate of the physical memory device (bit memory in Fig)

Y : outcome of measurement for X, coarse-grained state of the device





M_i : resultant memory state in i-th measurement

M_0 : unused memory state

$X \longrightarrow_Y X'$: state change with fixed protocol Y, post-measurement process

Probability of measurement outcome $\mathbf{y}$ given system state $\mathbf{q}$

$$\rho(\mathbf{y}|\mathbf{q}) = \frac{1}{(2\pi\sigma)^{d/2}} e^{-\frac{1}{2\sigma}(\mathbf{y}-\mathbf{q})^2}$$

Szilard engine

$$x,\ y = 0\ \text{(LEFT)},\ 1\ \text{(RIGHT)}$$

$$\rho(y|x) = \begin{cases} 1-\epsilon & y=1\ x=1 \\ \epsilon & y=0\ x=1 \\ 1-\epsilon & y=0\ x=0 \\ \epsilon & y=1\ x=0 \end{cases}$$

Post-measurement process by feedback control

$\mathbf{y}$: external protocol

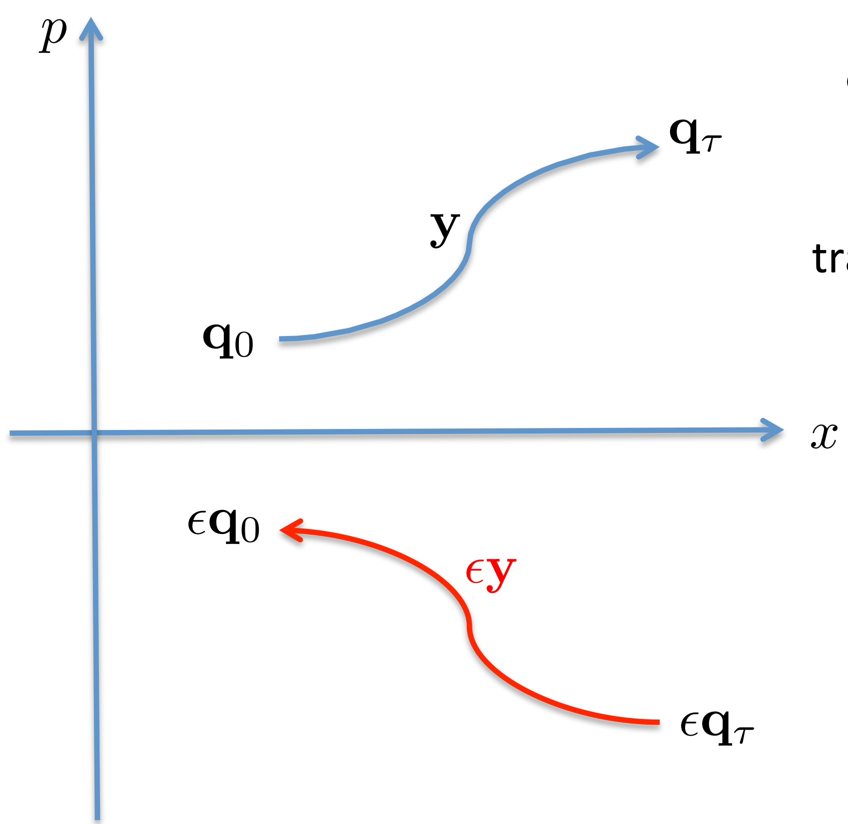
$$\dot{\mathbf{p}} = -\gamma\frac{\mathbf{p}}{m} + \mathbf{f}(\mathbf{x},t) + \mathbf{g}(\mathbf{y}) + \mathbf{z}(t)$$

$$\langle z_i(t) z_j(t')\rangle = 2\gamma T\delta(t-t')$$

Schnackenberg, 1976, master equation

Environmental entropy production

$$e^{\Delta S_{env}} = \frac{P[\mathbf{q}(t), 0 < t < \tau; \mathbf{y}]}{P[\epsilon\mathbf{q}(\tau - t), 0 < t < \tau; \epsilon\mathbf{y}]} = \frac{P[\mathbf{q}_0 \rightarrow \mathbf{q}_\tau; \mathbf{y}]}{P[\epsilon\mathbf{q}_0 \leftarrow \epsilon\mathbf{q}_\tau; \epsilon\mathbf{y}]}$$



