Statistical Mechanics of Self-Gravitating Systems

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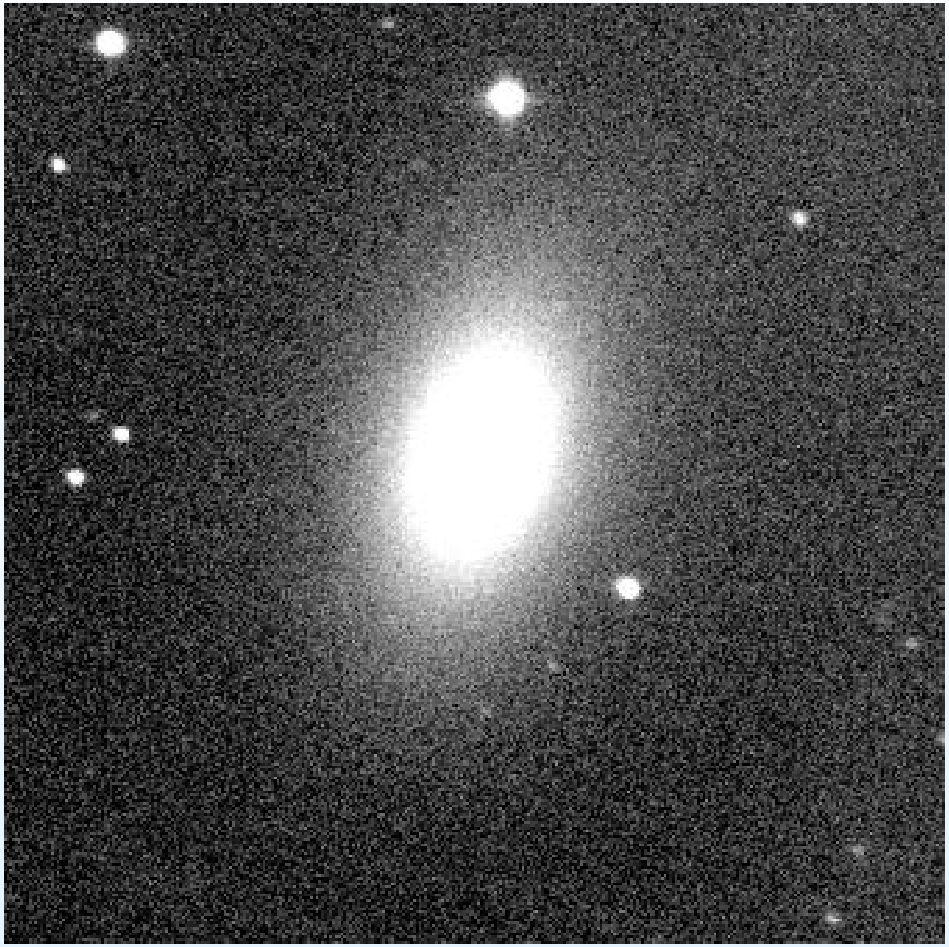
Elliptical Galaxies



Levin et al. Phys. Rev. E 78, 021130 (2008); Mon. Not. R. As. Soc. 417, L21 (2011).

Vlasov Equation

$$\frac{Df}{Dt} = 0 \to \frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} - \nabla\psi\frac{\partial f}{\partial \vec{v}} = 0$$

Casimir invariants

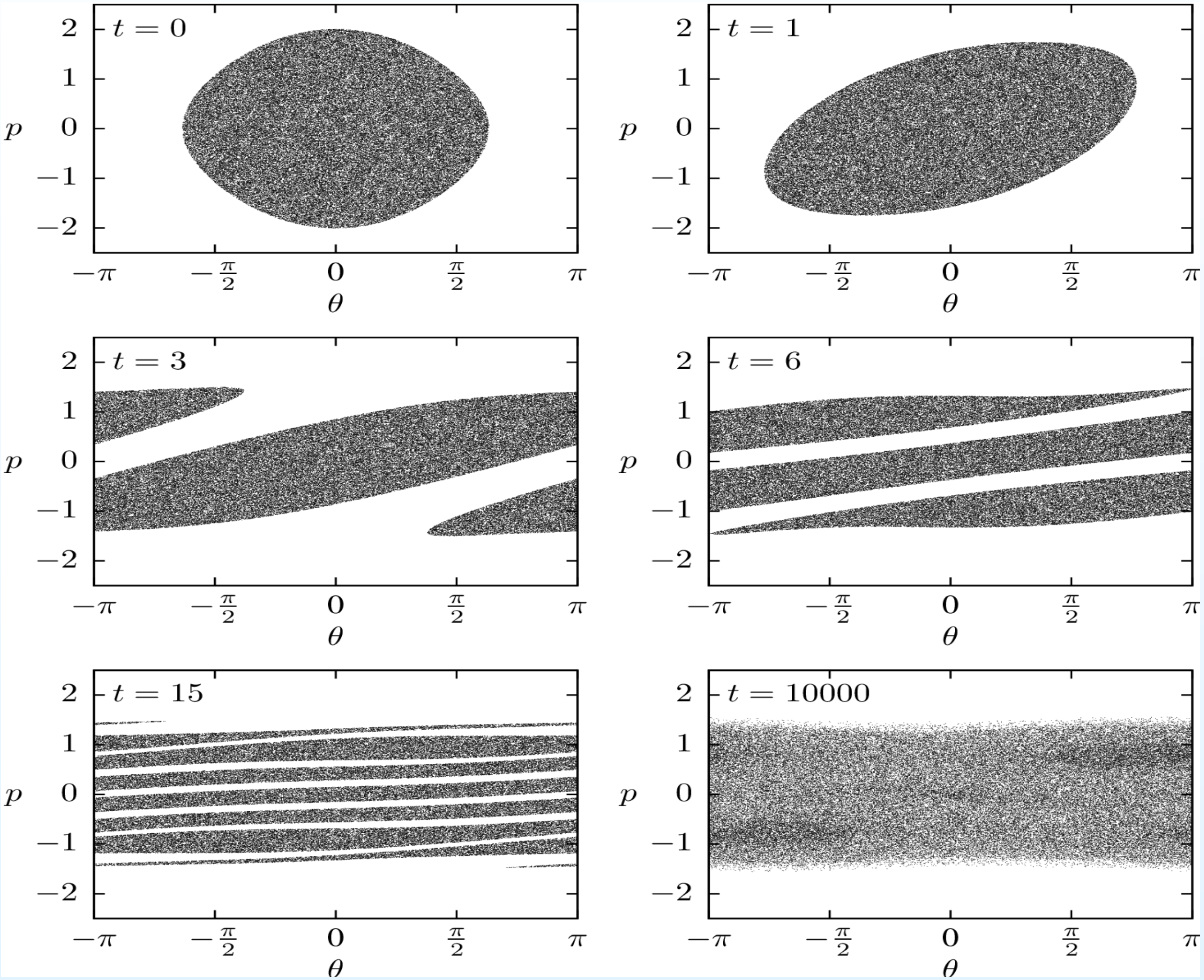
$$C_s[f] = \int s(f(\mathbf{r}, \mathbf{v}, t))d^d\mathbf{r}d^d\mathbf{v}$$

$s(f)$ is an arbitrary function of f

$s(f) = f$, $s(f) = f^N$, $s(f) = f\ ln\ f$, ...

