Everlasting initial memory threshold for rare events

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Equilibration process

Hot coffee

Equilibration process

$t = 0$
(initial state)

finite t
(transient region)

$t = \infty$
(asymptotic region)

Hot coffee

Equilibration process

$t = 0$
(initial state)

finite t
(transient region)

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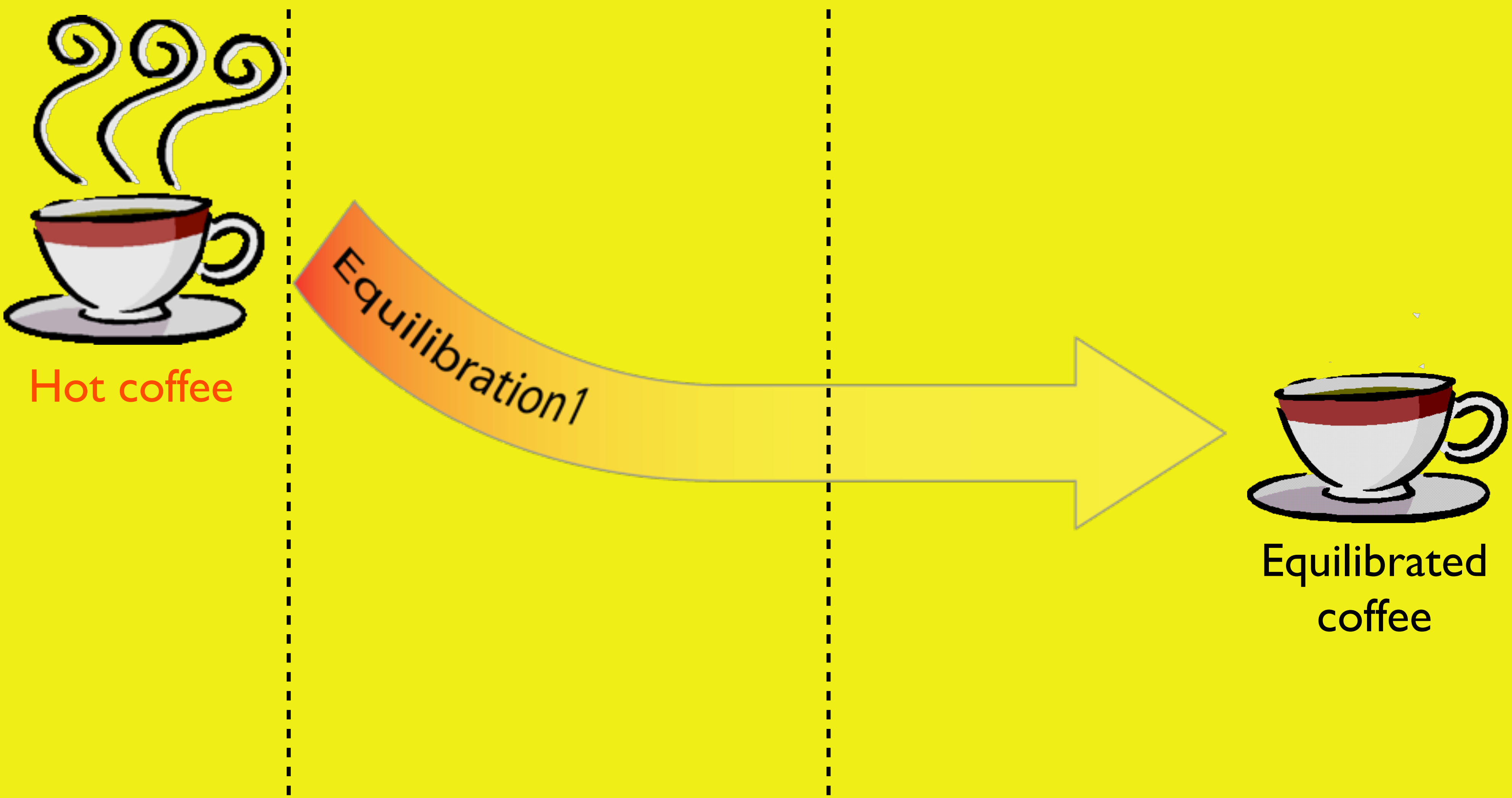
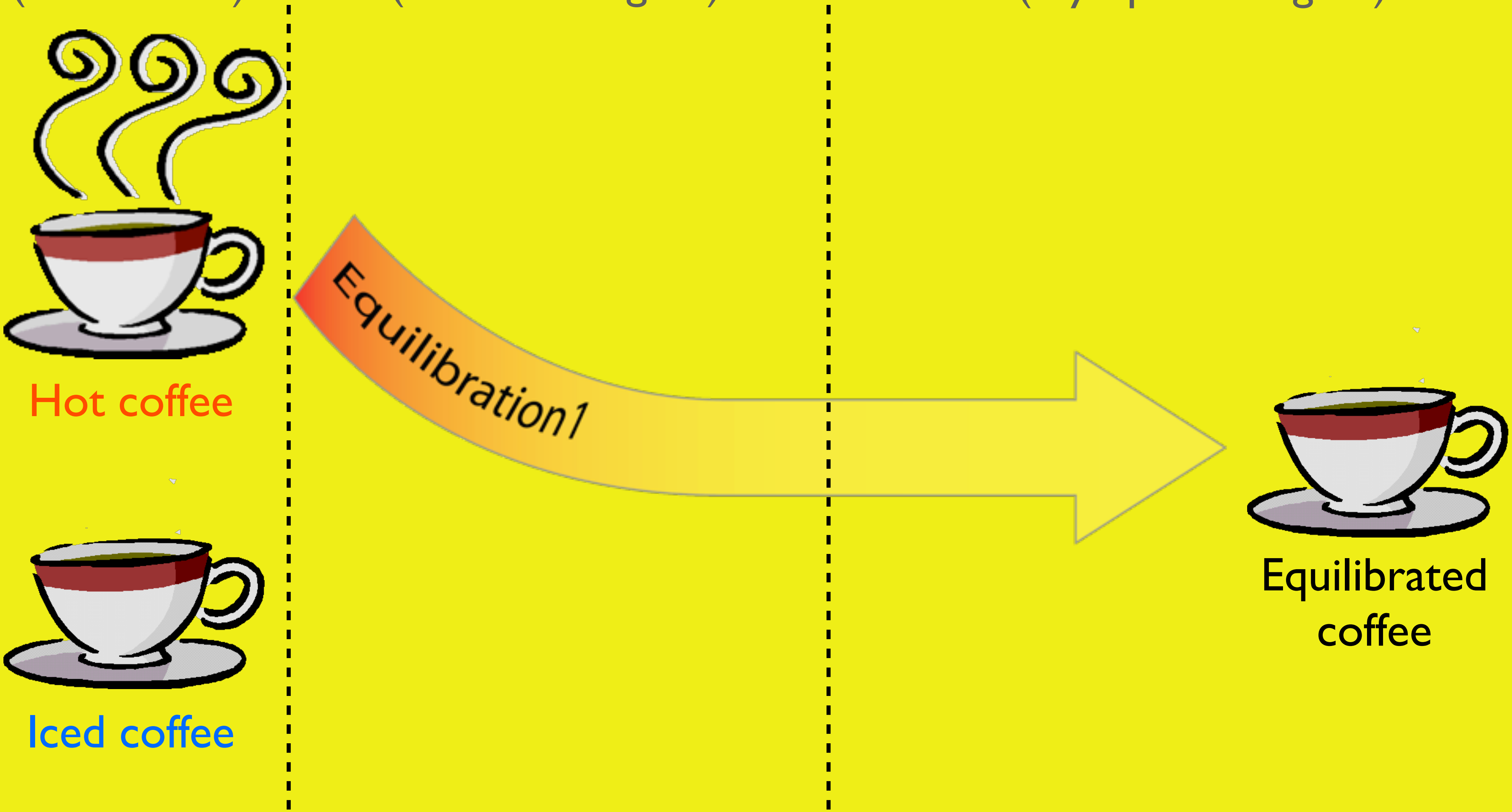
(asymptotic region)



Hot coffee

Equilibrated
coffee

Equilibration process



$t = 0$
(initial state)

finite t
(transient region)

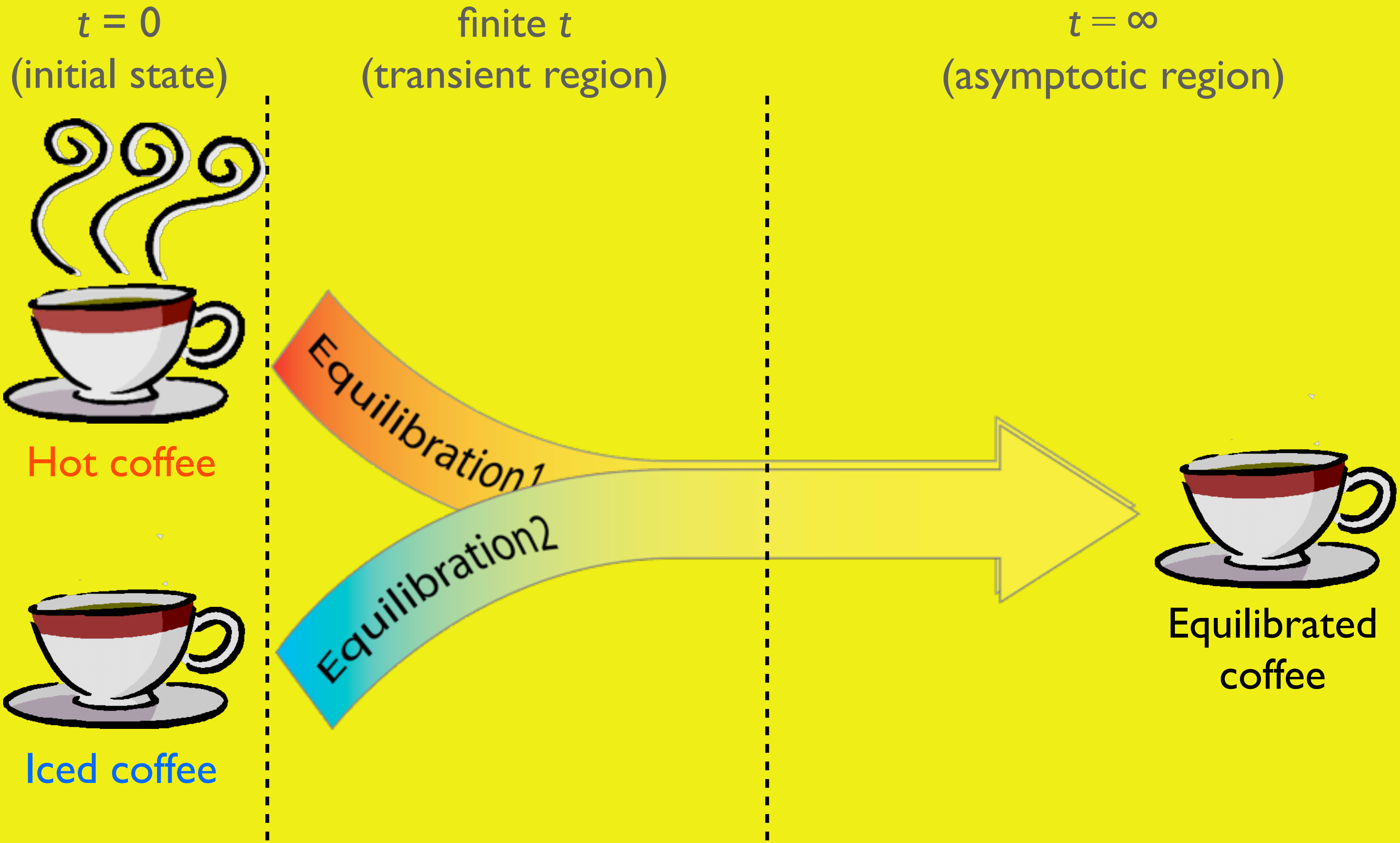
$t = \infty$
(asymptotic region)

Hot coffee

Iced coffee

Equilibrated
coffee

Equilibration process



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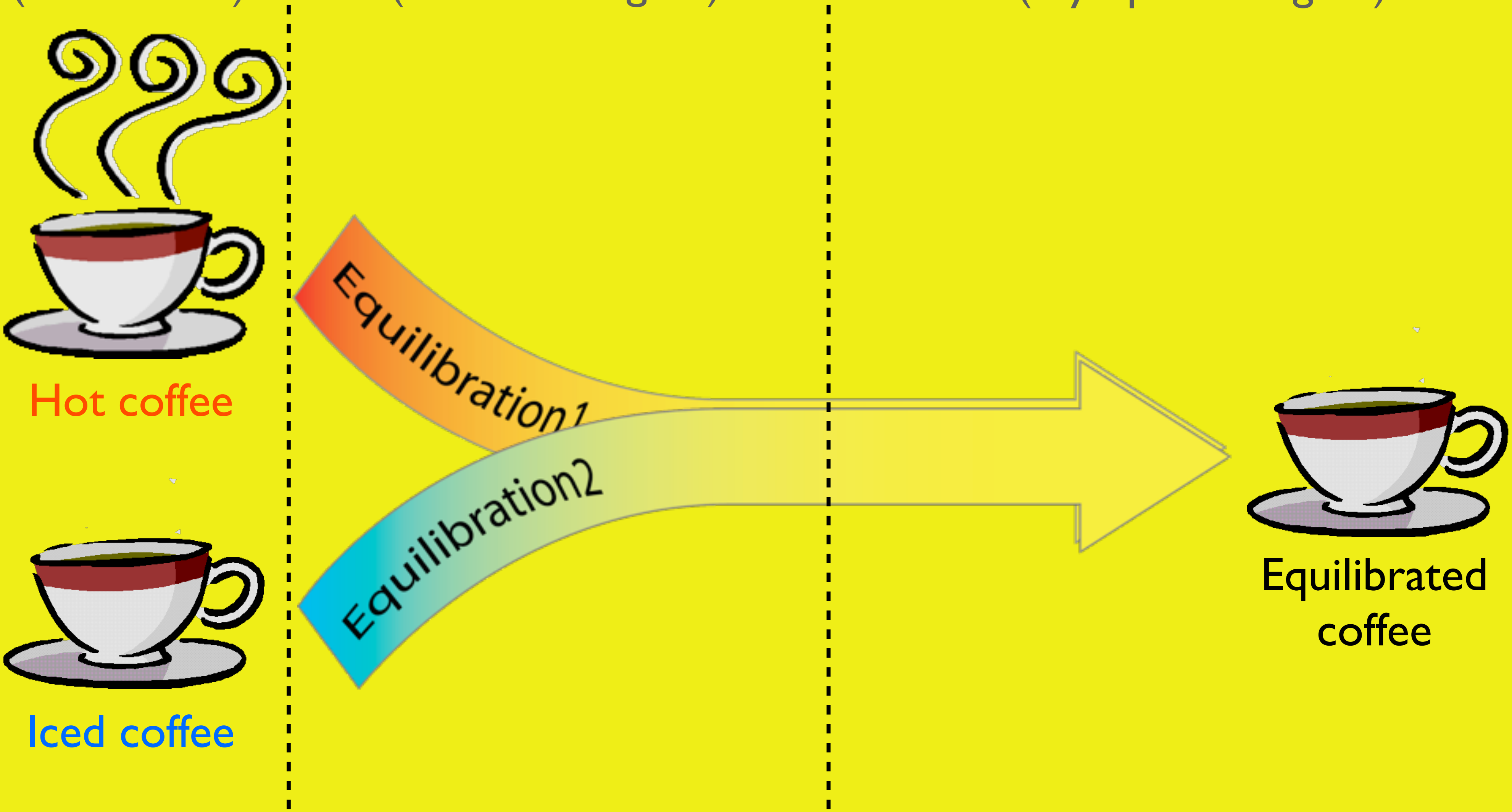