ϵ : parity operator

trajectory $\mathbf{q}(t) = (\mathbf{x}(t), \mathbf{p}(t))$

time-reverse trajectory

$$\epsilon\mathbf{q}(\tau - t) = (\mathbf{x}(\tau - t), -\mathbf{p}(\tau - t))$$

$$\Delta S_{env} = \frac{Q}{T} + \Delta S_{odd}$$

anomalous entropy production
due to odd-parity force

$$\mathbf{g}(-\mathbf{y}) = -\mathbf{g}(\mathbf{y})$$

Total entropy production

$$e^{\Delta S_{tot}} = \frac{\rho(\mathbf{q}_0)\rho(\mathbf{y}|\mathbf{q}_0)P[\mathbf{q}_0 \to \mathbf{q}_\tau; \mathbf{y}]}{P[\epsilon\mathbf{q}_0 \leftarrow \epsilon\mathbf{q}_\tau; \epsilon\mathbf{y}]\rho(\mathbf{q}_\tau, \mathbf{y})} \quad = \frac{P^F}{P^R}$$

$$\rho(\mathbf{q}_\tau, \mathbf{y}) = \int d\mathbf{q}_0 \int D\mathbf{q}(t)\rho(\mathbf{q}_0)\rho(\mathbf{y}|\mathbf{q}_0)P[\mathbf{q}_0 \to \mathbf{q}_\tau; \mathbf{y}]$$
$$= \rho(\mathbf{q}_\tau)\rho_c(\mathbf{y}|\mathbf{q}_\tau)$$

Fluctuation Theorem

$$\langle e^{-\Delta S_{tot}} \rangle = 1 \quad \longrightarrow \quad \langle \Delta S_{tot} \rangle \geq 0$$

$$\sum_{\mathbf{q}_0, \mathbf{q}(t)} P^F \frac{P^R}{P^F} = 1$$

$$\Delta S_{tot} = -\ln \frac{\rho(\mathbf{q}_\tau)}{\rho(\mathbf{q}_0)} + \Delta S_{env} - \ln \frac{\rho_c(\mathbf{y}|\mathbf{q}_\tau)}{\rho(\mathbf{y}|\mathbf{q}_0)}$$
$$= -\ln \frac{\rho(\mathbf{q}_\tau)}{\rho(\mathbf{q}_0)} + \Delta S_{env} - \ln \frac{\rho_c(\mathbf{y}|\mathbf{q}_\tau)}{\rho(\mathbf{y})} \frac{\rho(\mathbf{y})}{\rho(\mathbf{y}|\mathbf{q}_0)}$$
$$= \Delta S_{sys} + \Delta S_{env} - \Delta I$$

Mutual information

$$I(\mathbf{y} : \mathbf{q}_0) = \ln \frac{\rho(\mathbf{y}, \mathbf{q}_0)}{\rho(\mathbf{q}_0)\rho(\mathbf{y})} = \ln \frac{\rho(\mathbf{q}_0|\mathbf{y})}{\rho(\mathbf{q}_0)} = \ln \frac{\rho(\mathbf{y}|\mathbf{q}_0)}{\rho(\mathbf{y})}$$

$$I(\mathbf{y} : \mathbf{q}_\tau) = \frac{\rho_c(\mathbf{y}|\mathbf{q}_\tau)}{\rho(\mathbf{y})}$$

$$x-1 \geq \ln x \;\rightarrow\; \ln x \geq 1-\frac{1}{x}$$

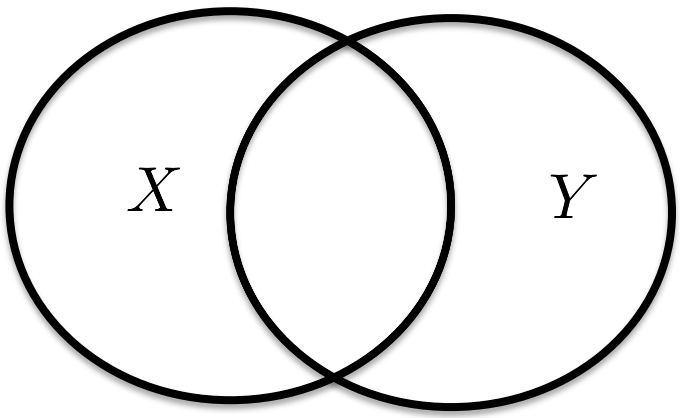
$$\int dx \int dy\; P(Y|X)P(X) \ln \frac{P(Y|X)}{P(Y)}$$

$$\geq \int dx \int dy\; P(Y|X)P(X) \left(1-\frac{P(Y)}{P(Y|X)}\right) = 0 \quad , \langle I \rangle \geq 0$$