The Boltzmann entropy is constant!

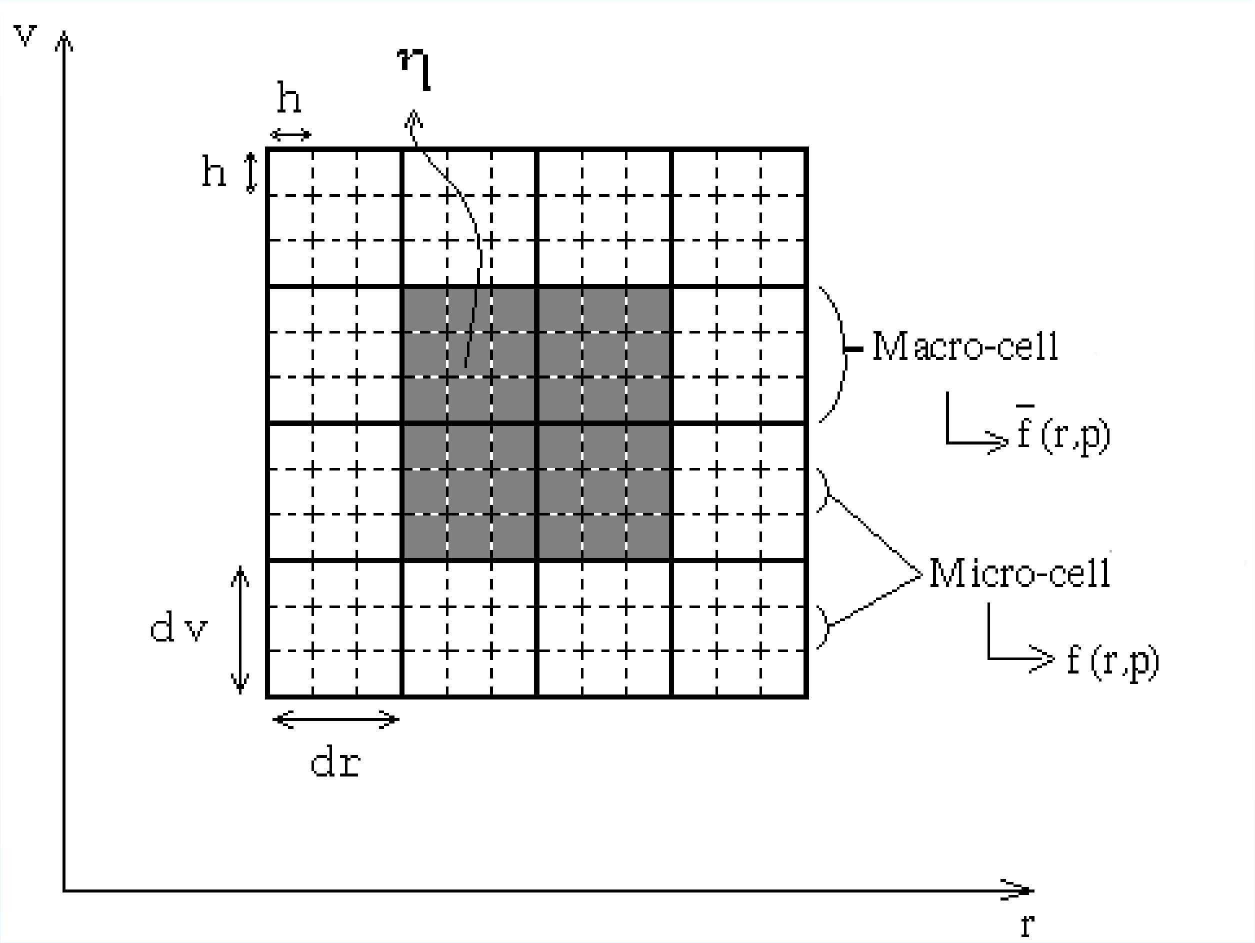
The Phase Space



*Levin et al., Physics Reports **535** 1 (2014)