$t = 0$
(initial state)

finite t
(transient region)

$t = \infty$
(asymptotic region)

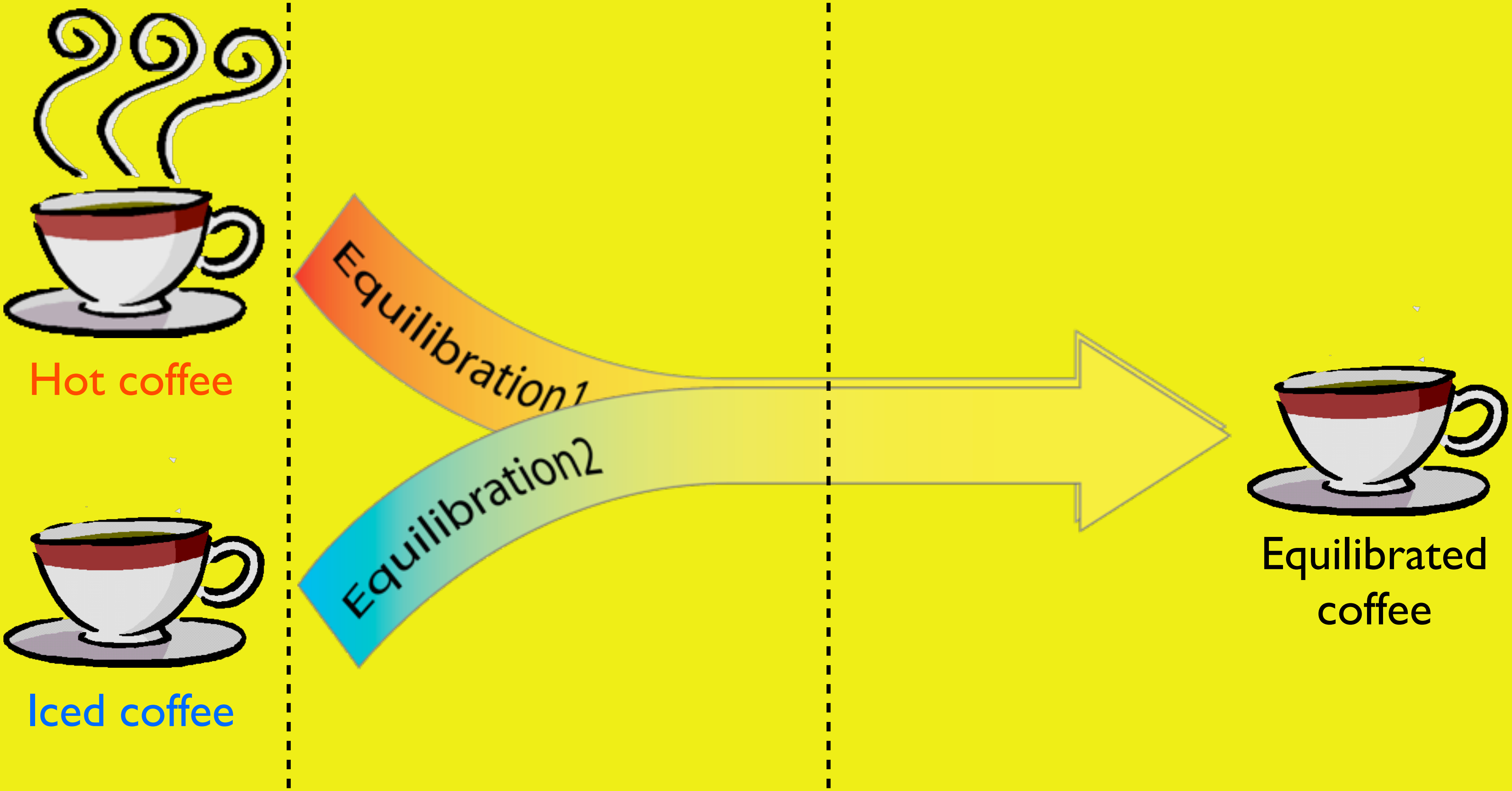
Hot coffee

Iced coffee

Equilibrated
coffee

Q. Does the memory of the initial-state difference *remain* even in $t = \infty$ limit?

Equilibration process



$t = 0$
(initial state)

finite t
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$\tau = \infty$
(asymptotic region)

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Equilibration process

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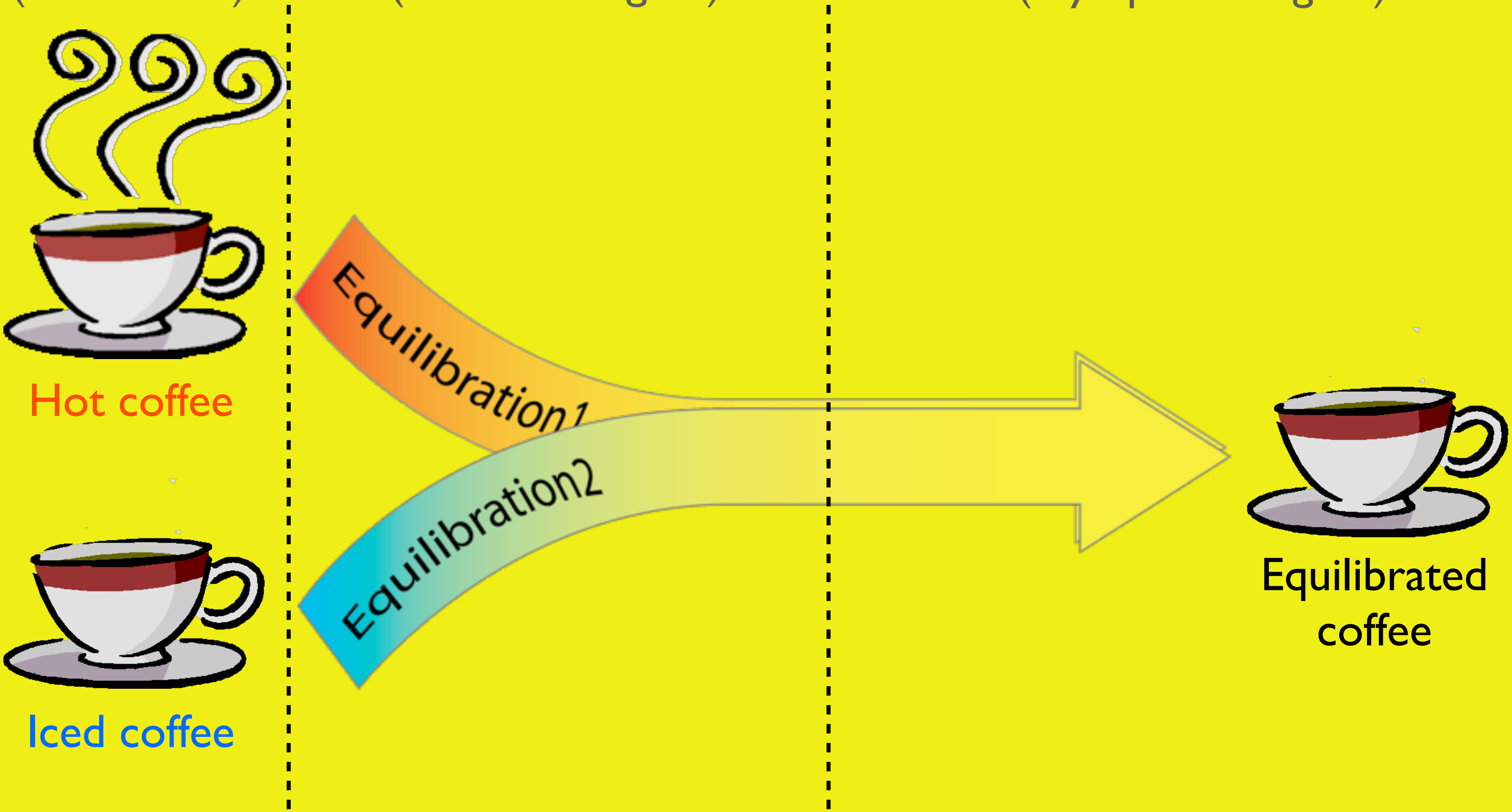


Hot coffee

Iced coffee

Equilibrated
coffee

Disappear

Equilibration process

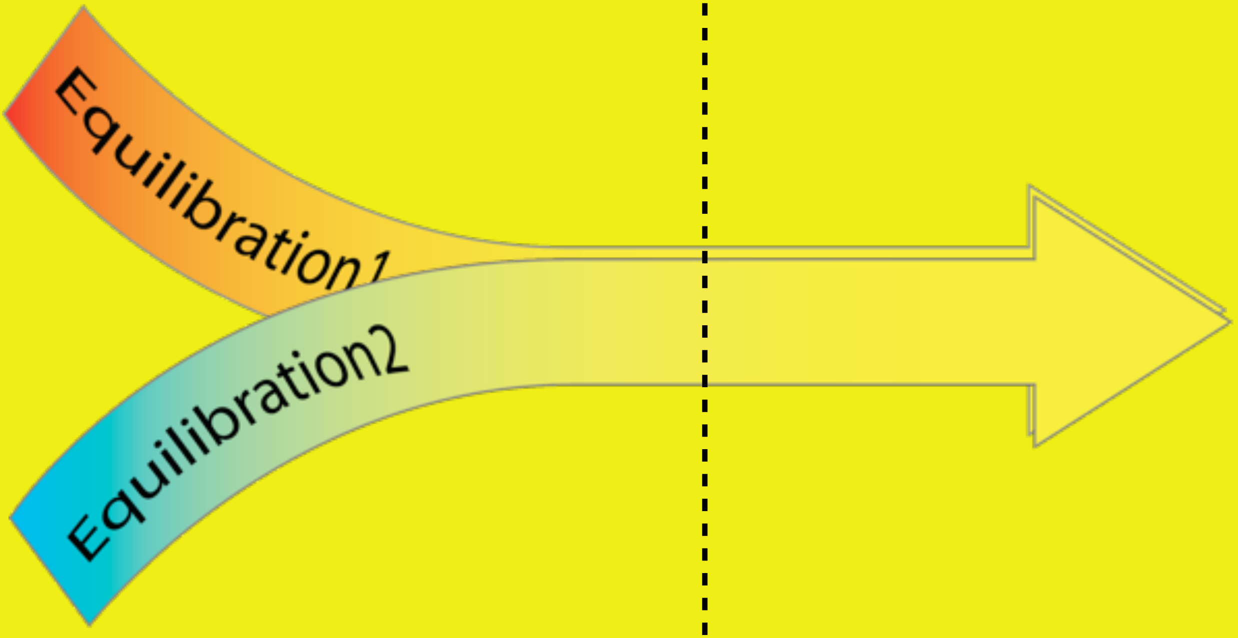
$t = 0$
(initial state)

finite t
(transient region)

$\tau = \infty$
(asymptotic region)

Hot coffee

Iced coffee



Equilibrated
coffee

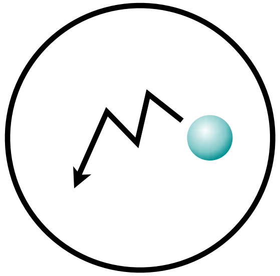
If we measure time-integrated quantities

Remain
Disappear

depending on the temperature difference between system and heat bath

The schematic diagram for this study

system
(Brownian particle)



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system
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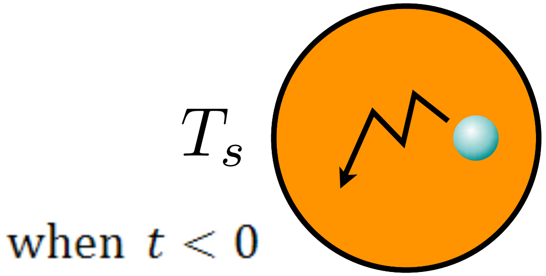
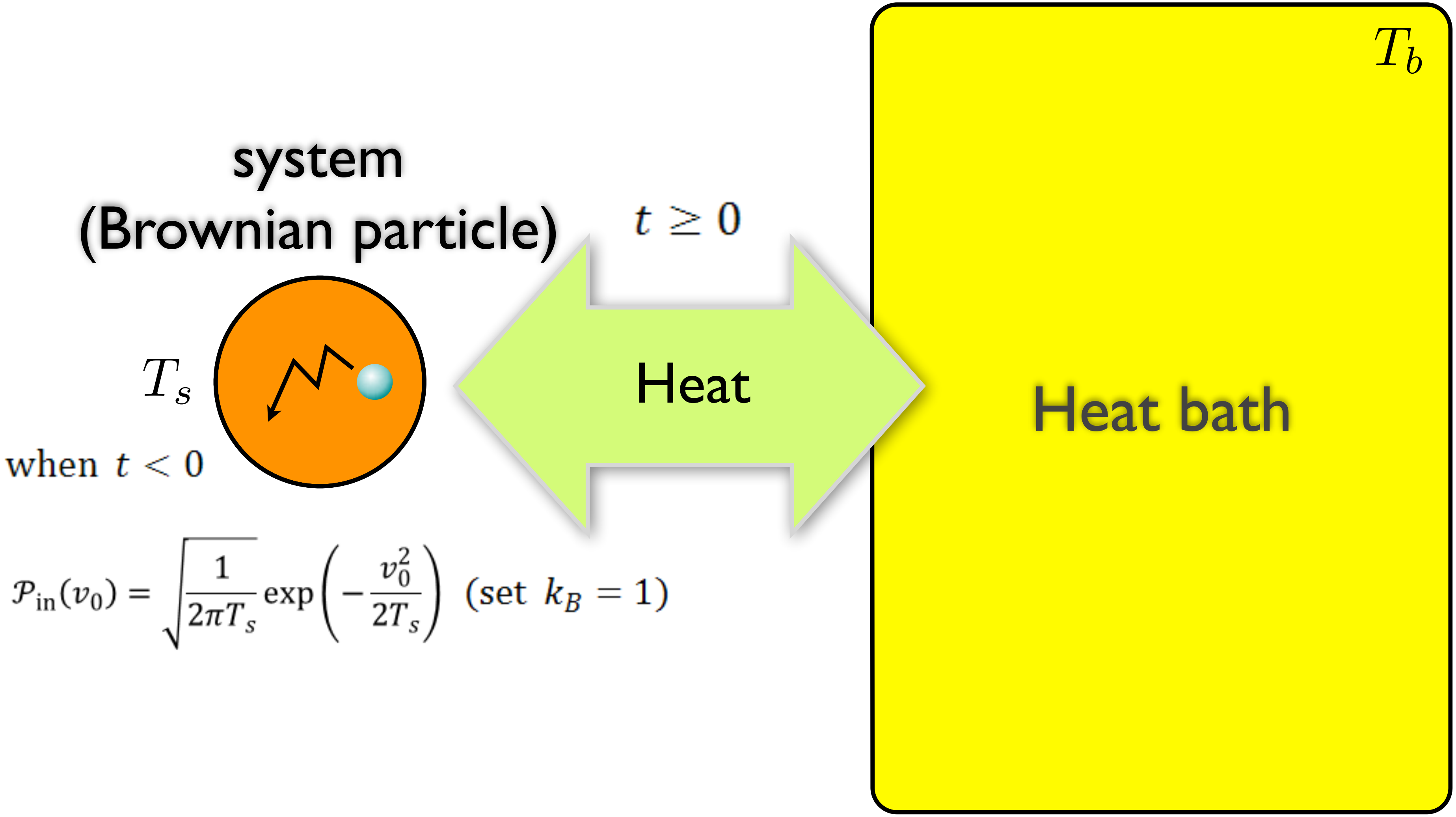


$$\mathcal{P}_{\text{in}}(v_0) = \sqrt{\frac{1}{2\pi T_s}} \exp\left(-\frac{v_0^2}{2T_s}\right) \quad (\text{set } k_B = 1)$$