Entropy of joint probability $P(X,Y)$ = the volume of $X \cup Y$

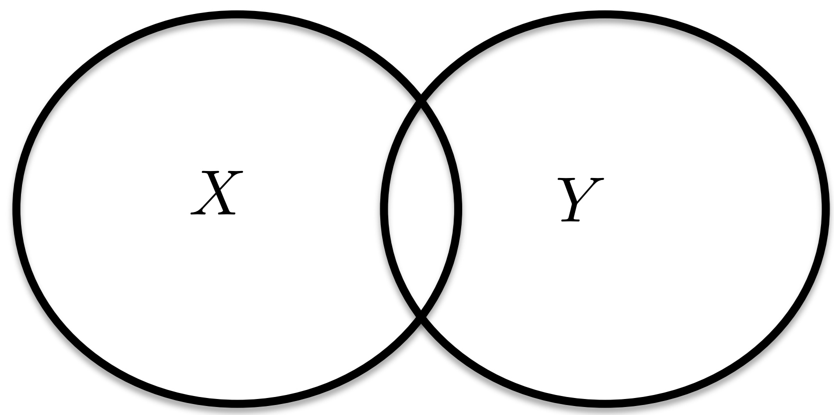
mutual information = the volume of $X \cap Y$

more correlation, more information



lower entropy
higher mutual information
at measurement

Less correlation, less information



higher entropy
lower mutual information
in post-measurement

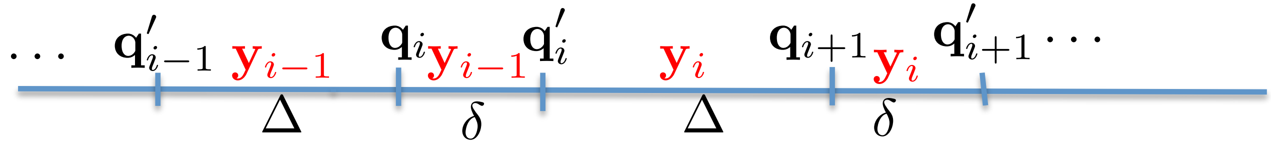
Entropy is increased by the amount $|\Delta I| = -\Delta I$

III. Multi-step feedback control with time delay

Measurement at time t_i : $\mathbf{y}_i$

Process in time delay δ : $\mathbf{q}_i \to \mathbf{q}_i'$ by previous protocol $\mathbf{y}_{i-1}$

Feedback process in time interval Δ : $\mathbf{q}_i' \to \mathbf{q}_{i+1}$ by new protocol $\mathbf{y}_i$



$$\cdots P[\mathbf{q}_{i-1}' \to \mathbf{q}_i, \Delta; \mathbf{y}_{i-1}]\rho(\mathbf{y}_i|\mathbf{q}_i)P[\mathbf{q}_i \to \mathbf{q}_i', \delta; \mathbf{y}_{i-1}]P[\mathbf{q}_i' \to \mathbf{q}_{i+1}, \Delta; \mathbf{y}_i]\rho(\mathbf{y}_{i+1}|\mathbf{q}_i)\cdots$$

process between t_i and t_{i+1}

Joint PDF at measurement i

$$\rho(\mathbf{q}_i, \mathbf{y}_{i-1}) = \int d\mathbf{q}_{i-1}' \int D[\mathbf{q}_{i-1}' \to \mathbf{q}_i]\rho(\mathbf{q}_{i-1}')P[\mathbf{q}_{i-1}' \to \mathbf{q}_i, \Delta; \mathbf{y}_{i-1}]$$

Joint PDF after measurement i

$$\rho(\mathbf{q}_i, \mathbf{y}_{i-1}, \mathbf{y}_i) = \rho(\mathbf{q}_i, \mathbf{y}_{i-1})\rho(\mathbf{y}_i|\mathbf{q}_i)$$

Joint PDF before measurement $i+1$

$$\rho(\mathbf{q}_{i+1}, \mathbf{y}_i, \mathbf{y}_{i-1}) = \int d\mathbf{q}_i \int D[\mathbf{q}_i \to \mathbf{q}_{i+1}]\rho(\mathbf{q}_i, \mathbf{y}_{i-1}, \mathbf{y}_i)$$
$$\times P[\mathbf{q}_i \to \mathbf{q}_i', \Delta; \mathbf{y}_{i-1}]P[\mathbf{q}_i' \to \mathbf{q}_{i+1}, \Delta; \mathbf{y}_i]$$

Total entropy production

$$e^{\Delta S_{tot}} = \frac{P^F}{P^R} =$$

$$\frac{\rho(\mathbf{q}_i, \mathbf{y}_{i-1}, \mathbf{y}_i) \quad P[\mathbf{q}_i \to \mathbf{q}_i', \delta; \mathbf{y}_{i-1}] P[\mathbf{q}_i' \to \mathbf{q}_{i+1}, \Delta; \mathbf{y}_i]}{P[\epsilon\mathbf{q}_i \to \epsilon\mathbf{q}_i', \delta; \mathbf{y}_{i-1}] P[\epsilon\mathbf{q}_i' \to \epsilon\mathbf{q}_{i+1}, \Delta; \epsilon\mathbf{y}_i] \rho(\mathbf{q}_{i+1}, \mathbf{y}_{i-1}, \mathbf{y}_i)}$$