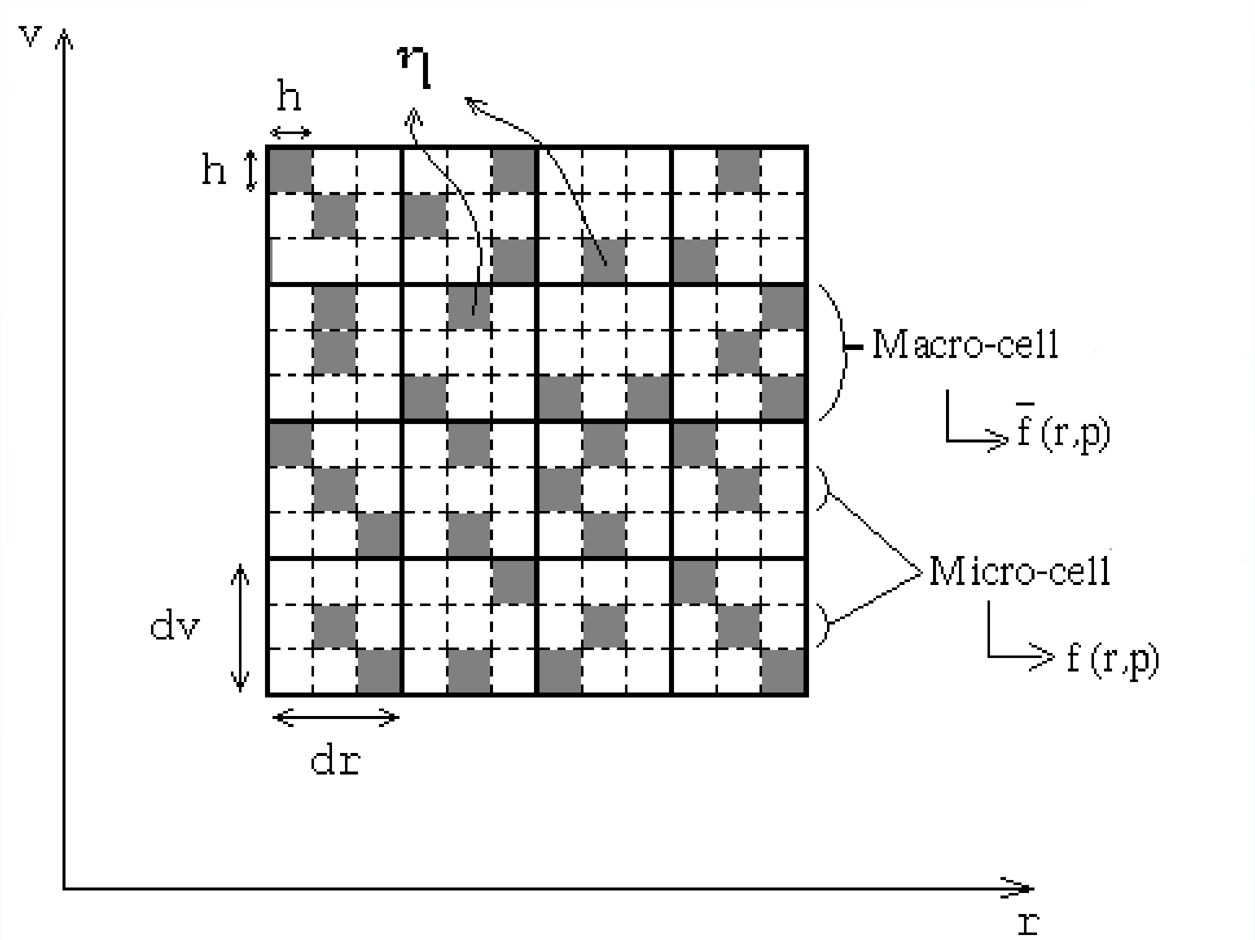
The Model

- Initial Distribution: $f_0(\mathbf{r}, \mathbf{v}) = \eta\Theta(r_m - r)\Theta(v_m - v)$



Incompressible Flow

- Final Distribution $f(\vec{r}, \vec{v}) = ?$

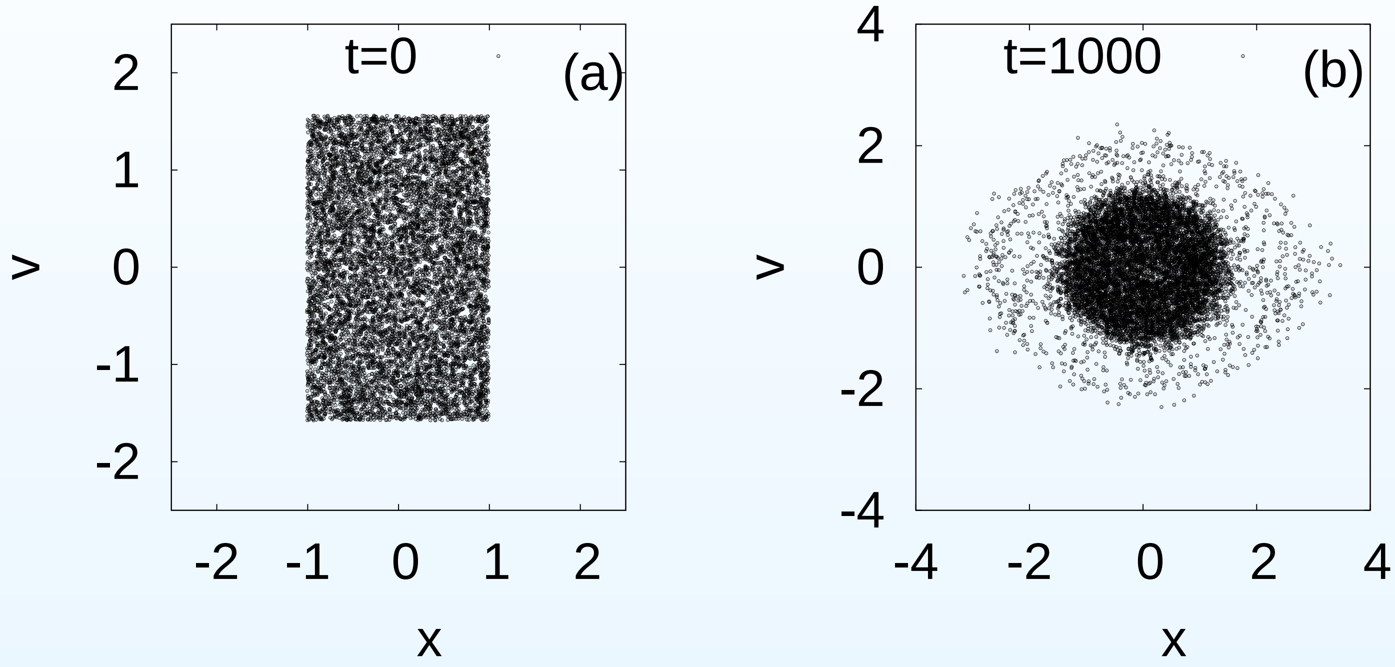


The Theory

The coarse-grained particle distribution must satisfy:

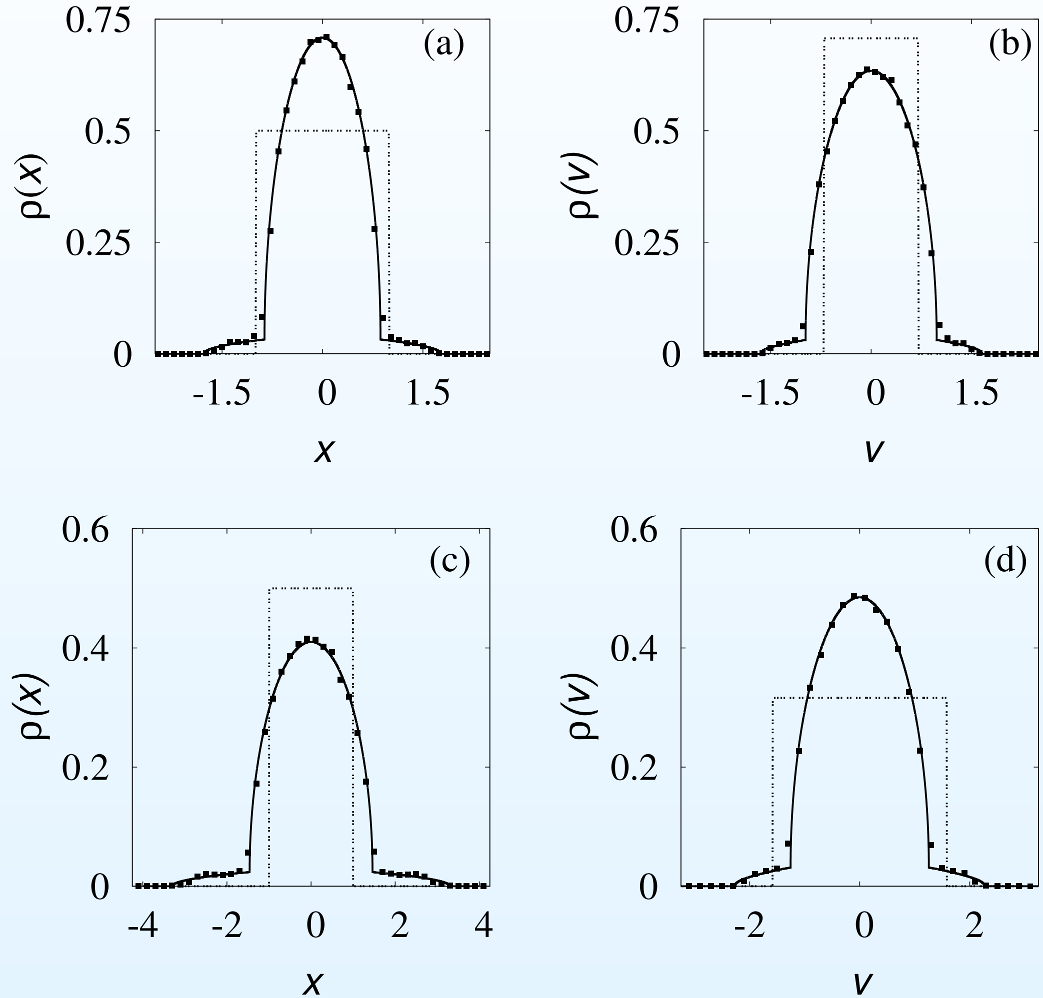
$$f(\vec{r}, \vec{v}) \leq \eta$$

1d Gravity



*Teles et al. MNRAS 417, L21 (2011)

Density and Velocity Distributions in 1d



*Teles et al. MNRAS 417, L21 (2011)

Relaxation of 2d Gravitational Systems

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \frac{\nabla \psi}{m} \cdot \nabla_{\mathbf{v}} f = 0\,.$$

and

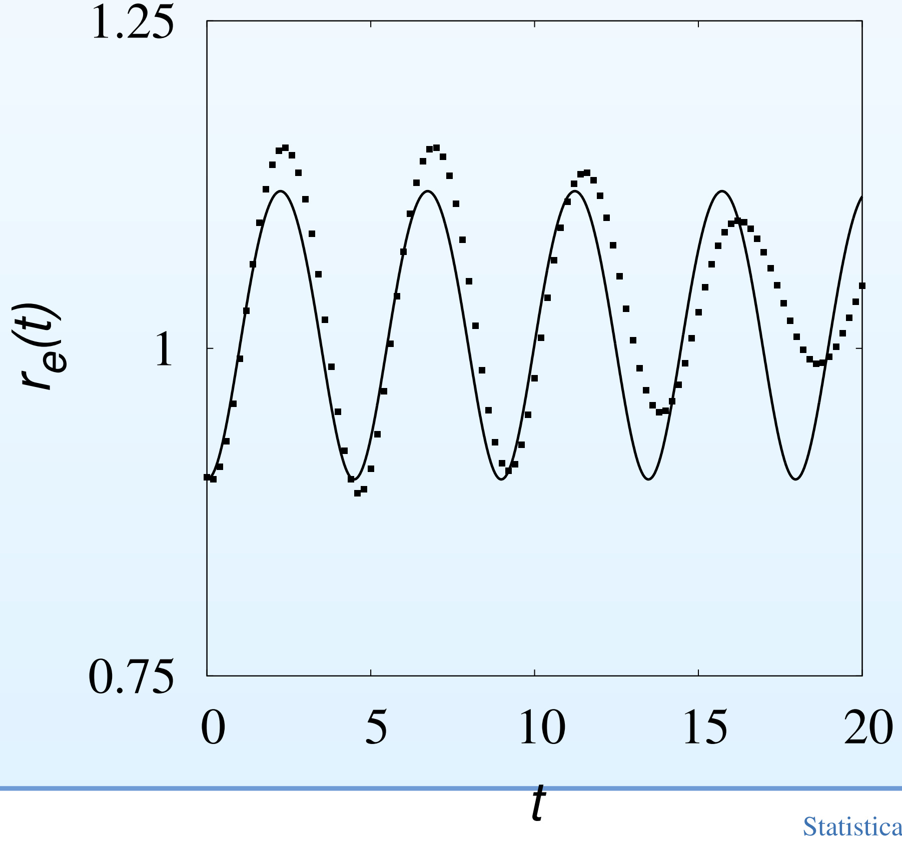
$$\nabla^2 \psi = 4\pi G m n(\mathbf{r})$$