The schematic diagram for this study



system
(Brownian particle)

T_s

when $t < 0$

$$\mathcal{P}_{\text{in}}(v_0) = \sqrt{\frac{1}{2\pi T_s}} \exp\left(-\frac{v_0^2}{2T_s}\right) \quad (\text{set } k_B = 1)$$

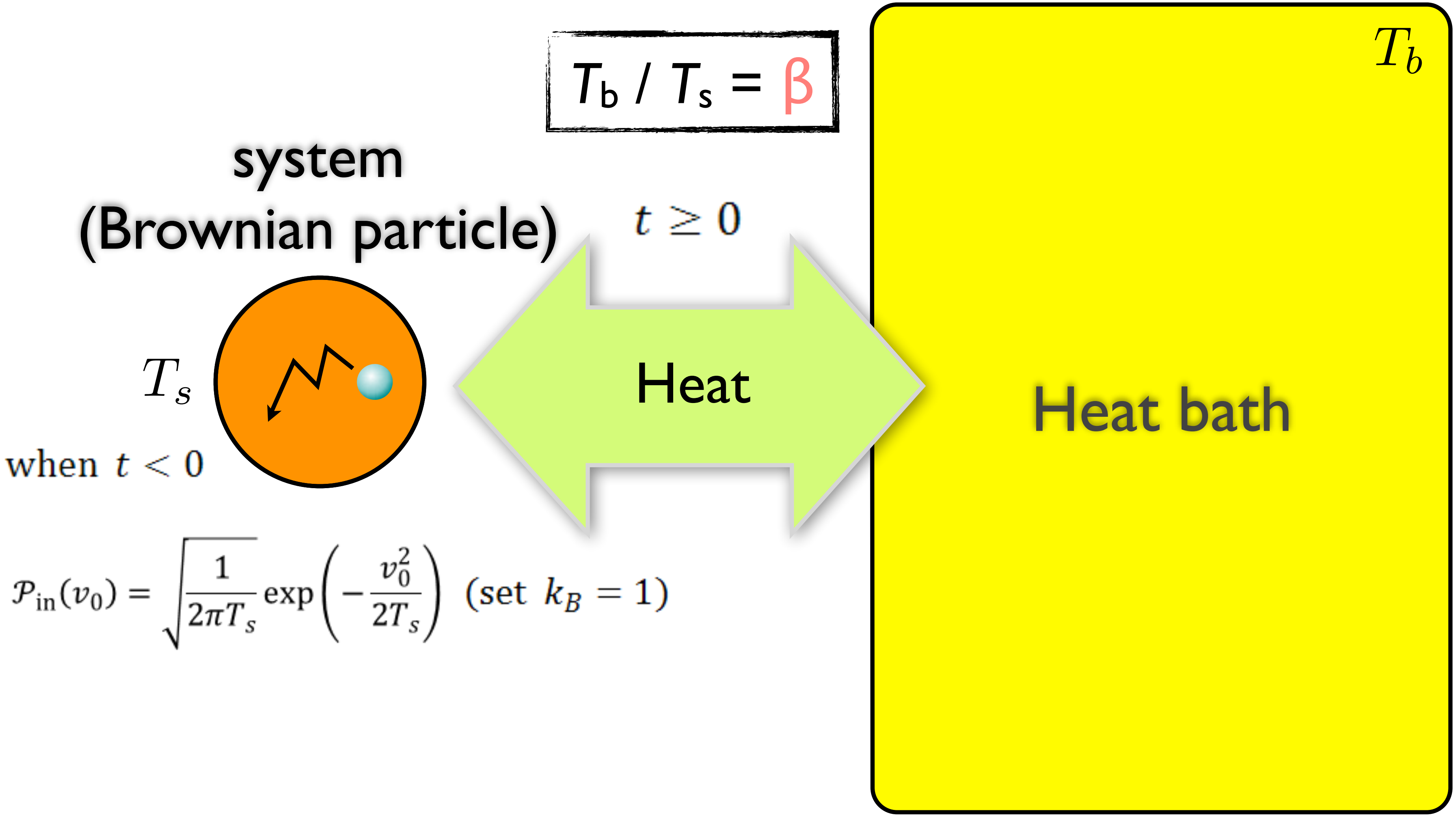
$t \geq 0$

Heat

T_b

Heat bath

The schematic diagram for this study



$T_b / T_s = \beta$

system
(Brownian particle)

T_s

when $t < 0$

$t \geq 0$

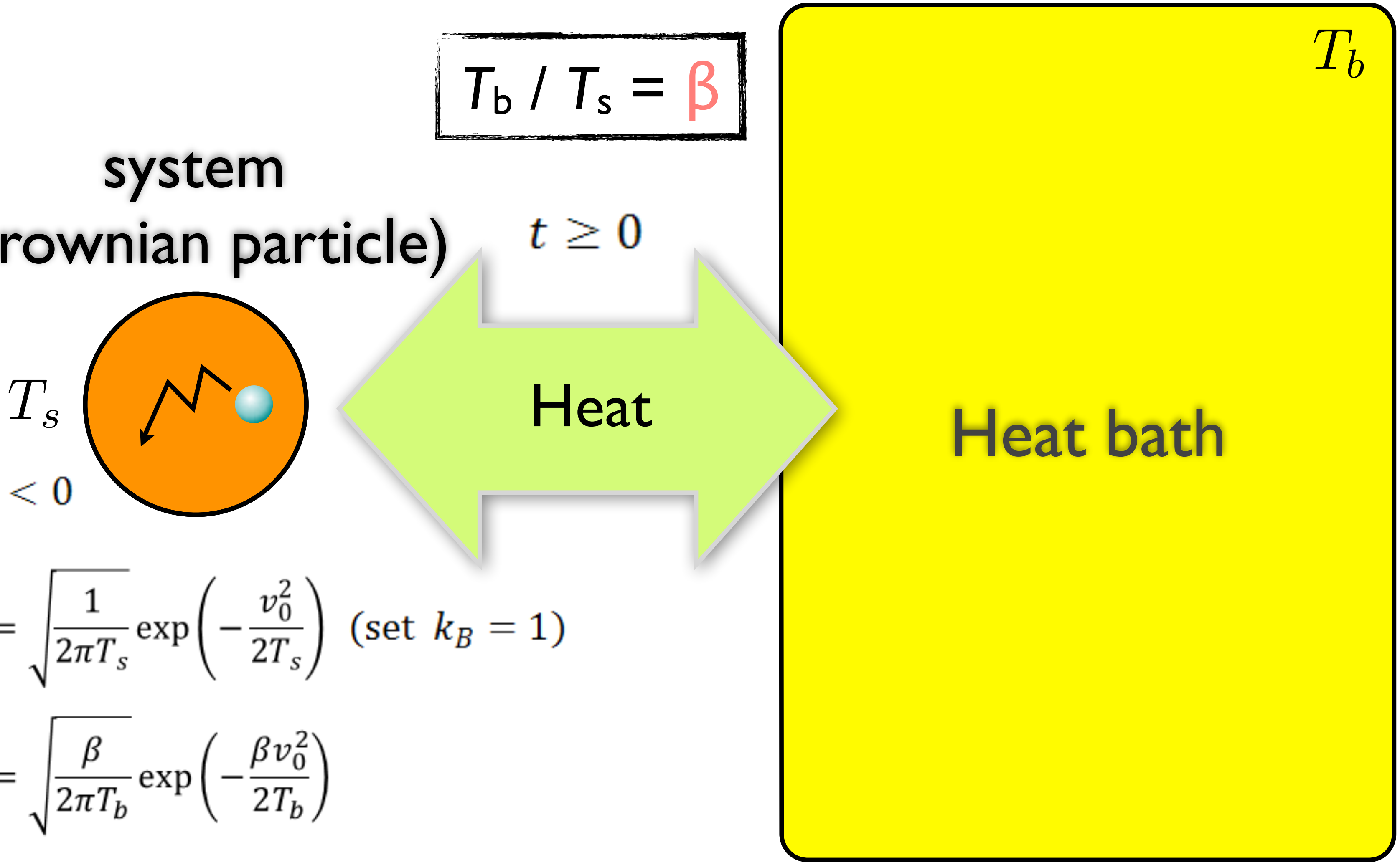
Heat

T_b

Heat bath

$$\mathcal{P}_{\rm in}(v_0) = \sqrt{\frac{1}{2\pi T_s}} \exp\left(-\frac{v_0^2}{2T_s}\right) \quad (\text{set } k_B = 1)$$

The schematic diagram for this study



$T_b / T_s = \beta$

system

(Brownian particle)

$t \geq 0$

T_b

T_s

Heat

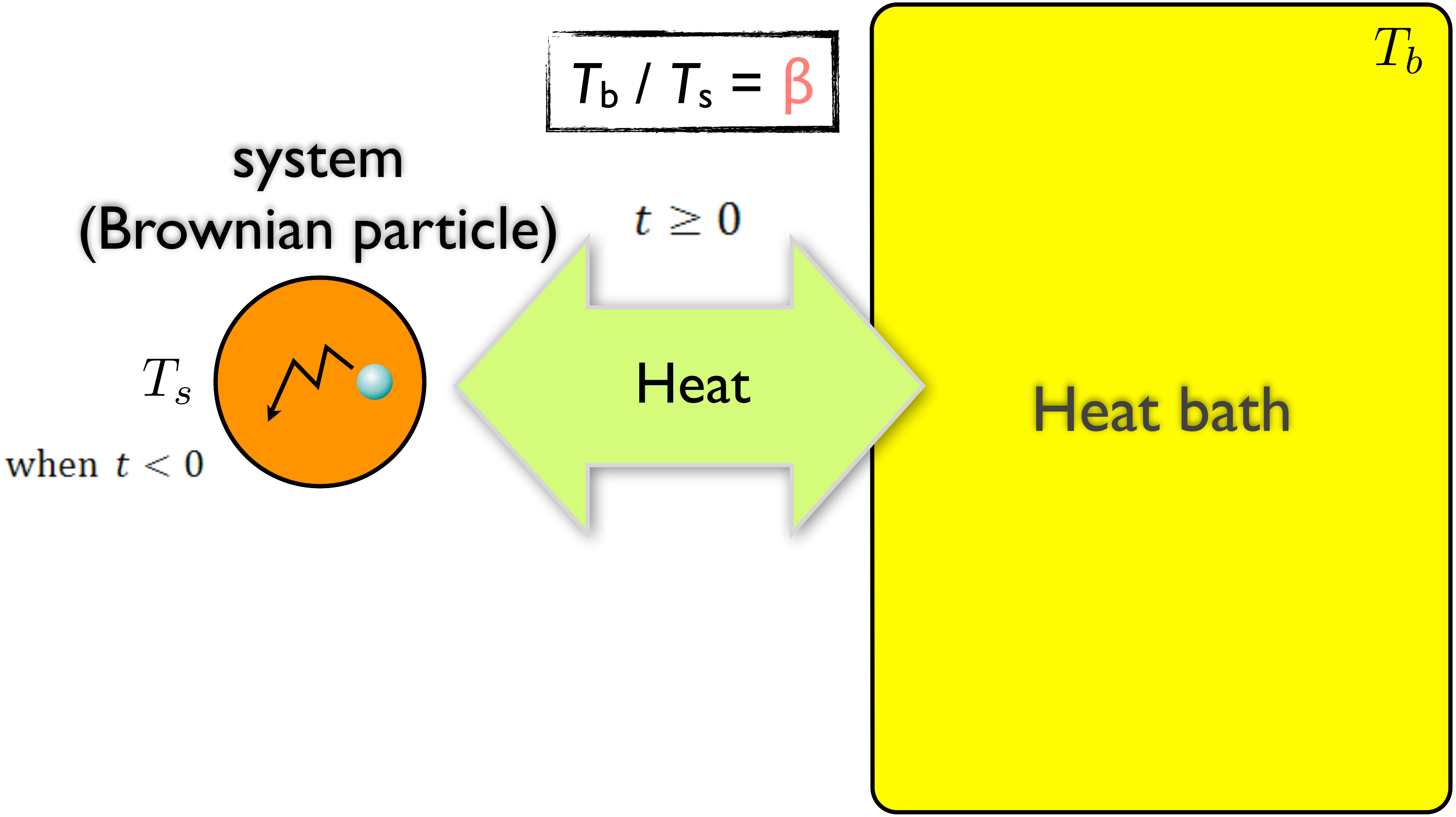
Heat bath

when $t < 0$

$$\mathcal{P}_{\text{in}}(v_0) = \sqrt{\frac{1}{2\pi T_s}} \exp\left(-\frac{v_0^2}{2T_s}\right) \quad (\text{set } k_B = 1)$$

$$= \sqrt{\frac{\beta}{2\pi T_b}} \exp\left(-\frac{\beta v_0^2}{2T_b}\right)$$

The schematic diagram for this study



$T_b / T_s = \beta$

system
(Brownian particle)

T_s

when $t < 0$

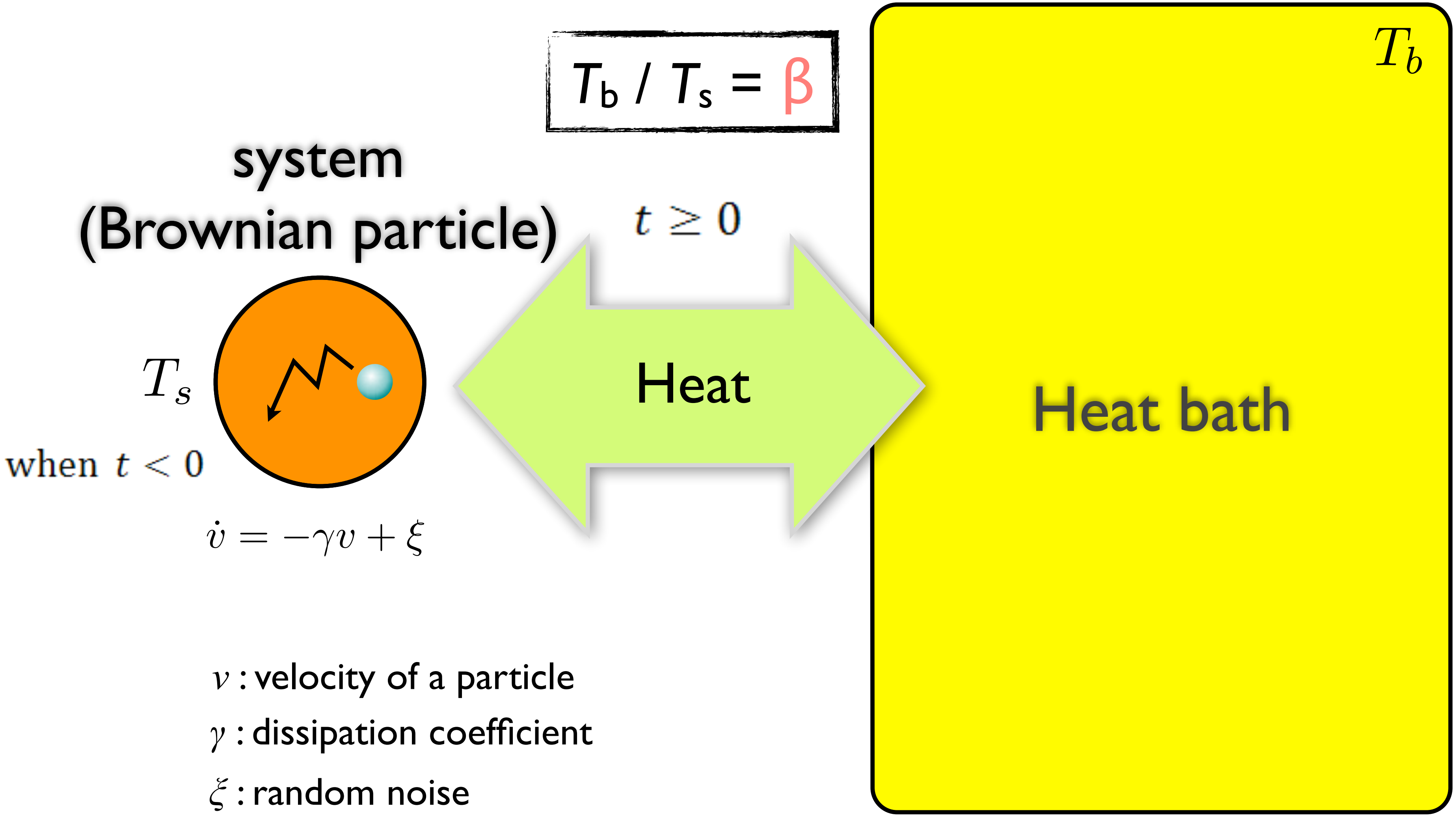
$t \geq 0$

Heat

Heat bath

T_b

The schematic diagram for this study



$T_b / T_s = \beta$

system

(Brownian particle)