Fluctuation theorem

$$\langle e^{-\Delta S_{tot}} \rangle = 1 \qquad \because \sum_{\mathbf{q}} P^F \cdot \frac{P^R}{P^F} = 1$$

Generalized 2nd law

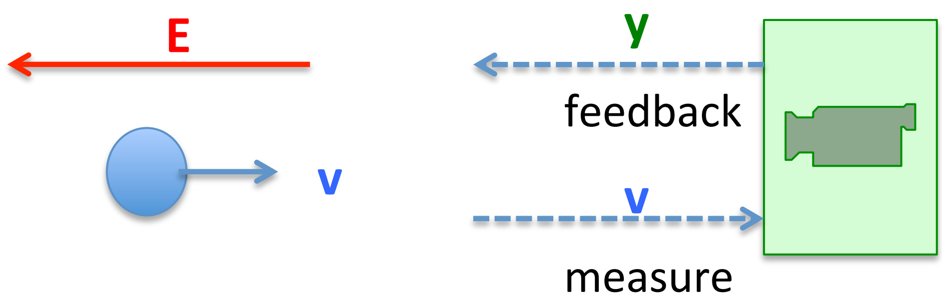
$$\langle \Delta S_{tot} \rangle \geq 0$$

$$\Delta S_{tot} = -\ln \frac{\rho(\mathbf{q}_i, \mathbf{y}_{i-1}, \mathbf{y}_i)}{\rho(\mathbf{q}_{i+1}, \mathbf{y}_{i-1}, \mathbf{y}_i)} + \Delta S_{env,\delta} + \Delta S_{env,\Delta}$$

$$= \Delta S_{sys} - \Delta I(\mathbf{q} : \mathbf{y}_{i-1}, \mathbf{y}_i) + \Delta S_{env}$$

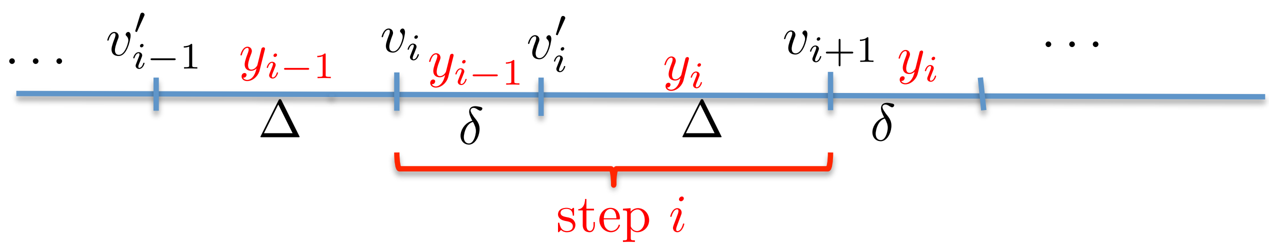
IV. Cold damping

Kim & Qian, PRL, 2007; Jourdan et al, Nanotechnology, 2007;
Ito & Sano, PRE, 2011



$$E = -\gamma' \frac{y}{q}, \quad g = -\gamma' y$$

$$\dot{v} = -\gamma v - \gamma' y + \xi(t), \quad \langle \xi(t)\xi(t') \rangle = 2\gamma T \delta(t - t'), \quad f(x) = 0, \; m = 1$$



$$u = v + \frac{\gamma'}{\gamma} y \quad \longrightarrow \quad \dot{u} = -\gamma u + \xi$$

Joint PDF at i

$$\rho(v_i, y_{i-1}) = \left[\frac{a_i c_i - b_i^2}{2\pi}\right]^{1/2} e^{-\frac{1}{2}(a_i v_i^2 - b_i v_i y_{i-1} + c_i y_{i-1}^2)}$$

$$\text{Temperature at } i \quad T_i = \frac{c_i}{a_i c_i - b_i^2}$$

Measurement probability $\quad \rho(y_i|v_i) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(y - v_i)^2}$