where

$$n(\mathbf{r}) = N \int f(\mathbf{r}, \mathbf{v}) d^2\mathbf{v}$$

Envelope Oscillations

$$\ddot{r}_e(t) + \frac{2GM}{r_e(t)} = \frac{v_m^2 r_m^2}{r_e^3(t)} \ .$$



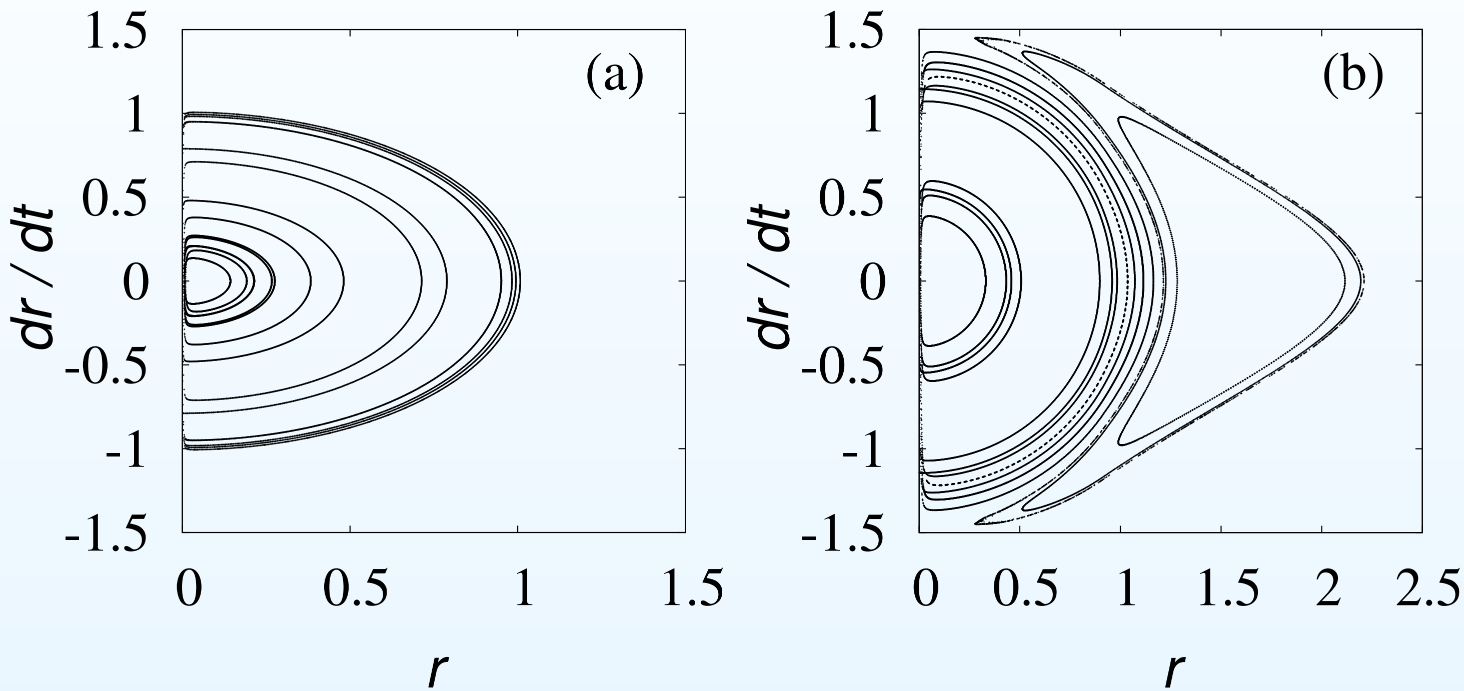
Test Particle

$$\ddot{r}_i(t) - \frac{L_i^2}{r_i^3(t)} = \begin{cases} -\frac{r_i(t)}{r_e^2(t)} & \text{for} \quad r_i(t) \leq r_e(t) \\ \\ -\frac{1}{r_i(t)} & \text{for} \quad r_i(t) \geq r_e(t) \end{cases}$$

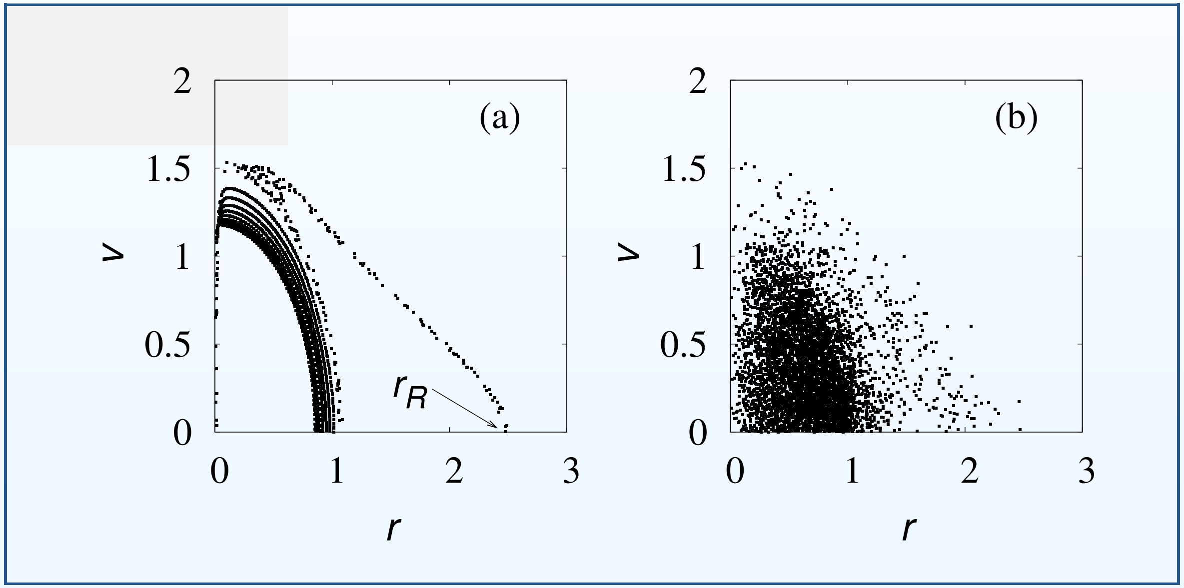
where

$$L_i = \left| \mathbf{r}_i \times \mathbf{v}_i \right|$$

Poincaré Section



Resonance and Simulations

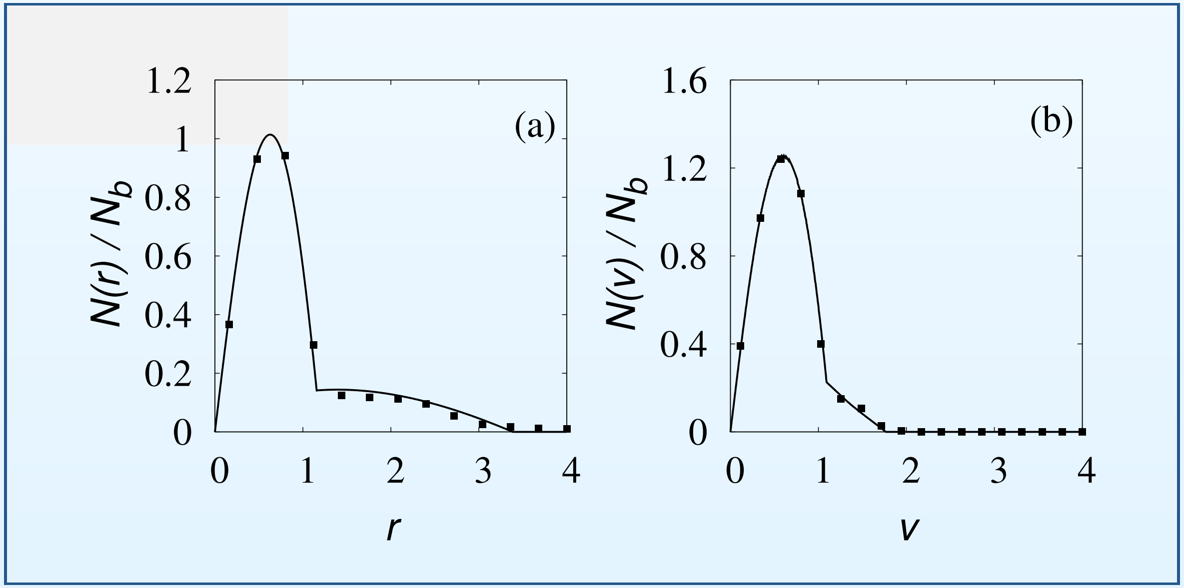


$$f(\mathbf{r},\mathbf{v}) = \eta\Theta\left(\epsilon_F - \epsilon(\mathbf{r},\mathbf{v})\right) + \chi\Theta(\epsilon(\mathbf{r},\mathbf{v}) - \epsilon_F)\Theta(\epsilon_R - \epsilon(\mathbf{r},\mathbf{v})) \quad (1)$$