$t \geq 0$

T_s

when $t < 0$

Heat

Heat bath

T_b

$$\dot{v} = -\gamma v + \xi$$

v : velocity of a particle

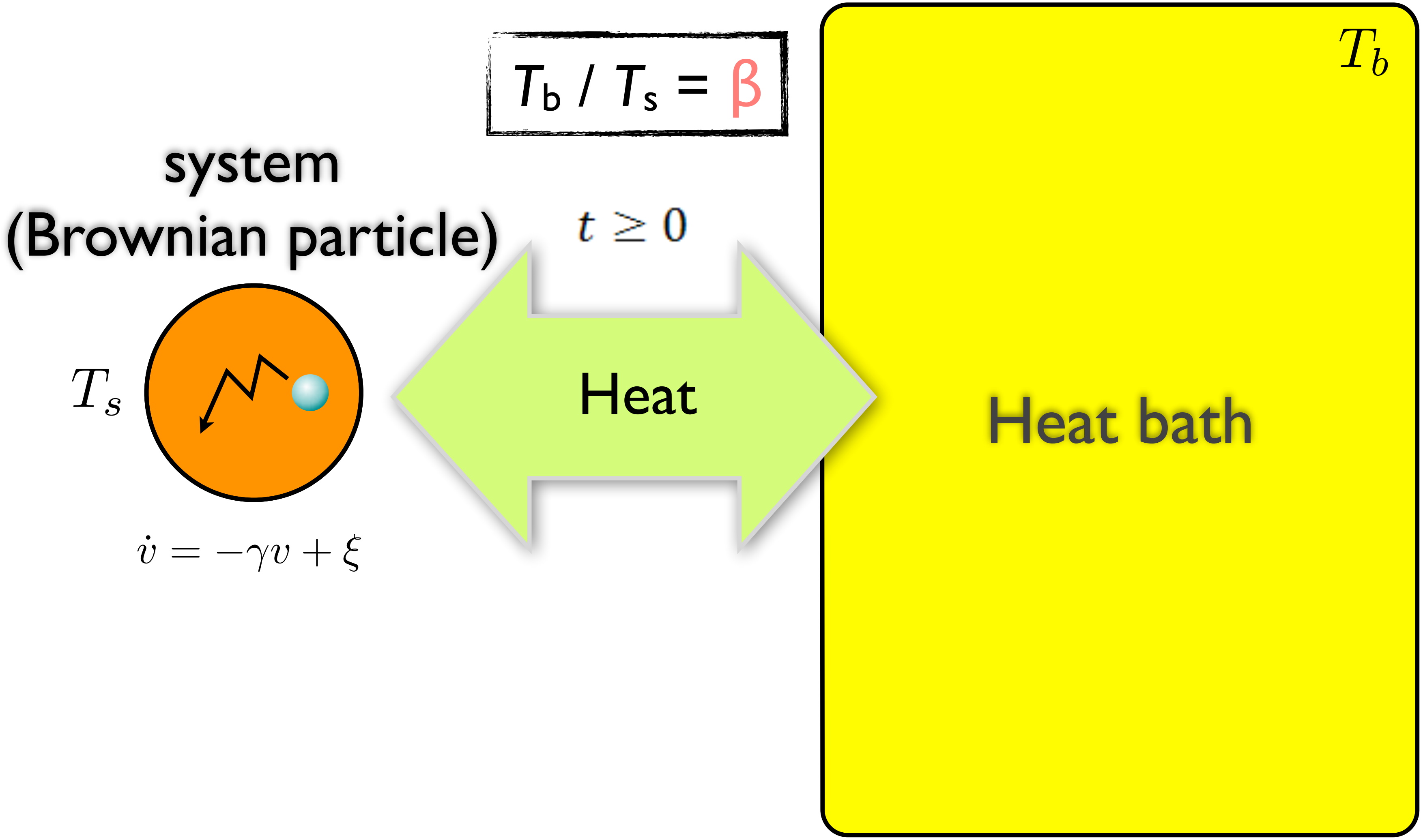
γ : dissipation coefficient

ξ : random noise

$$\langle \xi(\tau)\xi(\tau') \rangle = 2D\delta(\tau - \tau')$$

$$T_b = D/\gamma$$

The schematic diagram for this study



$T_b / T_s = \beta$

system
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T_s

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$t \geq 0$

Heat

Heat bath

T_b

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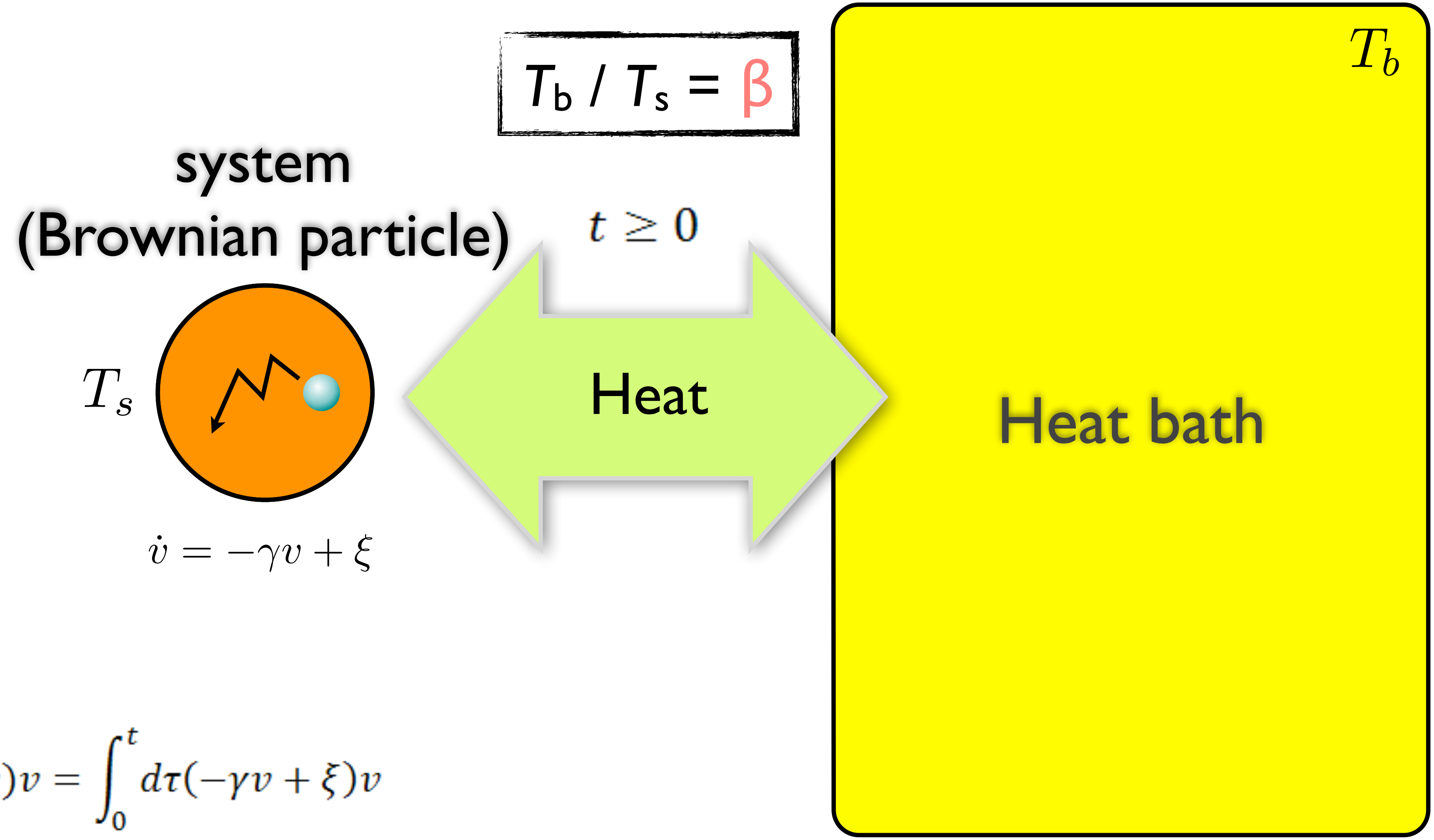


T_b / T_s = β

T_b

system
(Brownian particle)

$t \geq 0$

T_s

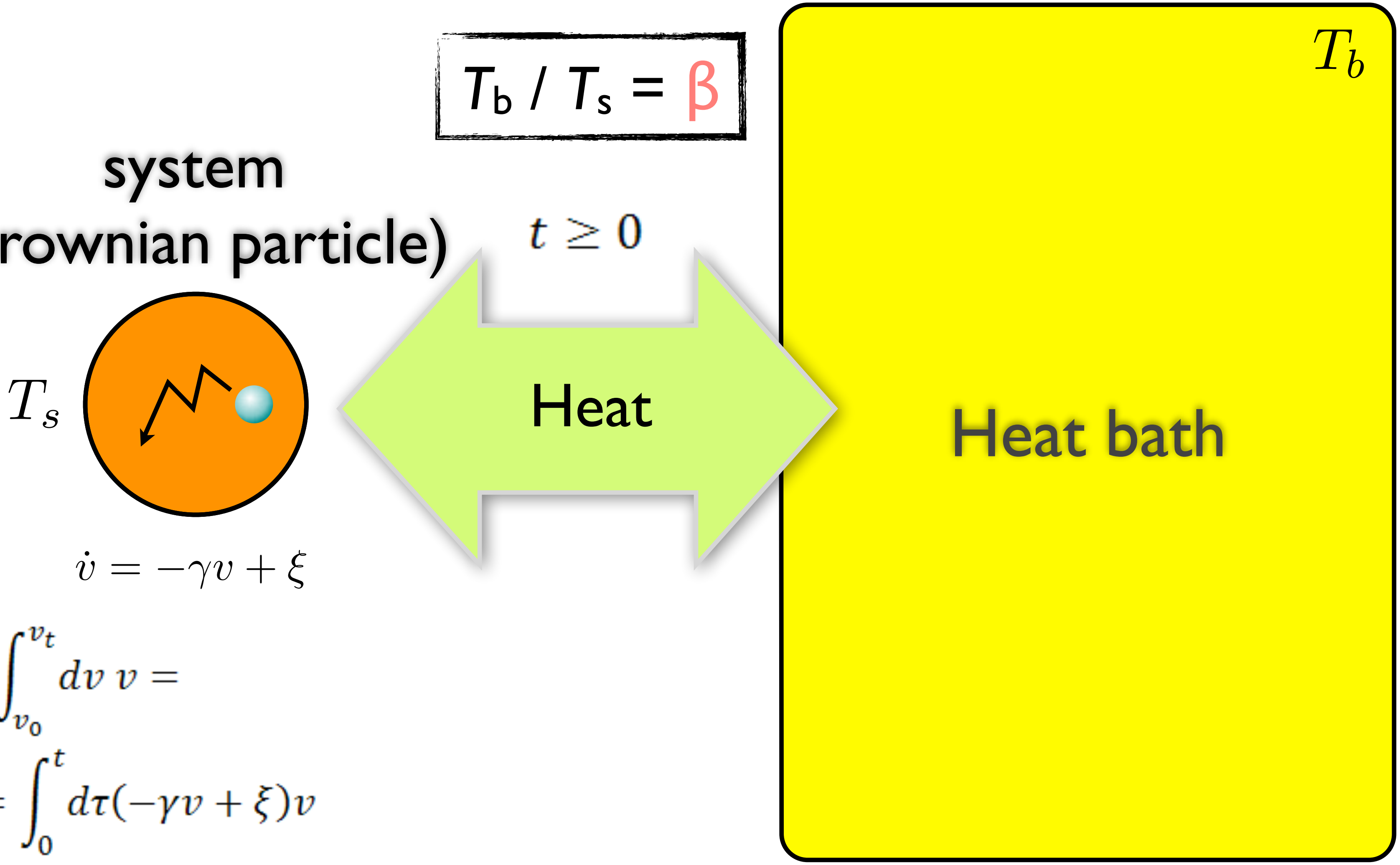
Heat

Heat bath

$$\dot{v} = -\gamma v + \xi$$

$$\int_0^t d\tau (\dot{v}) v = \int_0^t d\tau (-\gamma v + \xi) v$$

The schematic diagram for this study



$T_b / T_s = \beta$

$t \geq 0$

system
(Brownian particle)