Propagator for $v_i \to v_i' \to v_{i+1}$

$$P(v_i \to v_i' \to v_{i+1}) = \left[\frac{A_\delta}{2\pi}\right]^{1/2} e^{-\frac{A_\delta}{2}(u_i' - e^{-\gamma\delta} u_i)} \left[\frac{A_\Delta}{2\pi}\right]^{1/2} e^{-\frac{A_\Delta}{2}(u_{i+1} - e^{-\gamma\Delta} u_i')}$$

$$\text{where } u_i = v_i + \frac{\gamma'}{\gamma} y_{i-1},\ u_i' = v_i' + \frac{\gamma'}{\gamma} y_{i-1},\ u_{i+1} = v_{i+1} + \frac{\gamma'}{\gamma} y_i$$

$$A_\delta = \beta(1 - e^{-2\gamma\delta})^{-1},\ A_\Delta = \beta(1 - e^{-2\gamma\Delta})^{-1}$$

$$\beta = T^{-1} \ \text{ inverse temperature of heat bath}$$

Joint PDF for $v_i, y_{i-1},\ v_i',\ v_{i+1},\ y_i$: Gaussian with 5 variables

$$P(v_i, y_{i-1},\ v_i',\ v_{i+1},\ y_i) = \rho(v_i, y_{i-1})\rho(y_i|v_i)P(v_i \to v_i' \to v_{i+1})$$

Joint PDF for $v_{i+1},\ y_i$

$$\rho(v_{i+1}, y_i) = \int dv_i \int dv_i' \int dy_{i-1} P(v_i, y_{i-1},\ v_i',\ v_{i+1},\ y_i)$$

$$= \left[\frac{a_{i+1}c_{i+1} - b_{i+1}^2}{2\pi}\right]^{1/2} e^{-\frac{1}{2}(a_{i+1}v_{i+1}^2 - b_{i+1}v_{i+1}y_i + c_{i+1}y_i^2)}$$

Temperature at $i+1$ $\quad T_{i+1} = \dfrac{c_{i+1}}{a_{i+1}c_{i+1} - b_{i+1}^2}$

Recursion relation $\quad a_i,\ b_i,\ c_i,\ T_i \quad \rightarrow \quad a_{i+1},\ b_{i+1},\ c_{i+1},\ T_{i+1}$

Plot T_i, starting from $T_0 = T = 1$

$$\gamma = 1, \quad \gamma' = 0.4,\ \sigma = 0.1,$$

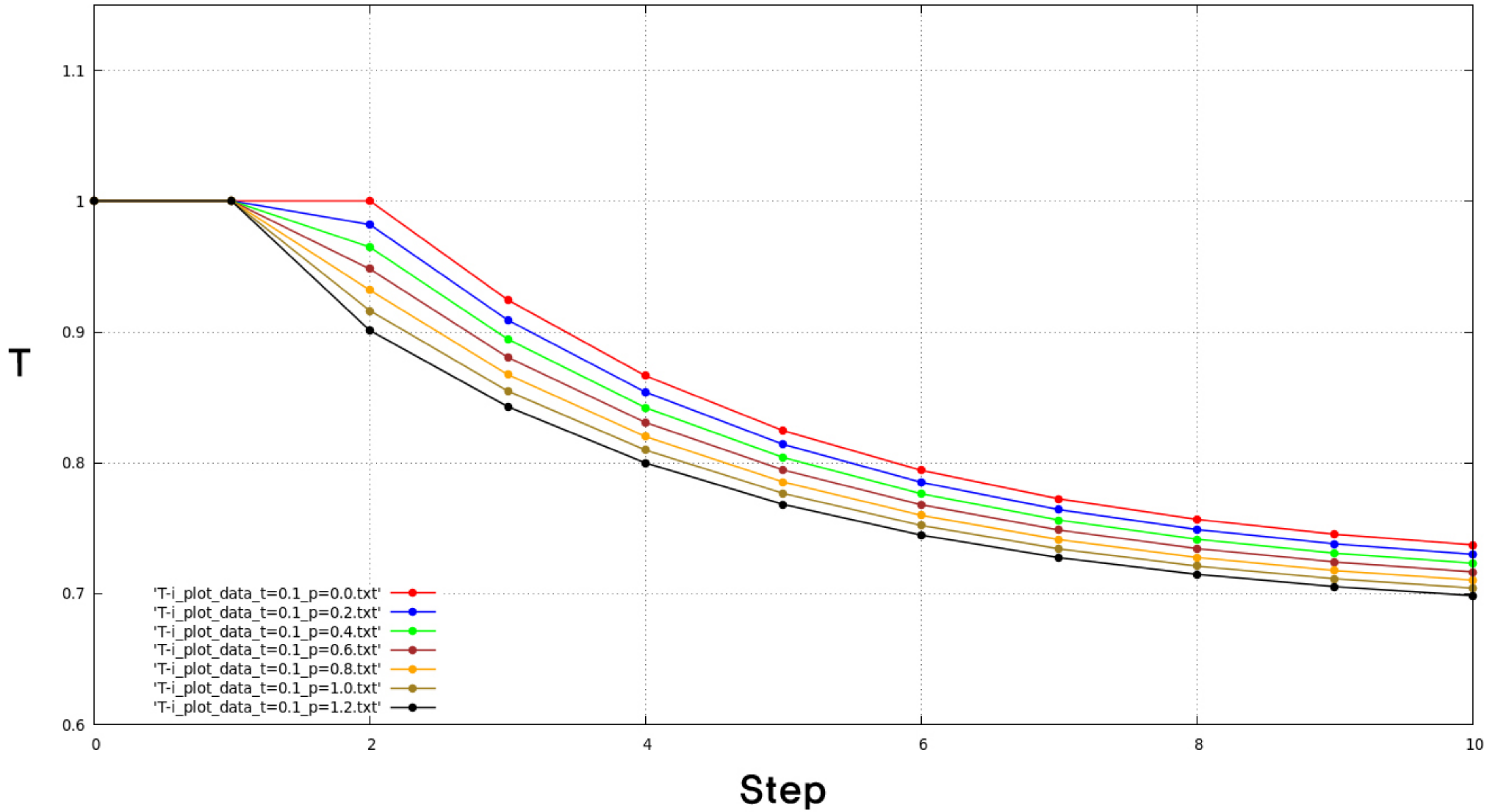
$$\delta + \Delta = t, \quad \delta = pt$$