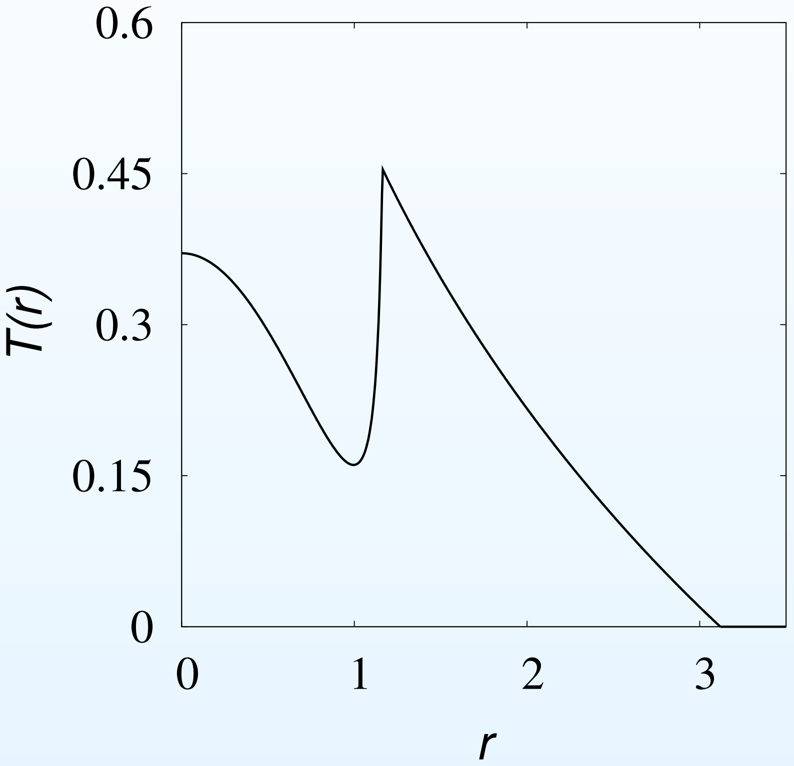
Density and Velocity Distributions

$$\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}^2\mathbf{v}\,\left[\frac{\mathbf{v}^2}{2}+\frac{\psi(r)}{2}\right] f(\mathbf{r},\mathbf{v}) = \epsilon_0$$

$$\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}^2\mathbf{v}\, f(\mathbf{r},\mathbf{v}) = 1 \ , \qquad (2)$$



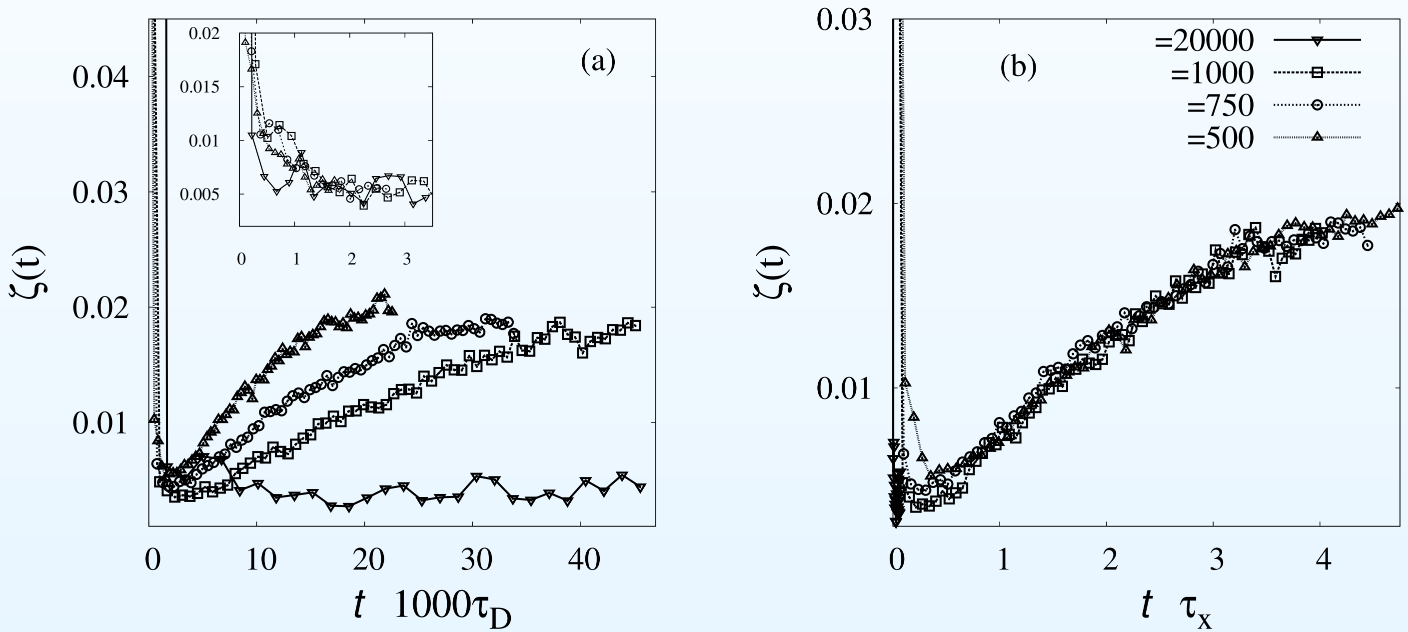
Kinetic Temperature Distributions



*Teles et al. J. Stat. Mech. P05007 (2010)

Crossover to Boltzmann Distribution

- $\chi(t) = \frac{1}{N^2} \int_0^\infty [N(r,t) - N_{ch}(r)]^2 \, d^2\mathbf{r}$



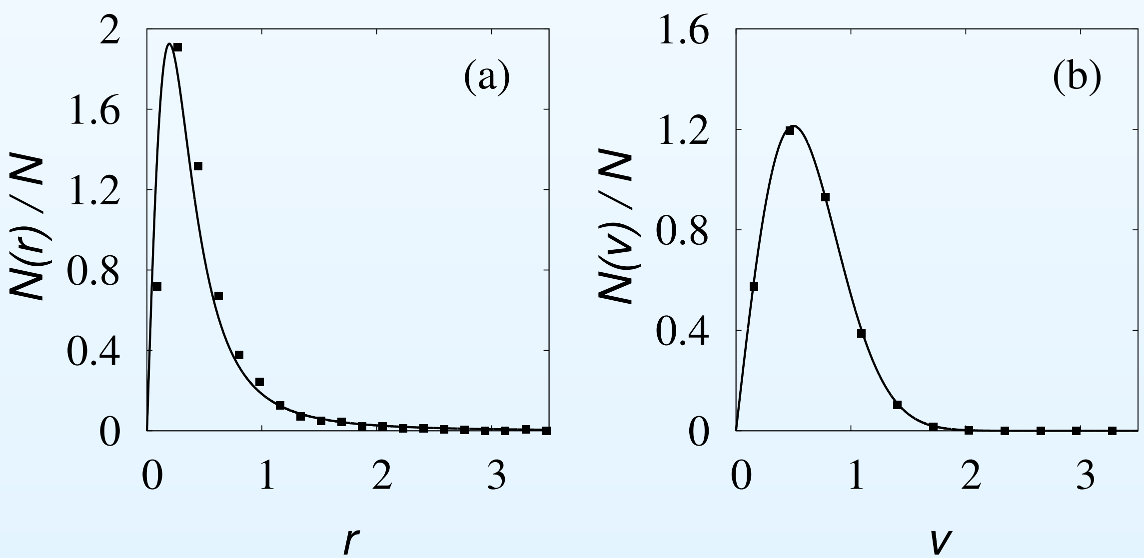
- collapse time $\tau_\times \approx N^{1.35} \tau_D$, where $\tau_D = r_m / \sqrt{2GM}$