T_s

Heat

Heat bath

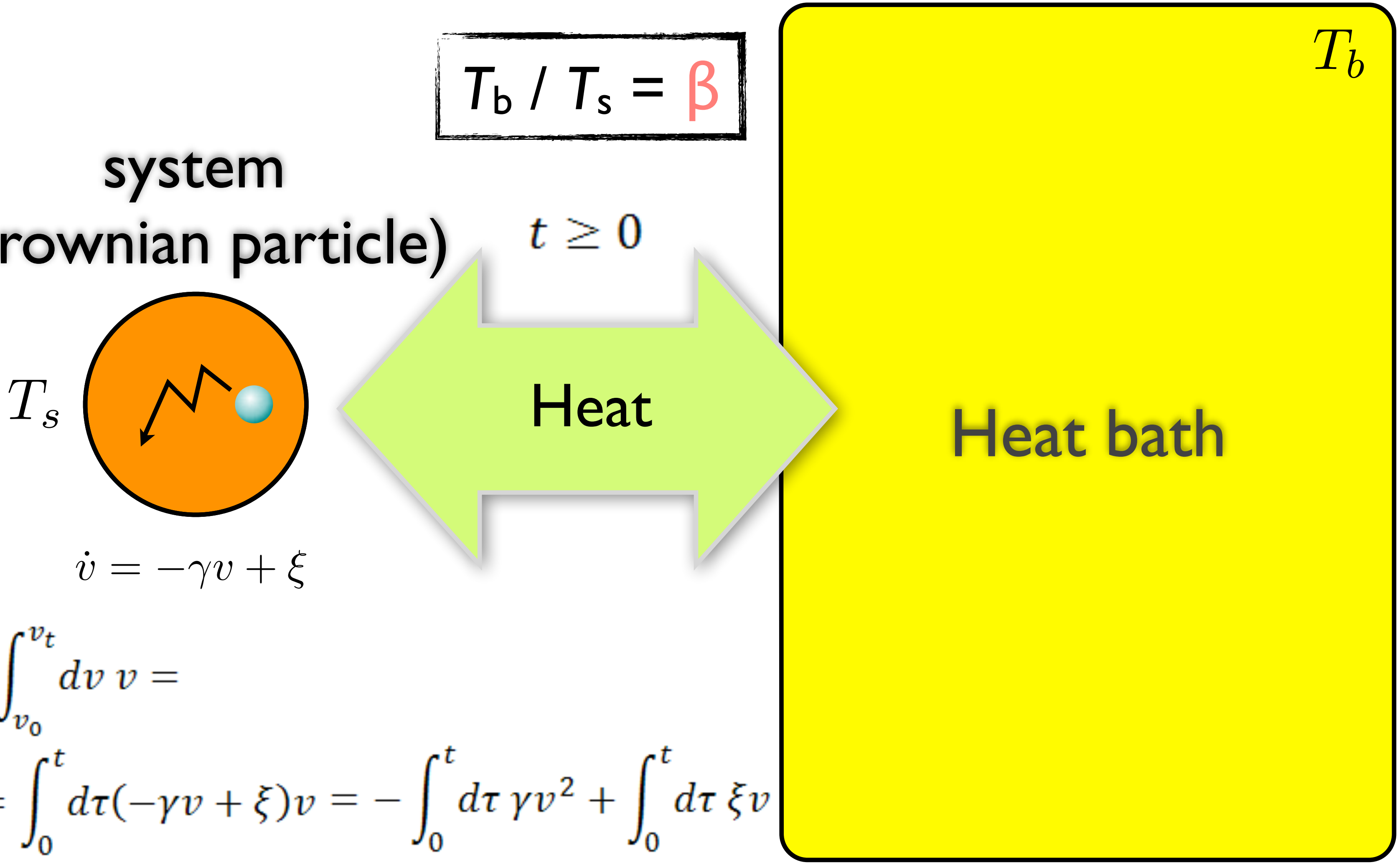
T_b

$$\dot{v} = -\gamma v + \xi$$

$$Q = \Delta E = \int_{v_0}^{v_t} dv\, v =$$

$$\int_0^t d\tau(\dot{v})v = \int_0^t d\tau(-\gamma v + \xi)v$$

The schematic diagram for this study



$T_b / T_s = \beta$

system
(Brownian particle)

$t \geq 0$

T_s

Heat

Heat bath

T_b

$$\dot{v} = -\gamma v + \xi$$

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$$\int_0^t d\tau (\dot{v}) v = \int_0^t d\tau (-\gamma v + \xi) v = -\int_0^t d\tau\, \gamma v^2 + \int_0^t d\tau\, \xi v$$

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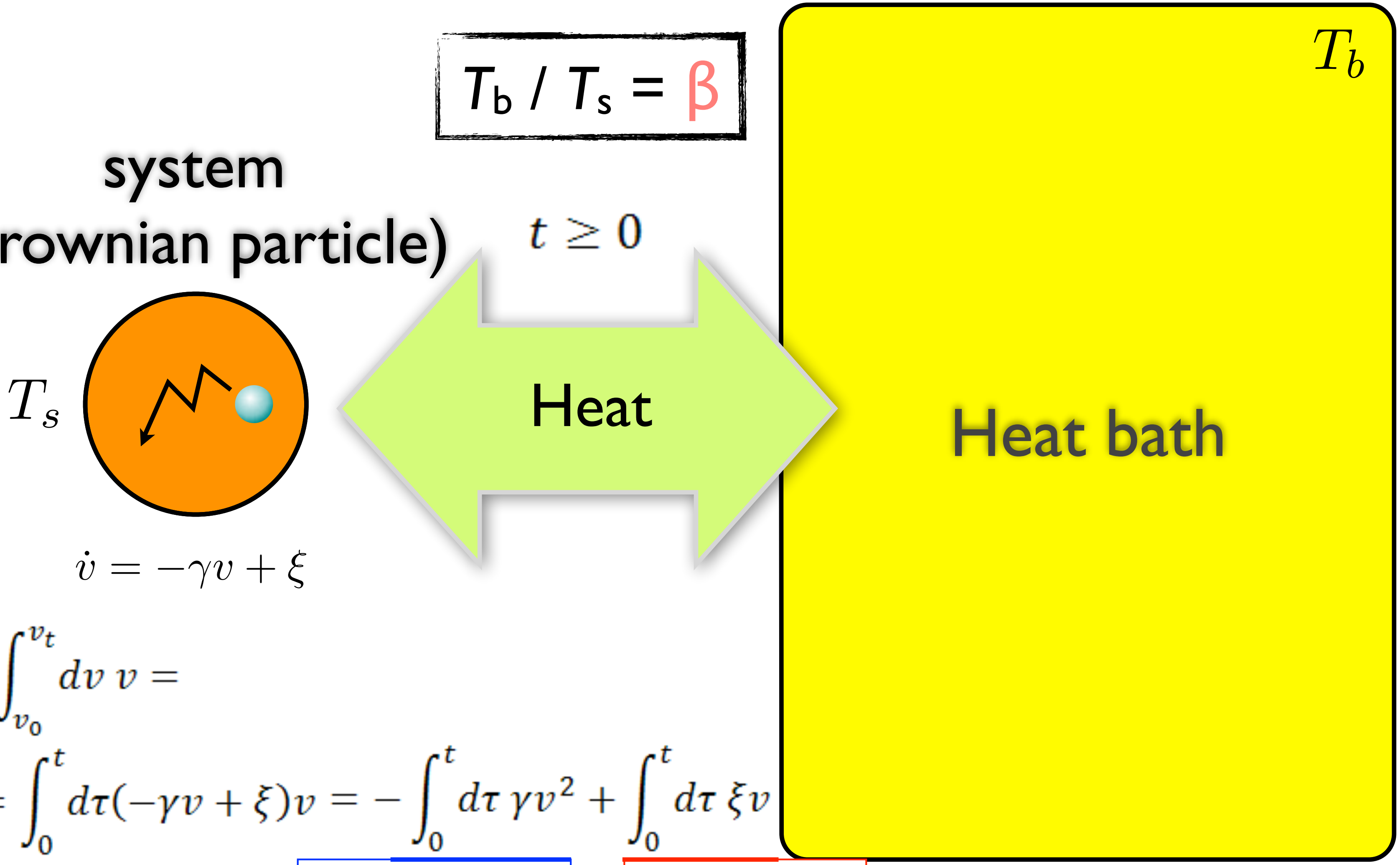

T_b / T_s = β

system
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$t \geq 0$

T_s

Heat

Heat bath

T_b

$$\dot{v} = -\gamma v + \xi$$

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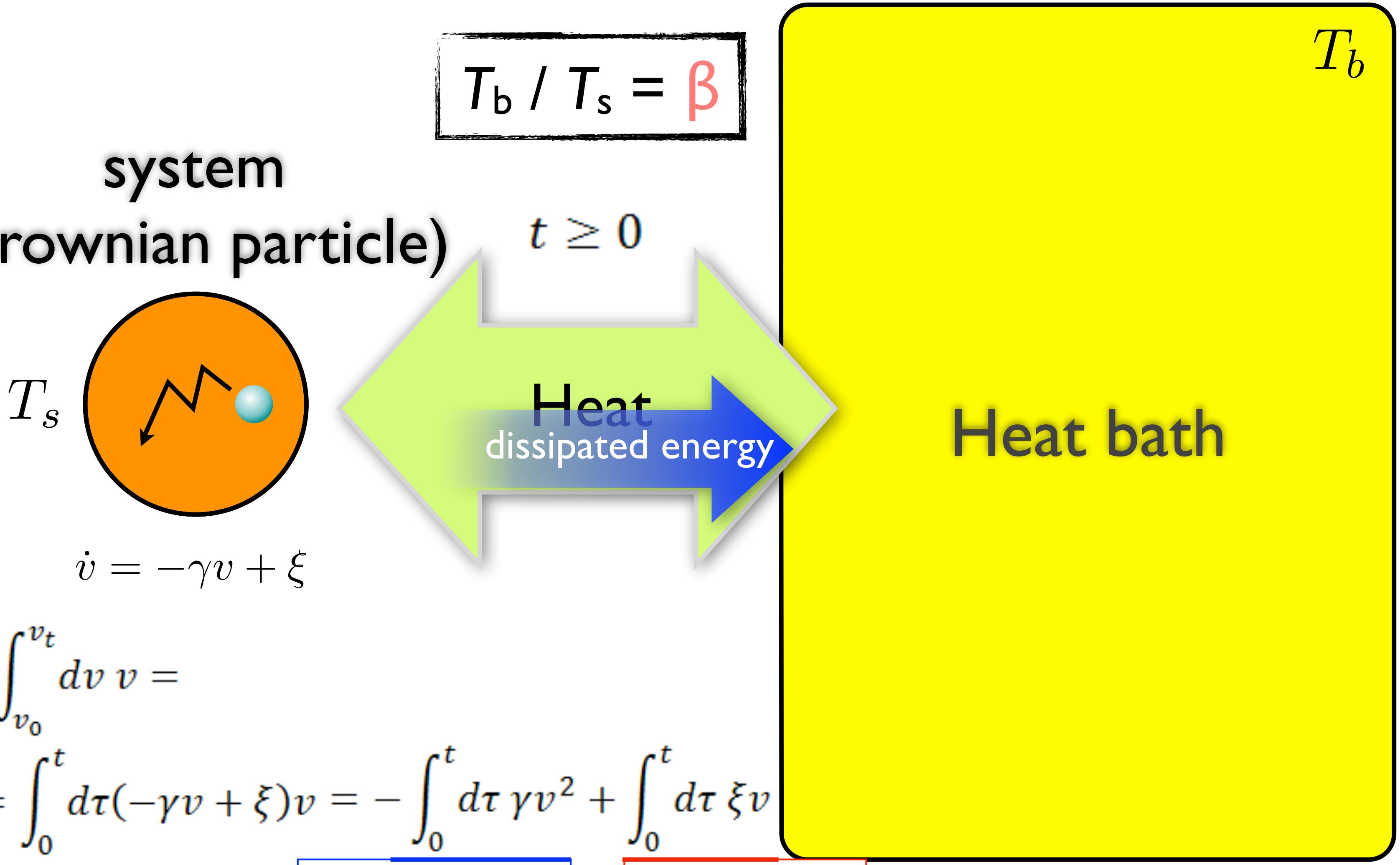
$$\boxed{Q_d = \int_0^t d\tau\, \gamma v^2} \qquad \boxed{Q_i = \int_0^t d\tau\, \xi v}$$

Farago, J. Stat. Phys. 107 781 (2002)

The schematic diagram for this study



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dissipated energy flow

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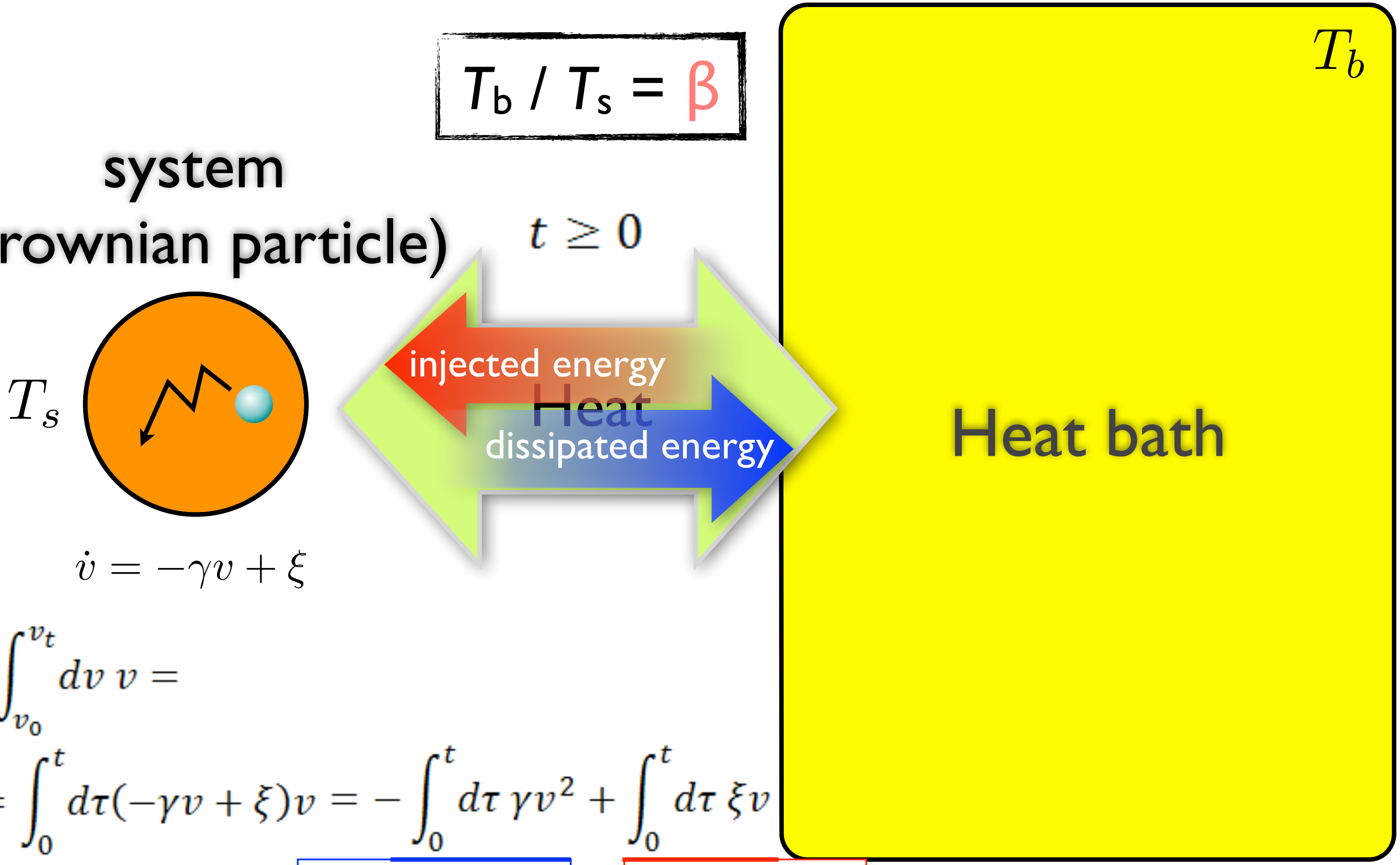

$T_b / T_s = \beta$

system
(Brownian particle)

T_s

$t \geq 0$

injected energy

Heat

dissipated energy

Heat bath

T_b

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dissipated energy flow $\boxed{Q_d = \int_0^t d\tau\, \gamma v^2}$