$$t = 0.1,\ 0.5,\ 1,\ 1.2$$

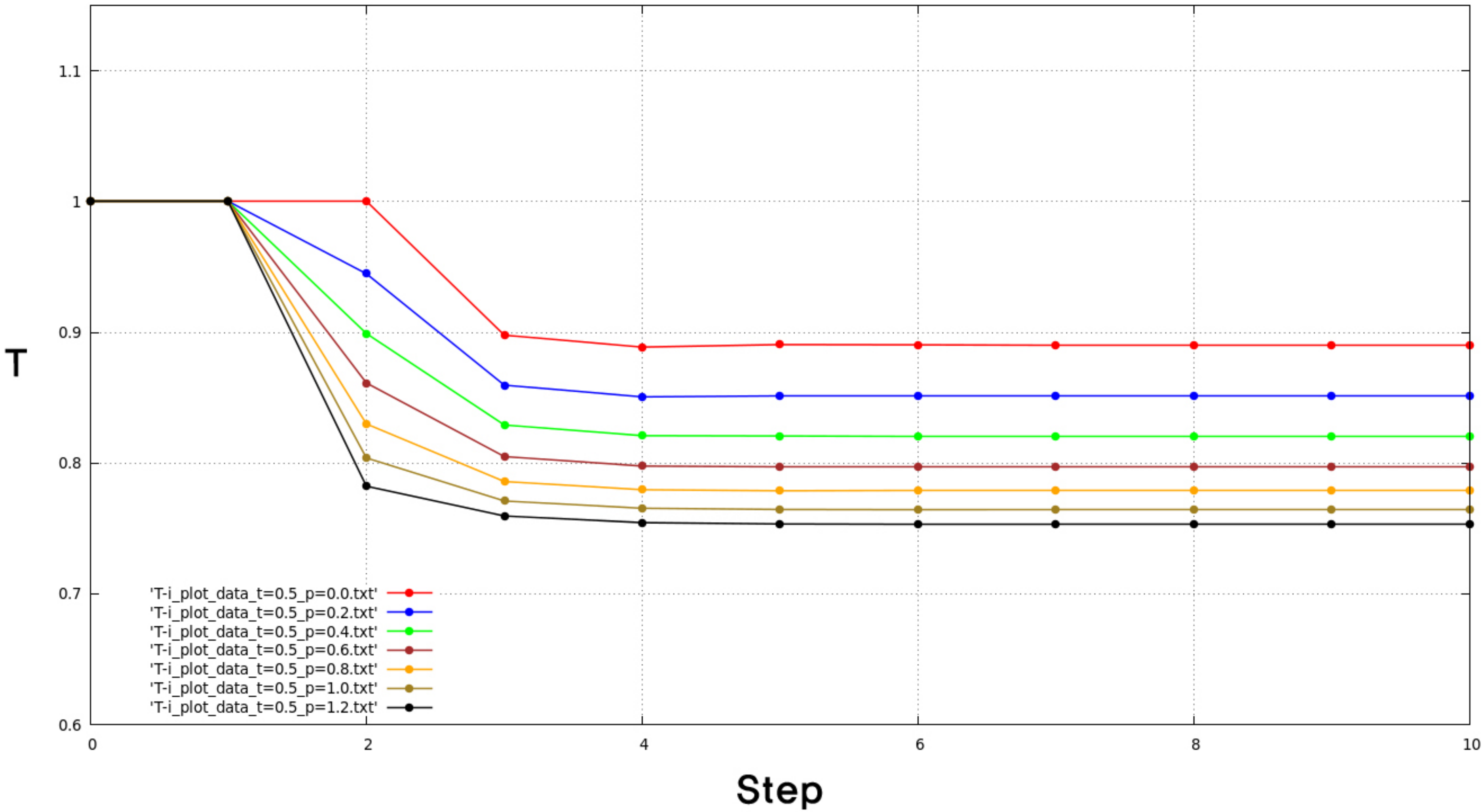
relaxation time $\sim \dfrac{1}{\gamma} = 1$