*Teles et al. J. Stat. Mech. P05007 (2010)

Finite N: Boltzmann Distribution

- $f_{MB} = Ce^{-\beta(\mathbf{v}^2/2+\psi(\mathbf{r}))}$
- Poisson-Boltzmann equation:

$$\nabla^2\psi = \frac{4\pi^2 C}{\beta} e^{-\beta\psi}$$



Symmetry Breaking in d-Dimensions

- Initial distribution:

$$f_0(\mathbf{r}, \mathbf{v}) = \frac{d^2}{C_d^2 r_m^d v_m^d} \Theta(r_m - r)\Theta(v_m - v)$$

- Evelope in position:

$$X_i(t) = \sqrt{(d+2)\langle x_i^2 \rangle}$$

- Envelope in velocity:

$$V_i(t) = \sqrt{(d+2)\langle v_i^2 \rangle}$$

Envelope Equation

$$\dot{X}_i^2 + X_i \ddot{X}_i = V_i^2 - (d+2)\left\langle x_i \frac{\partial \psi}{\partial x_i} \right\rangle$$

and

$$V_i \dot{V}_i = -(d+2)\left\langle \dot{x}_i \frac{\partial \psi}{\partial x_i} \right\rangle$$

where

$$\nabla^2 \psi = C_d \, n(\mathbf{r}, t)$$

Envelope Equation

$$\ddot{X}_i = \frac{\epsilon_i^2}{X_i^3} - \frac{d}{2}\, X_i\, g_i(X_1, \cdots, X_d)$$

where emittance

$$\epsilon_i^2(t) = X_i^2 V_i^2 - \dot{X}_i^2 X_i^2 = \epsilon_i^2(0)$$

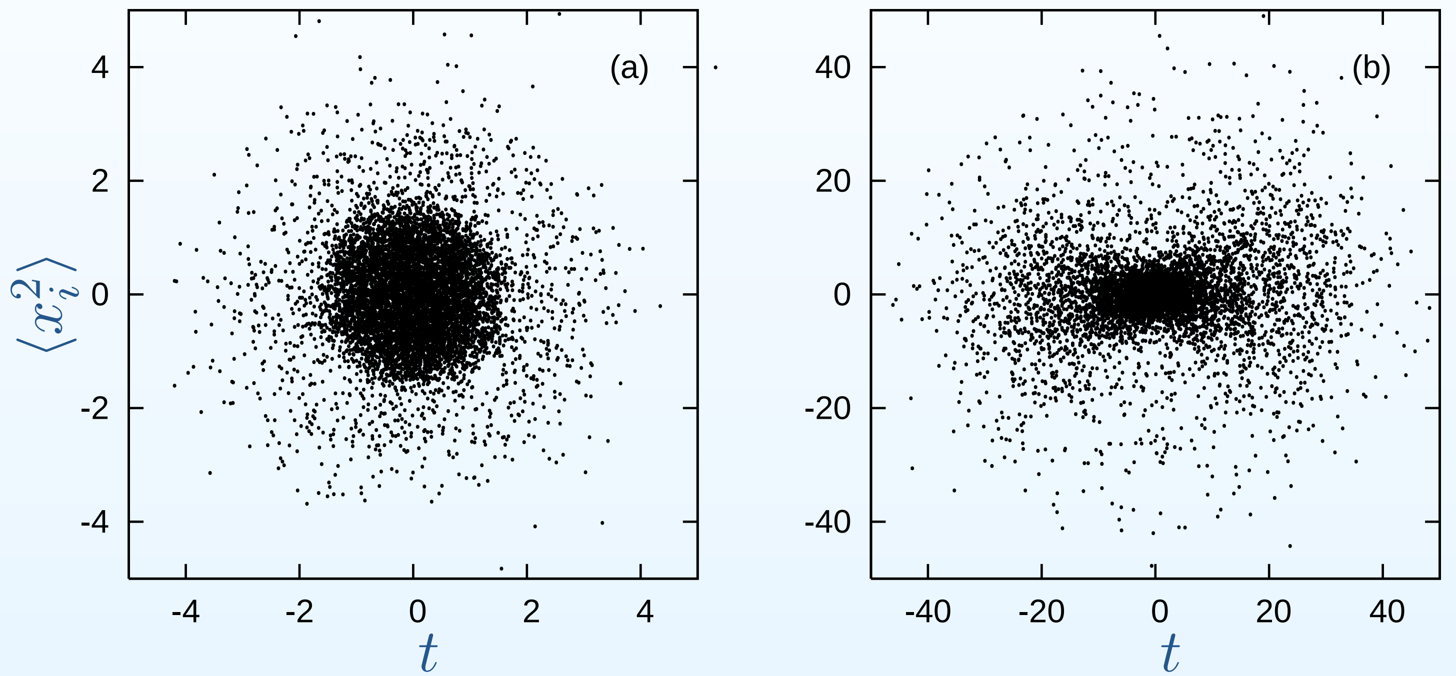
and

$$g_i(X_1, \cdots, X_d) = \int_0^\infty \frac{ds}{(X_i^2 + s) \prod_{j=1}^d (X_j^2 + s)^{1/2}}$$

Symmetry Breaking

- Virial number $\mathcal{R}_0 \equiv \frac{2K}{(2-d)U} = v_m^2 = \epsilon^2(0)$
- Symmetric mode $\bar{X} = (\sum_i X_i)/d$ is unstable for $d > 3$
- Oscillations of $\bar{X}$ result in the growth of antisymmetric mode.
- Instability in $d = 2$ for $\mathcal{R}_0 < 0.255893...$ and $\mathcal{R}_0 > 2.55819...$
- Instability in $d = 3$ for $\mathcal{R}_0 < 0.388666...$ and $\mathcal{R}_0 > 1.61133...$

Symmetry Breaking in d=2



2D system system: Panel (a) $\mathcal{R}_0 = 2$; Panel (b) $\mathcal{R}_0 = 6.25$

Pakter, Marcos, and Y. Levin, Phys. Rev. Lett. **111**, 230603 (2013).