$\boxed{Q_i = \int_0^t d\tau\, \xi v}$ injected energy flow

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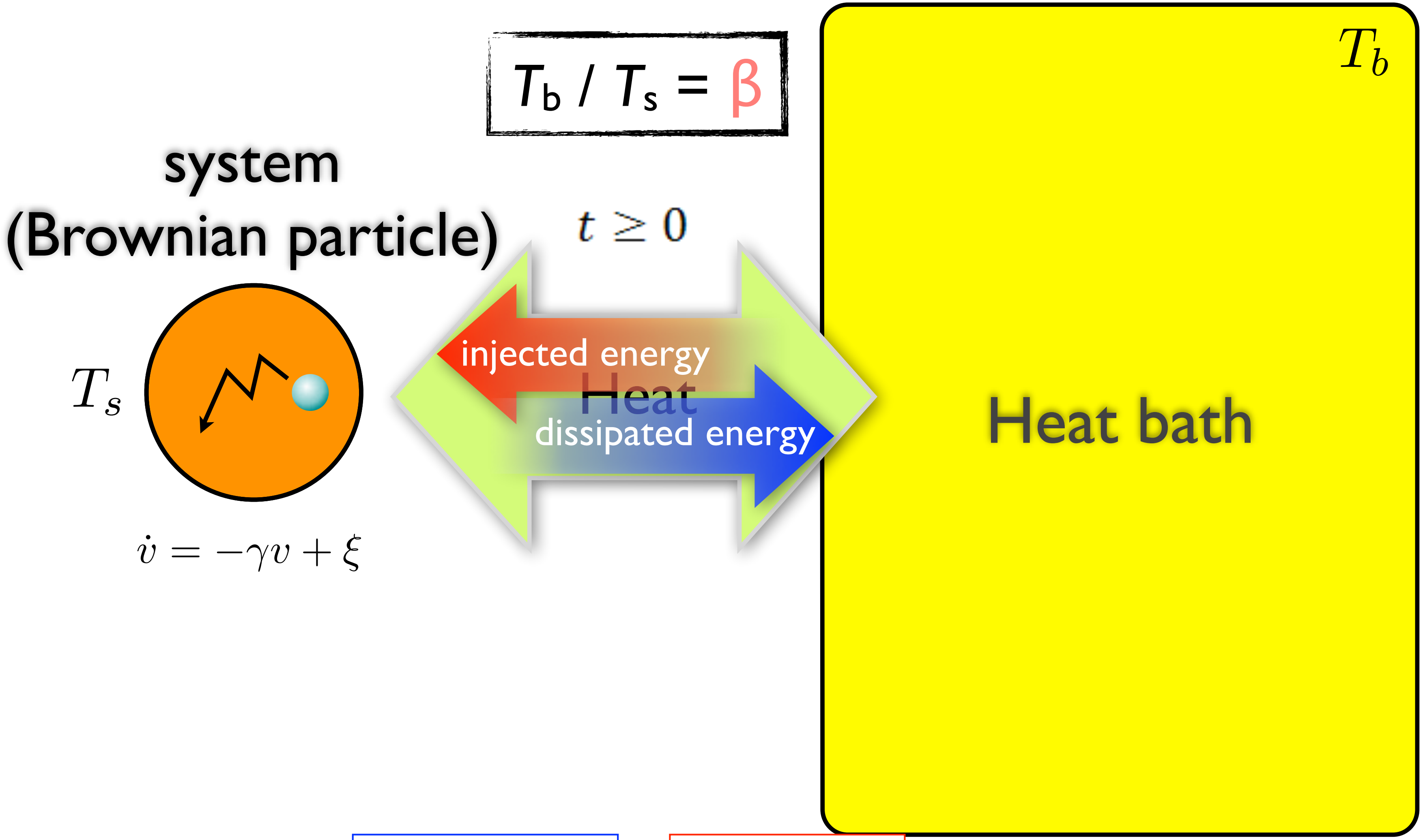

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$\dot{v} = -\gamma v + \xi$

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injected energy

Heat

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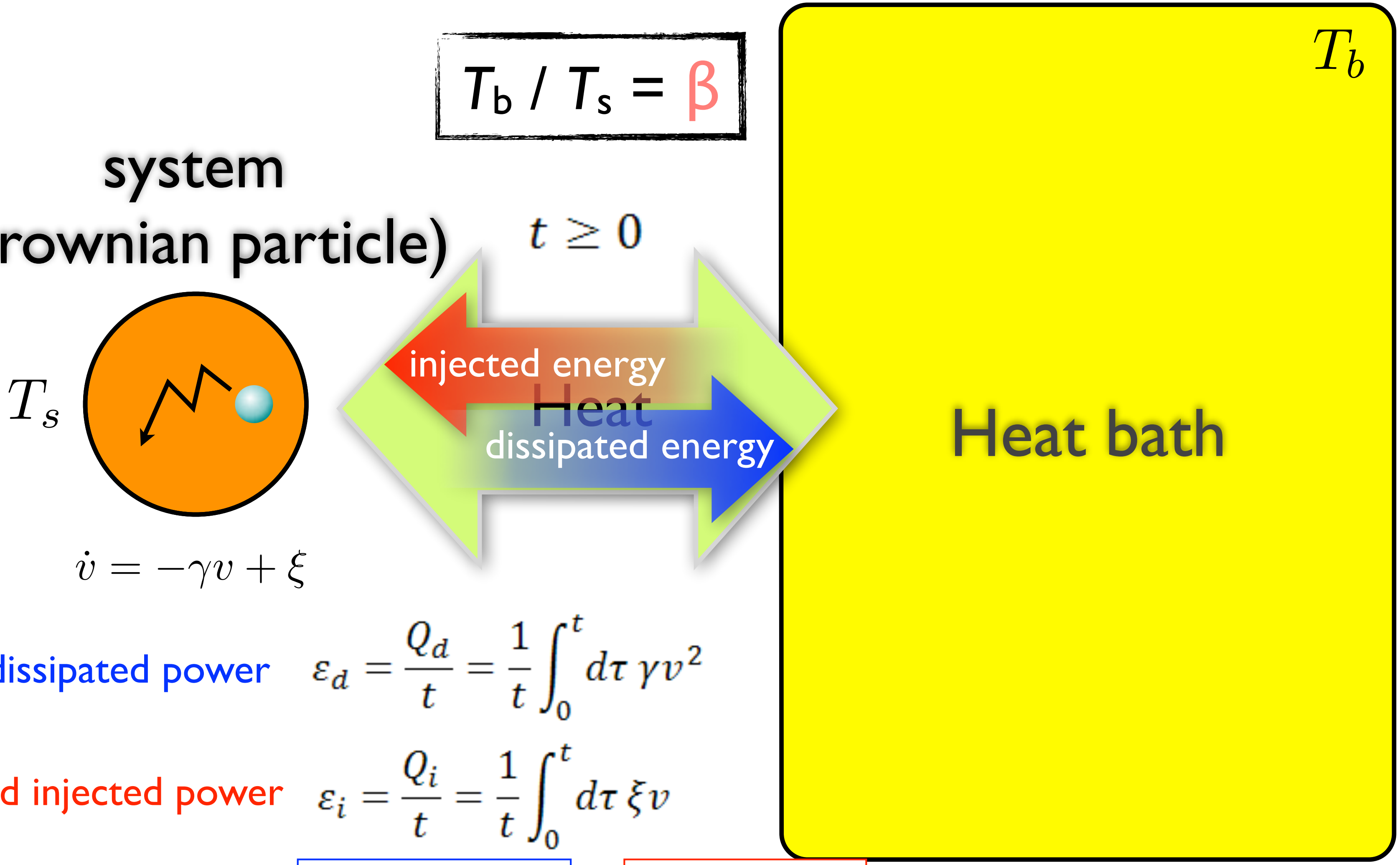

$T_b / T_s = \beta$

system
(Brownian particle)

$t \geq 0$

T_s

injected energy

Heat

dissipated energy

Heat bath

T_b

$$\dot{v} = -\gamma v + \xi$$

averaged dissipated power $\varepsilon_d = \frac{Q_d}{t} = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

averaged injected power $\varepsilon_i = \frac{Q_i}{t} = \frac{1}{t}\int_0^t d\tau\, \xi v$

dissipated energy flow $\boxed{Q_d = \int_0^t d\tau\, \gamma v^2}$ $\boxed{Q_i = \int_0^t d\tau\, \xi v}$ injected energy flow

Farago, J. Stat. Phys. 107 781 (2002)

What we study

Langevin equation

$$\dot{v} = -\gamma v + \xi$$

initial temperature : T_s when $t < 0$

heat bath temperature : $T_b = D/\gamma$ when $t \geq 0$ (T_b / T_s = β)

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Dissipated Power

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Injected Power

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Probability density function (PDF)

$P(\ \varepsilon_\mathrm{d}\ )$

$P(\ \varepsilon_\mathrm{i}\ )$

What we study

Langevin equation

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initial temperature : T_s when $t < 0$

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Injected Power

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Probability density function (PDF)

$P(\varepsilon_d) \sim \exp(t\, h(\varepsilon_d))$ (for large t)

$P(\varepsilon_i) \sim \exp(t\, h(\varepsilon_i))$ (for large t)

$h(\varepsilon)$: large deviation function

What we study

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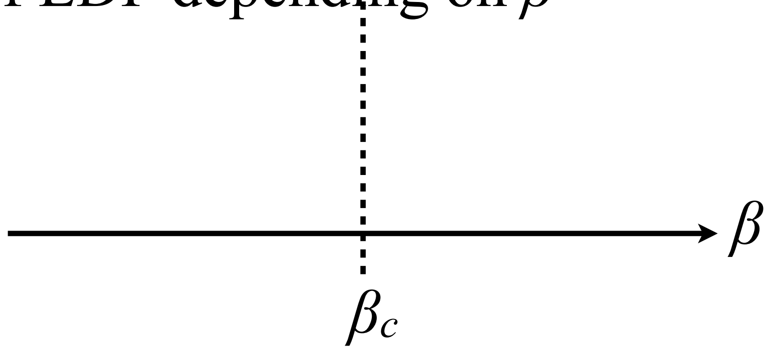
Sharp transition of LDF depending on β



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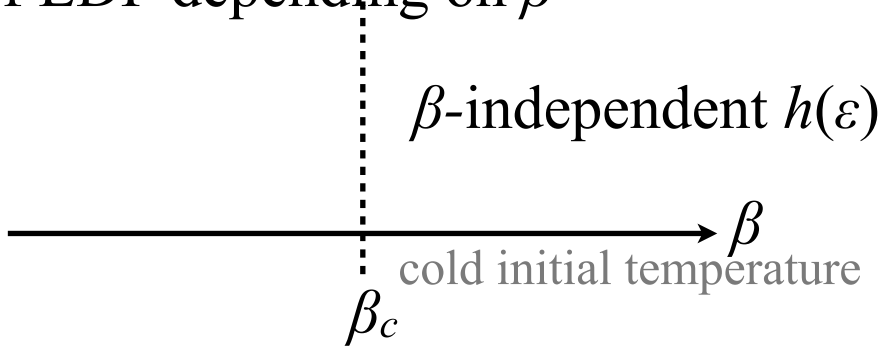

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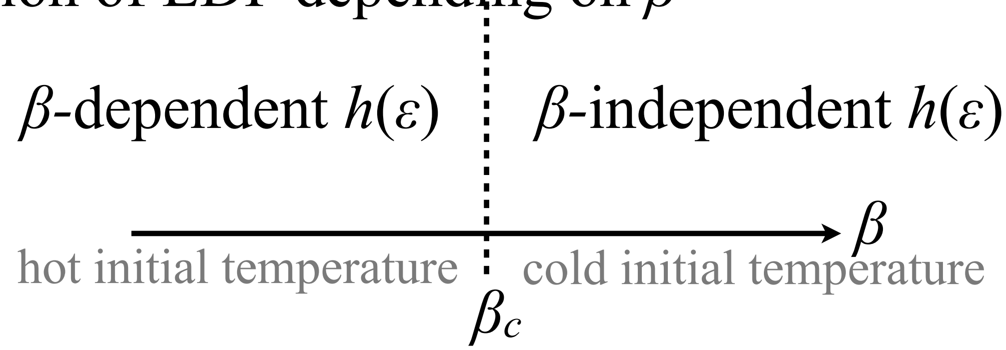

β-dependent $h(\varepsilon)$

β-independent $h(\varepsilon)$

Calculation method (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

Objective : Calculate the probability density function (PDF) of ε_d

1) Calculation of the generating function

$$\pi_d(\lambda) = \langle e^{-\lambda t \varepsilon_d} \rangle = \int_{-\infty}^{\infty} d\varepsilon_d P(\varepsilon_d) e^{-\lambda t \varepsilon_d} = \hat{P}(-i\lambda t)$$

where $\hat{P}(k)$ is the Fourier transform of $P(\varepsilon_d)$.

2) Inverse Fourier transform

$$P(\tilde{\varepsilon}_d) = \frac{\gamma t}{4\pi i}\int_{-i\infty}^{i\infty} d\tilde{\lambda}\pi_d(\gamma\tilde{\lambda}/2D) \exp\left[\frac{\gamma\tilde{\varepsilon}_d\tilde{\lambda}t}{2}\right]$$

where $\widetilde{\varepsilon}_d = \varepsilon/D$

Calculation method (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\,\gamma v^2$

1) Calculation of the generating function

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$\mathbb{P}[v(t)]$: the probability for a given velocity path with fixed initial $v(0) = v_0$ and final $v(t) = v_t$ end points

$$\pi_d(\lambda)_{v_0} = \int_{-\infty}^{\infty} dv_t \int_{v_0}^{v_t} \mathcal{D}v\, \mathbb{P}[v(t)] \exp\left(-\lambda \int_0^t d\tau\, \gamma v^2\right)$$

Calculation method (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

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$$\pi_d(\lambda) = \int_{-\infty}^{\infty} dv_0 \mathcal{P}_{\text{in}}(v_0) \boxed{\pi_d(\lambda)_{v_0}}$$

Calculation method (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

1) Calculation of the generating function

$$\pi_d(\lambda) = \langle e^{-\lambda t \varepsilon_d} \rangle = \int_{-\infty}^{\infty} d\varepsilon_d P(\varepsilon_d) e^{-\lambda t \varepsilon_d} = \hat{P}(-i\lambda t)$$

where $\hat{P}(k)$ is the Fourier transform of $P(\varepsilon_d)$.