$$p = 0,\ 0.2,\ 0.4, \cdots,\ 1$$

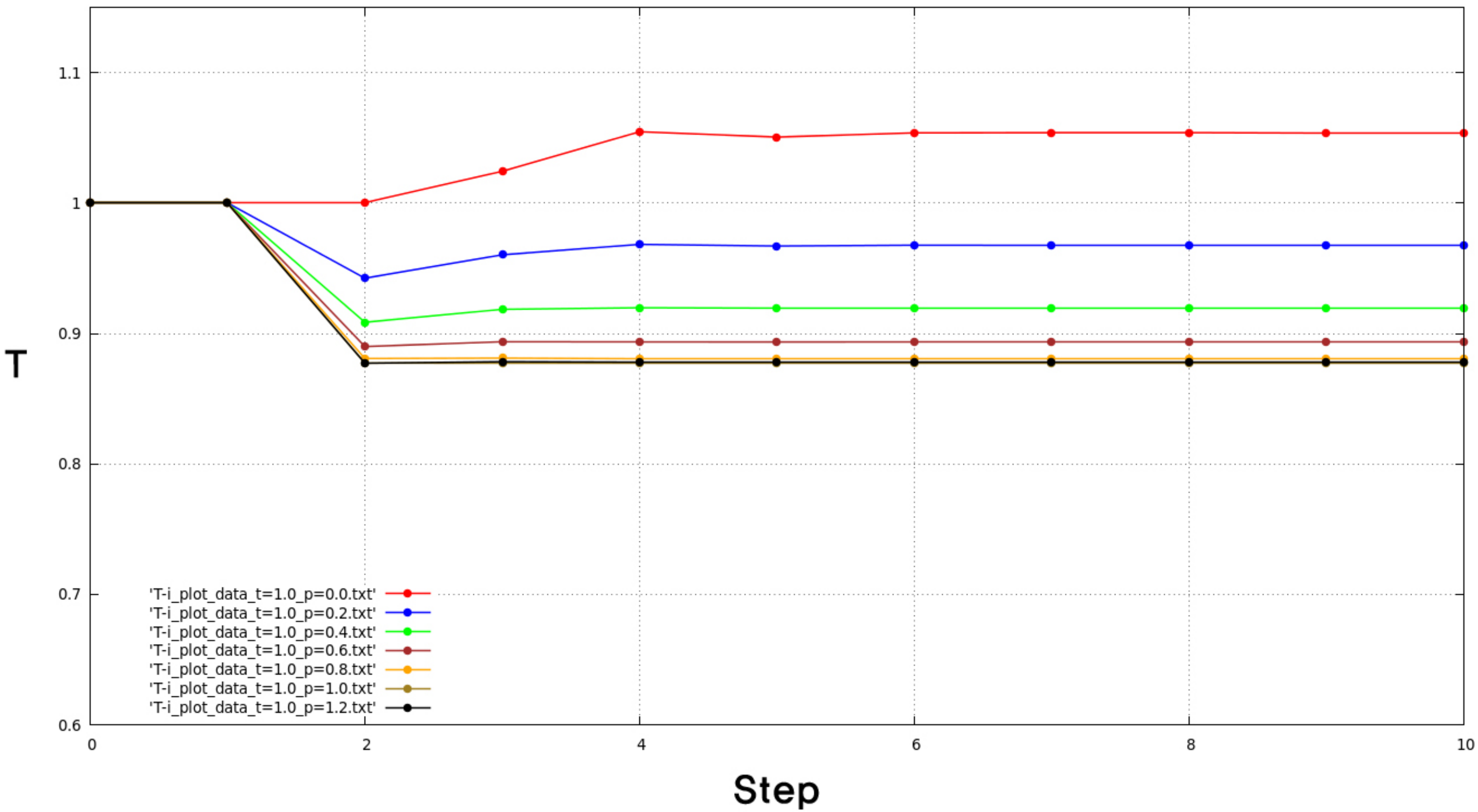
$$t = 0.1,\ 0 \leq p \leq 1$$



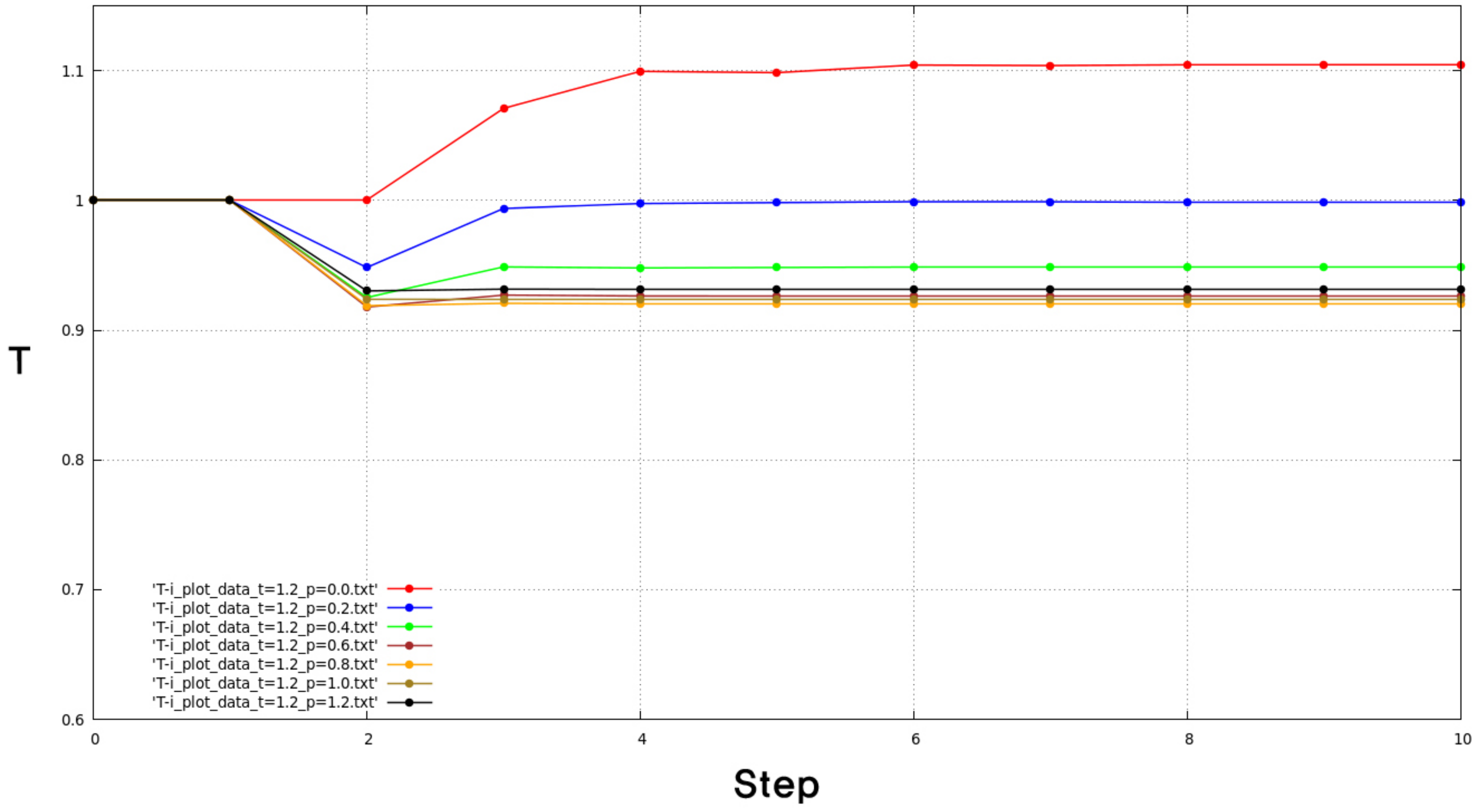
$$t = 0.5,\ 0 \leq p \leq 1$$



$$t = 1.0,\ 0 \leq p \leq 1$$



$$t = 1.2,\ 0 \leq p \leq 1$$



V. Summary

- The paradox of Maxwell's demon is resolved by the information thermodynamics of the feedback control.
- The observable post-measurement process satisfies the fluctuation theorem, which is the true exorcism of Maxwell's demon, with no knowledge about hidden process in Maxwell's demon or memory device such as measurement, erasure.
- The multi-step feedback control with time delay is studied:

$$\langle \Delta S_{total} \rangle = \langle \Delta S_{sys} - \Delta I(\mathbf{q} : \mathbf{y}_{i-1}, \mathbf{y}_i) + \frac{Q}{T} + \Delta S_{odd} \rangle$$

- The cold damping problem is revisited. Temperature depends on time delay and time interval between measurement steps.
- An experimental study is expected.