3d Gravity with $\mathcal{R}_0 = 1$

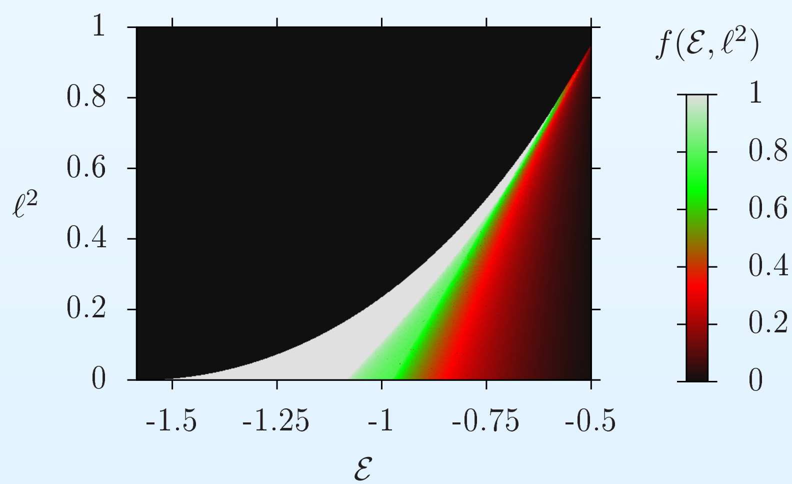
- Small envelope oscillations.
- Quasi-static mean-field potential $\psi(r)$.
- Coarse-grained distribution function:

$$f(\mathcal{E}, \ell^2) = \frac{n\left(\mathcal{E}, \ell^2\right)}{g\left(\mathcal{E}, \ell^2\right)}$$

Distribution Function

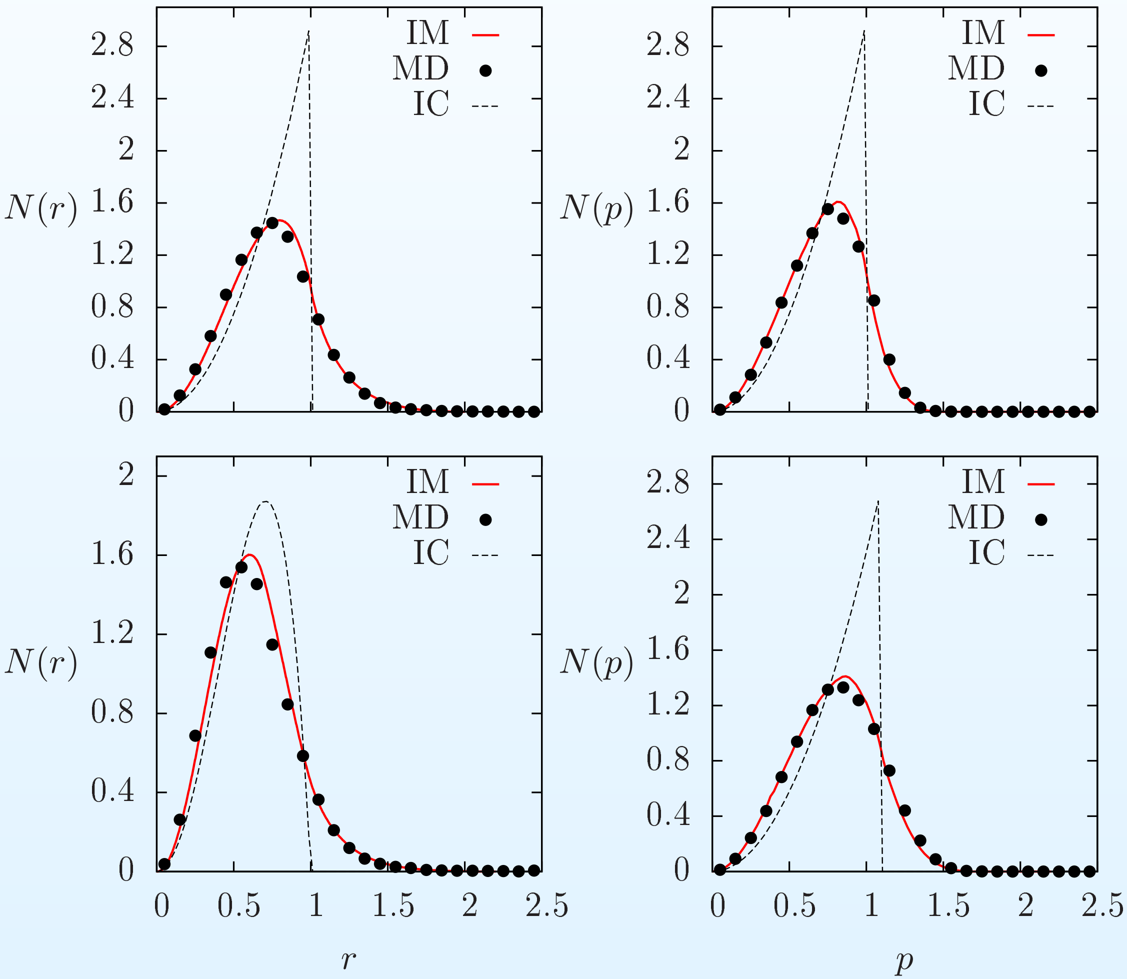
- Coarse-grained distribution function:

$$f(\mathcal{E},\ell^2) = \frac{\int \mathrm{d}r\, f_0\left[r, \sqrt{2\big(\mathcal{E}-\psi(r)\big)}\right] \frac{\Theta\left[\mathcal{E}-\frac{\ell^2}{2r^2}-\psi(r)\right]}{\sqrt{\mathcal{E}-\frac{\ell^2}{2r^2}-\psi(r)}}}{\int \mathrm{d}r\, \frac{\Theta\left[\mathcal{E}-\frac{\ell^2}{2r^2}-\psi(r)\right]}{\sqrt{\mathcal{E}-\frac{\ell^2}{2r^2}-\psi(r)}}}$$



Marginal distributions

- Initial waterbag and parabolic initial distributions



Conclusions

- Systems with long-range interactions lack ergodicity and mixing.
- The proposed core-halo distribution describes plasmas, 1d and 2d self-gravitating systems, HMF, etc.
- There is a significant degree of universality.
- Non-symmetric QSS are possible for gravitational systems.
- 3d gravity is very complex because of particle evaporation, but some progress is being made...
- **Review:** Levin, Pakter, Rizzato, Telles, Benetti, *Nonequilibrium statistical mechanics of systems with long-range interactions*, Phys. Rep. **535**, 1 (2014).