$\mathbb{P}[v(t)]$: the probability for a given velocity path with fixed initial $v(0) = v_0$ and final $v(t) = v_t$ end points

$$\boxed{\pi_d(\lambda)_{v_0}} = \int_{-\infty}^{\infty} dv_t \int_{v_0}^{v_t} \mathcal{D}v\, \mathbb{P}[v(t)] \exp\left(-\lambda \int_0^t d\tau\, \gamma v^2\right)$$

$$\pi_d(\lambda) = \int_{-\infty}^{\infty} dv_0 \underset{\uparrow}{\mathcal{P}_{\text{in}}(v_0)} \boxed{\pi_d(\lambda)_{v_0}}$$

using $\mathcal{P}_{\text{in}}(v_0) = \sqrt{\frac{\beta\gamma}{2D\pi}} \exp\left(-\frac{\beta\gamma v_0^2}{2D}\right)$

Calculation method (1. Dissipated power)

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$$\pi_d(\lambda) = \int_{-\infty}^{\infty} dv_0 \mathcal{P}_{\mathrm{in}}(v_0) \boxed{\pi_d(\lambda)_{v_0}} = e^{\gamma t/2}\left(\cosh \eta\gamma t + \frac{1+\tilde{\lambda}/\beta}{\eta} \sinh \eta\gamma t\right)^{-1/2}$$

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Calculation method (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

2) Inverse Fourier transform

$$\pi_d(\lambda) = \int_{-\infty}^{\infty} dv_0 \mathcal{P}_{\text{in}}(v_0)\pi_d(\lambda)_{v_0} = e^{\gamma t/2}\left(\cosh \eta\gamma t + \frac{1+\tilde{\lambda}/\beta}{\eta}\sinh \eta\gamma t\right)^{-1/2}$$

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PDF of : $\varepsilon_{\rm d}$

$$P(\widetilde{\varepsilon}_d) = \frac{\gamma t}{4i\pi}\int_{-i\infty}^{i\infty} d\widetilde{\lambda}\boxed{\pi_d(\gamma\widetilde{\lambda}/2D)}\exp\left(\frac{\gamma t\widetilde{\varepsilon}_d\widetilde{\lambda}}{2}\right)$$

where $\widetilde{\varepsilon}_d = \varepsilon_d/D$: dimensionless

$$= \frac{\gamma t}{4\pi i}\int_{-i\infty}^{i\infty} d\tilde{\lambda}\; e^{\frac{\gamma t}{2}(\tilde{\varepsilon}_d\tilde{\lambda}+1-\eta)}\left(\frac{1+e^{-2\eta\gamma t}}{2}\right)^{-1/2} \times \left(1+\frac{1+\tilde{\lambda}/\beta}{\eta}\tanh\eta\gamma t\right)^{-1/2}$$

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$$\simeq \int_C d\lambda\, \phi(\lambda)e^{tH(\lambda;\varepsilon)}$$

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$$\simeq \int_C d\lambda\, \phi(\lambda)e^{tH(\lambda;\varepsilon)}$$

Here, we consider the large deviation function (LDF) for the PDF in the long time limit.
=> Use the saddle point method.

$$h(\widetilde{\varepsilon}_d) = \lim_{t\to\infty}(\ln P(\widetilde{\varepsilon}_d))/t$$

Analytic and Numerical Results (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

Large deviation function of ε_d

(T_b / T_s = β)

For $\beta > 1/2$ $\quad h(\widetilde{\varepsilon}_d) = -\frac{\gamma}{4\widetilde{\varepsilon}_d}(\widetilde{\varepsilon}_d - 1)^2$ $\quad$ where $\widetilde{\varepsilon}_d = \varepsilon_d / D$

cold initial system
($T_\mathrm{s} < 2T_\mathrm{b}$)

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cold initial system
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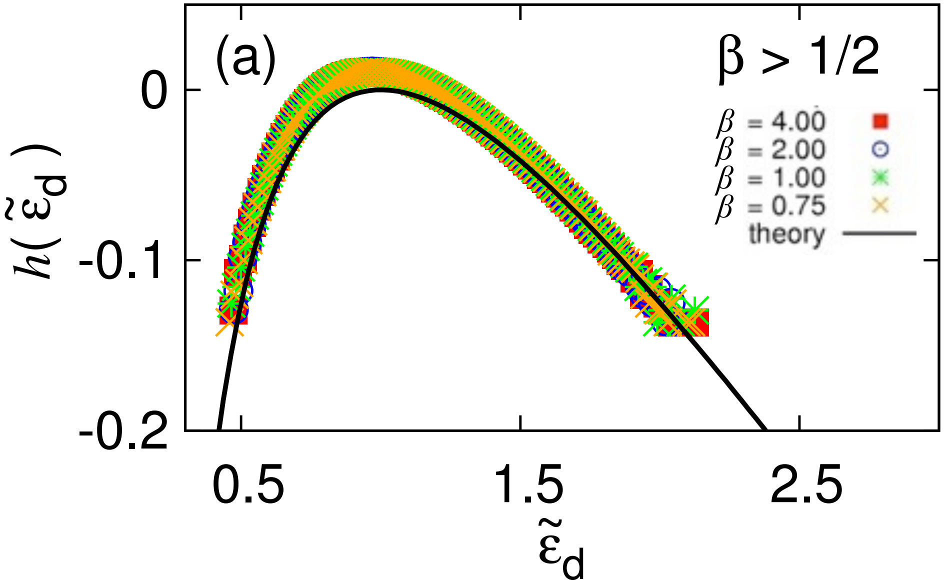
The LDF of ε_d does not depend on β
but depends only on the heat bath properties (γ, D).



Analytic and Numerical Results (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

Large deviation function of ε_{d} ($T_{\mathrm{b}} / T_{\mathrm{s}} = \beta$)

For $\beta > 1/2$ $h(\widetilde{\varepsilon}_d) = -\frac{\gamma}{4\widetilde{\varepsilon}_d}(\widetilde{\varepsilon}_d - 1)^2$ where $\widetilde{\varepsilon}_d = \varepsilon_d / D$

cold initial system

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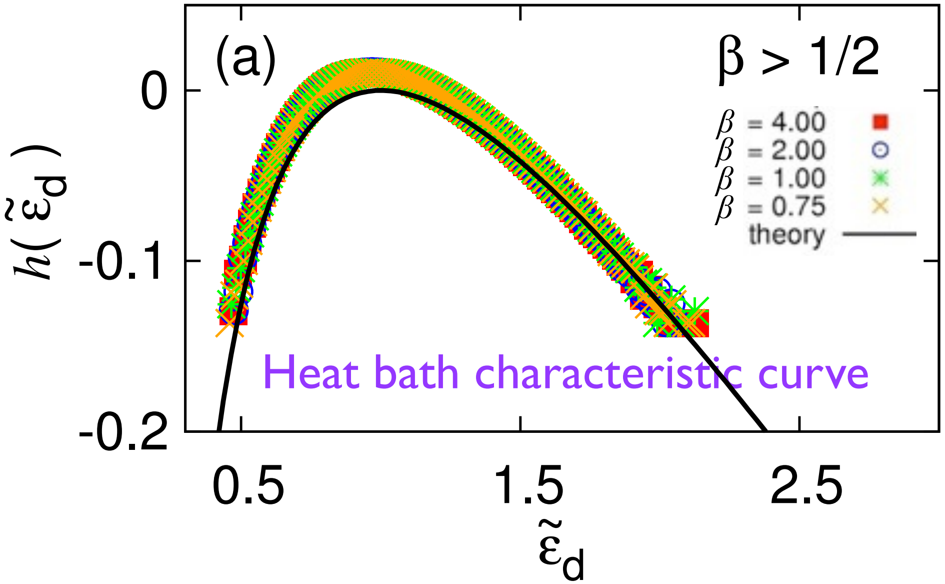
=> Heat bath characteristic curve



Heat bath characteristic curve

Analytic and Numerical Results (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

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cold initial system

($T_{\rm s} < 2T_{\rm b}$)

The LDF of $\varepsilon_{\rm d}$ does not depend on β

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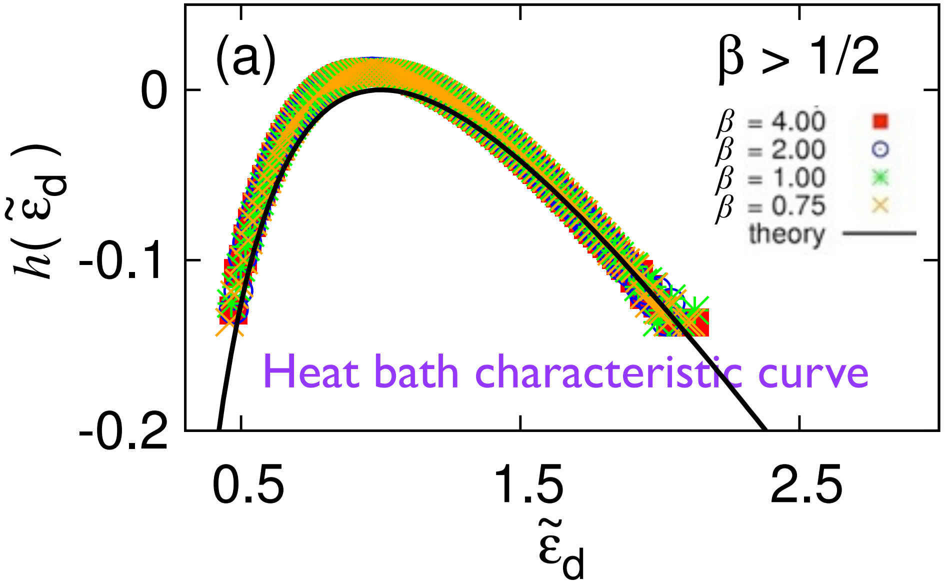
=> Heat bath characteristic curve

The initial memory does not remain in the $t = \infty$ limit for $2T_{\rm b} > T_{\rm s}$.



Analytic and Numerical Results (1. Dissipated power)

Definition : $\varepsilon_d = \frac{1}{t}\int_0^t d\tau\, \gamma v^2$

Large deviation function of ε_d (T_b / T_s = β)

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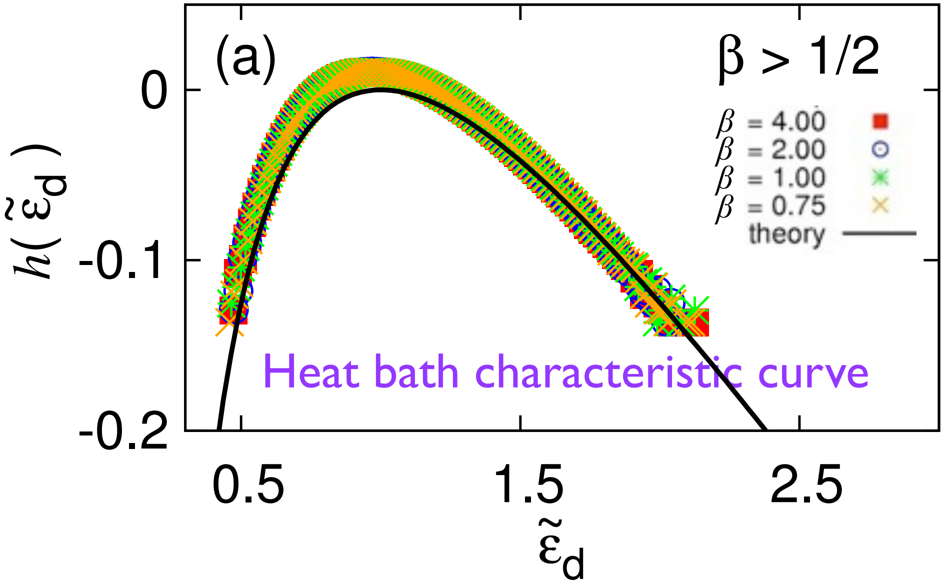
(T_s < 2T_b)

For $\beta < 1/2$ $h(\widetilde{\varepsilon}_d) = \begin{cases} -\frac{\gamma}{4\widetilde{\varepsilon}_d}(\widetilde{\varepsilon}_d - 1)^2, & \widetilde{\varepsilon}_d < \frac{1}{1-2\beta} \\ -\gamma\beta\left[(1-\beta)\widetilde{\varepsilon}_d - 1\right], & \widetilde{\varepsilon}_d > \frac{1}{1-2\beta} \end{cases}$

hot initial system
(T_s > 2T_b)



Heat bath characteristic curve

Analytic and Numerical Results (1. Dissipated power)

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Large deviation function of ε_d (T_b / T_s = β)

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cold initial system
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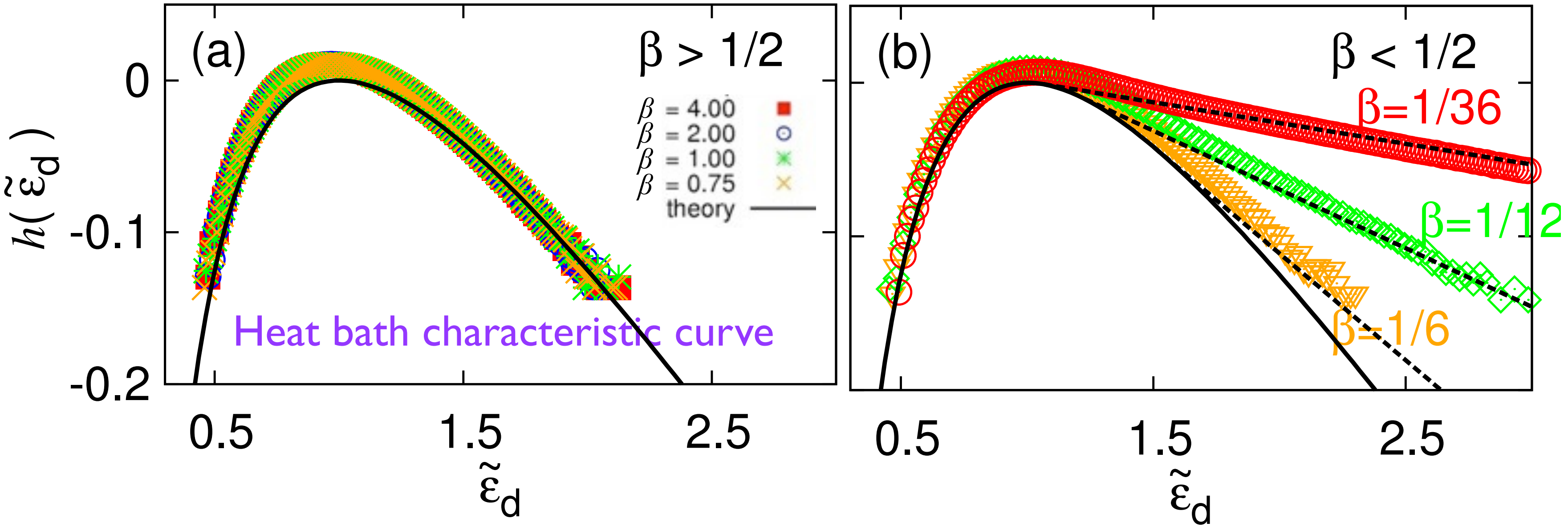
The positive tail of LDF depends on β.



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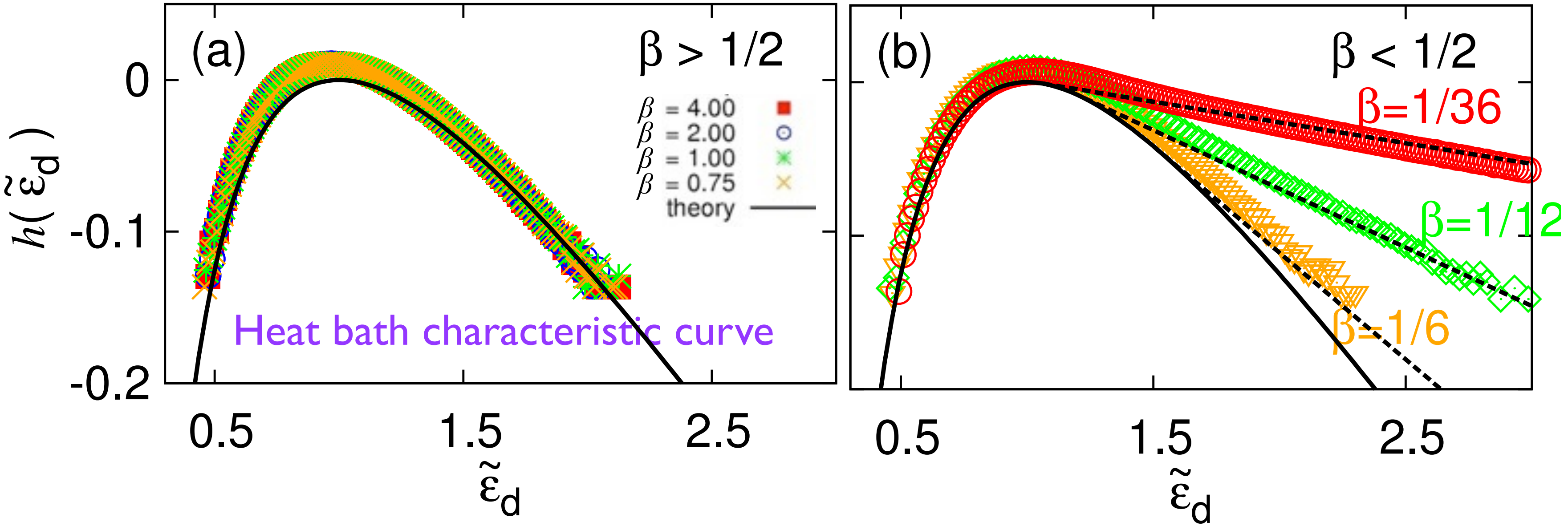


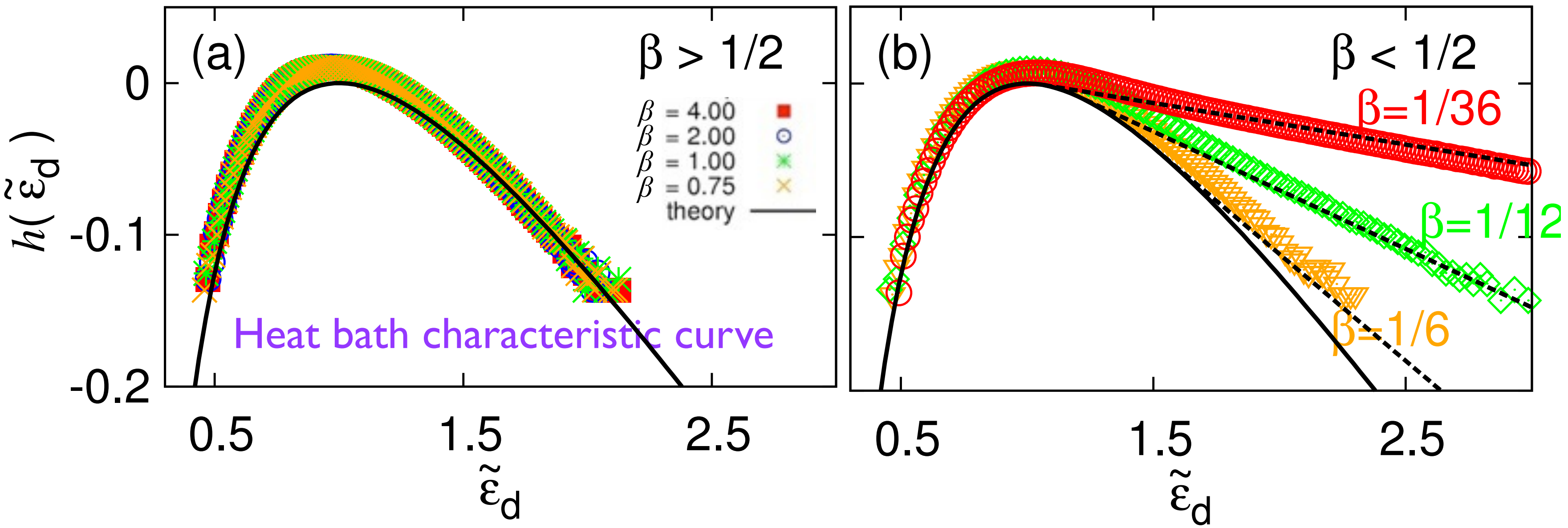
Origin for the LDF transition

LDF transition occurs in positive tail or rare event region (dominated by exponentially rarely high energetic particles).

Generation mechanism of high energetic particle

- high energetic particle can be generated by kicks of a random force of a heat bath
- high energetic particle can exist from the initial distribution

β-dependence is determined by which one is dominant mechanism.



Analytic and Numerical Results (2. Injected power)

Definition : $\varepsilon_i = \frac{1}{t}\int_0^t d\tau \xi v = \frac{1}{t}\int_0^t d\tau (\dot{v}v + \gamma v^2)$

Analytic and Numerical Results (2. Injected power)

Definition : $$\varepsilon_i = \frac{1}{t}\int_0^t d\tau \xi v = \frac{1}{t}\int_0^t d\tau(\dot{v}v + \gamma v^2)$$

Probability density function of ε_i (for large t)   (T_b / T_s = a)

For $\beta > 1/4$ cold initial system ($T_s < 4T_b$)

$$h(\widetilde{\varepsilon}_i) = \begin{cases} -\gamma\sqrt{\beta}\left[1-(1+\sqrt{\beta})\widetilde{\varepsilon}_i\right], & \widetilde{\varepsilon}_i < \frac{1}{1+2\sqrt{\beta}} \\ -\frac{\gamma}{4\widetilde{\varepsilon}_i}(\widetilde{\varepsilon}_i - 1)^2, & \widetilde{\varepsilon}_i > \frac{1}{1+2\sqrt{\beta}} \end{cases}$$

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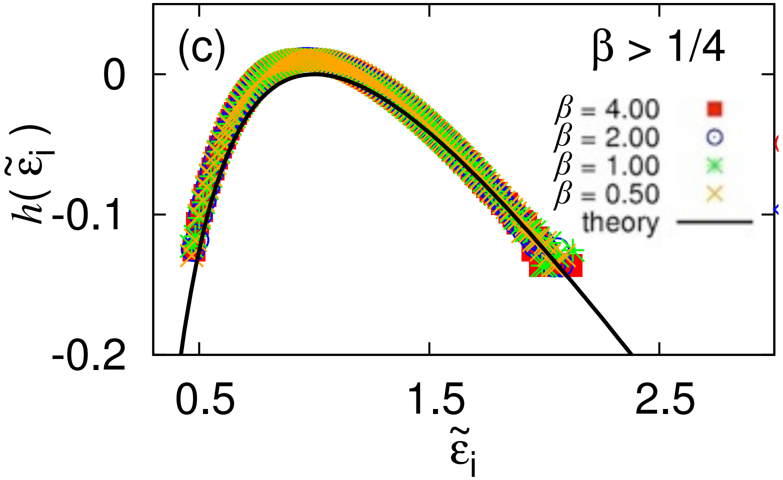

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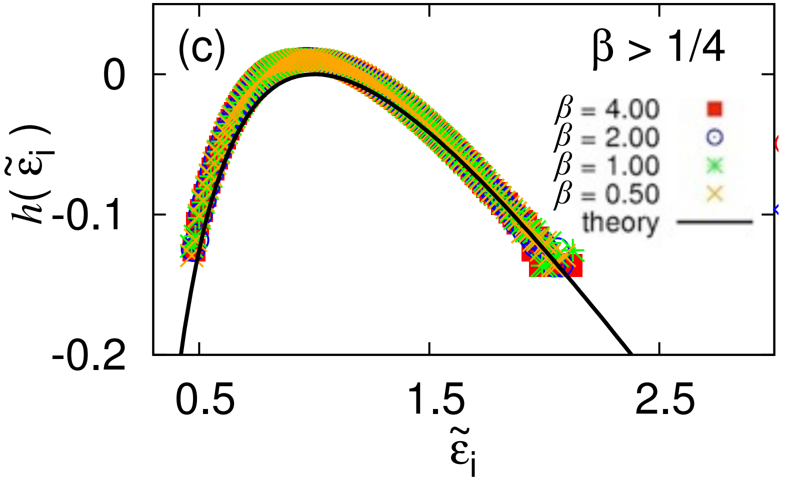

For $\beta < 1/4$ hot initial system ($T_s > 4T_b$)

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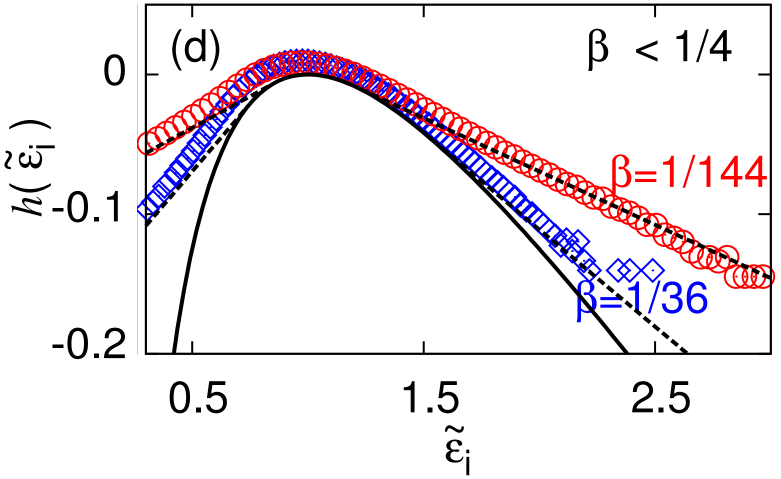

Summary of Part I *PRE* **87**, 020104R (2013)

- We studied an equilibration process of a Brownian particle.
- Heat —› Two heat flows: dissipated and injected powers

- Transition of LDF of the dissipated power occurs at $\beta_c = 1/2$.
- Transition of LDF of the injected power occurs at $\beta_c = 1/4$.

- LDF transition occurs due to the competition between
  - * probability of high energetic particles produced by kick of heat bath random force
  - * probability of high energetic particles come from the initial distribution
- β-dependence is general feature in non-equilibrium phenomena
  - * when a random force is applied to a system Sabhapandit PRE **85**, 021108 (2012)
  - * when a Brownian particle is dragged by a harmonic potential

Van Zon & Cohen PRL 91 110601 (2003)

Generating function

generating function associated with $P(\varepsilon;\tau)$ is defined as

$$G(\lambda;\tau) = \langle e^{-\lambda\tau\varepsilon}\rangle_\tau = \int d\varepsilon\ P(\varepsilon;\tau)e^{-\lambda\tau\varepsilon}$$

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for large τ

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C: steepest descent contour passing through a *conventional* saddle point $\lambda_0^*(\varepsilon)$

solution of $H'(\lambda;\varepsilon) = 0$ with $H' = dH/d\lambda$

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$$P(\varepsilon;\tau) \simeq \sqrt{\frac{2\pi}{\tau|H''(\lambda_0^*;\varepsilon)|}}\,\phi(\lambda_0^*)e^{i\delta}e^{\tau H(\lambda_0^*;\varepsilon)} \quad \text{where } H'' = d^2H/d\lambda^2$$

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When the prefactor $\phi(\lambda)$ has no singularity

⟶ correct LDF: $h(\varepsilon) = \lim_{\tau\to\infty}\frac{1}{\tau}\ln P(\varepsilon;\tau) = \boldsymbol{H(\lambda_0^*;\varepsilon)}$

⟶ correct finite-time correction $\sqrt{\dfrac{2\pi}{\tau|H''(\lambda_0^*;\varepsilon)|}}\ \phi(\lambda_0^*)$

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δ : angle b.t.w. contour path and real axis

When $\phi(\lambda) = \dfrac{g(\lambda)}{(\lambda-\lambda_B)^\alpha}$ with $\alpha > 0$ and $\lambda_0^*(\varepsilon_B) = \lambda_B$

—› *P* diverges due to the prefactor.

—› incorrect finite-time correction near the singular point

Physical Examples

1) van Zon and Cohen, PRL 91, 110601 (2003)

Heat flow from a heat bath to a brownian particle in a dragged harmonic potential

$P_\tau(Q_\tau)$: PDF of heat at time *t*

$$\langle e^{-\lambda Q_\tau}\rangle \equiv \int_{-\infty}^{\infty} dQ\, e^{-\lambda Q_\tau} P_\tau(Q_\tau) = \hat{P}_\tau(i\lambda)$$

$$\langle e^{-\lambda Q_\tau}\rangle = \frac{\exp[-w\lambda(1-\lambda)\{\tau + \frac{2\lambda^2(1-e^{-\tau})^2}{1-(1-e^{-2\tau})\lambda^2}\}]}{[1-(1-e^{-2\tau})\lambda^2]^{3/2}}$$

3/2 pole (for 3 dim.)

2) Sabhapandit, EPL 96, 20005 (2011)

Work done by a random force to brownian particle in a harmonic potential

$P(W_\tau)$: PDF of work at time *t*

$$P(W_\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} Z(\lambda,\tau)\, e^{\lambda W_\tau}\, \mathrm{d}\lambda$$

$Z(\lambda,\tau) \sim g(\lambda)\, e^{\tau\mu(\lambda)}$ where $g(\lambda) = \dfrac{2}{1+\eta(\lambda)-2\alpha\lambda} \times \dfrac{2\eta(\lambda)}{1+\eta(\lambda)+2\alpha\lambda}$

simple pole (for 1 dim.)

Physical Examples

3) Our work, PRE 87,020104 (2013)

dissipated and injected energy of a brownian particle

$P(\varepsilon_d)$: PDF of dissipated energy at time *t*

$$P(\widetilde{\varepsilon}_d) = \frac{\gamma t}{4\pi i} \int_{-i\infty}^{i\infty} d\widetilde{\lambda}\ \pi_d(\gamma\widetilde{\lambda}/2D) \exp\left[\frac{\gamma t \widetilde{\varepsilon}_d \widetilde{\lambda}}{2}\right]$$

$$\pi_d(\lambda) = e^{\gamma t/2}\left(\cosh\eta\gamma t + \frac{1+\widetilde{\lambda}/\beta}{\eta}\sinh\eta\gamma t\right)^{-1/2}$$

1/2 pole (for 1 dim.)

Objective

How to calculate the integral

$$P(\varepsilon;\tau) \simeq \int_C d\lambda \, \frac{g(\lambda)}{(\lambda-\lambda_B)^\alpha} \; e^{\tau H(\lambda;\varepsilon)}$$

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conventional saddle point solution of

Objective

How to calculate the integral

conventional saddle point solution of

modified saddle point solution of

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How to calculate the integral

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leading contribution is obtained at

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leading contribution is obtained at

: saddle point fixation

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finite-time correction

In the conventional method, contour should detour the branch cut.

One needs to calculate the integral for all segments C_1, C_2, C_3, and C_4.

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finite-time correction

In the conventional method, contour should detour the branch cut.

One needs to calculate the integral for all segments C_1, C_2, C_3, and C_4.

In the modified method, one needs to calculate only one saddle point integration near ⟶ much simpler

Objective

Comparison between LDF and LDF with finite-time correction

Dissipated power

Red circles: numerical calculation results at finite time

Black line: LDF only

blue line: LDF with finite-time correction

Summary of Part II

1. When there is a singularity in the prefactor function,

   conventional Gaussian integration gives incorrect result.

2. Modified saddle point , which is the solution of

   makes the integral much simpler.

3. If you are interested in the detailed calculation,
   see J. Stat. Mech. P11002 (2013).

Thank you for your attention!

What kind of particle leads to the high dissipated power ?

Two sources of high energy particle

1) particle with high initial energy → For $\beta < 1/2$

2) high energy particle due to injected energy from the heat bath → For $\beta > 1/2$

Probability to find a particle to (large) dissipated energy Q_d from source 2)

Probability to find a particle to (large) dissipated energy Q_d from source 1)

$\dot{v} = -\gamma v + \xi$

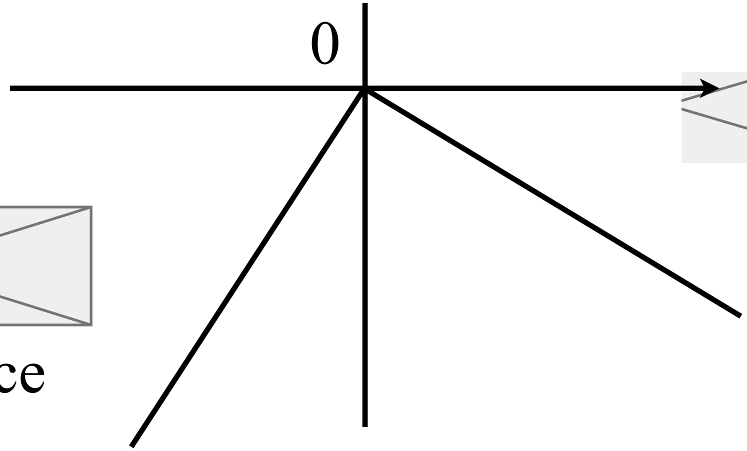
In the long time limit, initial energy is fully dissipated.

When → → heat bath dominant phase

When → → initial condition dominant phase

β-dependence for total heat

total heat



β-dependence

β-independent $h(\varepsilon)$