Large and very large deviations in disordered systems

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Large deviations in probability theory.

Large deviation for the ground state energy and the replica approach.

Large deviations and *very large* deviations in a simple example, the REM.

Infinite coordination models: The $p = 2$ spherical model, the Sherrington Kirkpatrick model.

Finite coordination,e.g, Bethe lattices: work in progress.

The central limit theorem

In disordered systems the observables depend on the random variables (J). Most predictions concern the average behaviour (that in many cases coincide with the most likely behaviour in the infinite volume limit), however there are many reasons for being interested to the tail of the probability distribution,

Let f_J be an intensive quantity that does not wildly fluctuate in systems with N components when N goes to infinity (for simplicity the free energy density or the ground state energy at zero temperature);

$$f_J = \frac{F_J}{N} \ .$$

We usually have that

$$\lim_{N \to \infty} P_N(f) = \delta(f - f^*) \ .$$

Large and very large deviations

Large deviations regime:

$$P_N(f) \approx \exp(-NL(f))$$

$$L(f) \geq 0 \ \text{ for } \ f \neq f^* \qquad L(f^*) = 0$$

Very large deviations regime

$$P_N(f) \approx \exp\left(-N^2 \mathcal{L}(f)\right) \quad \text{or} \quad P_N(f) \approx \exp\left(-N\log(N)\mathcal{L}(f)\right)$$

In some cases:

for $f < f^*$: large deviations

for $f > f^*$: very large deviations

General considerations

Let us call J the control parameters.

For studying large deviations it is convenient to consider the function

$$\overline{\exp(nNf_J)} = \int df\, P_N(f) \exp(Nnf) \equiv \exp(Nnf(n, N)) \ ,$$

where for large N $f(n, N)$ does not depend on N; it will be called $f(n)$.

$$f(0) = f^* \ .$$

The functions $f(n)$ and $L(f)$ are connected by Legendre transform.

For very large deviations is convenient to consider the function

$$\overline{\exp(\alpha N^2 f_J)} \equiv \exp(N^2 \alpha f(N\alpha, N)) \ .$$

Here, for $n = O(1)$, $f(n, N) = f(0)$ and $f(N\alpha, N)) = g(\alpha)$.

The replica method

If f_J is the free energy density, we have to compute

$$\overline{\exp(nNf_J)} = \overline{Z_J^{n/\beta}} \ .$$

This can be done with the replica method.

Everything is smooth in the limit where $\beta \to \infty$ and the free energy becomes the internal energy.

For $n > 0$ we explore the region where $f < f^*$.

For $n < 0$ we explore the region where $f > f^*$ (where sometimes the very large deviations lurk).

The Random Energy Model

The REM was introduced by firstly by Cabibbo (but never published) and nearly simultaneously and independently by Derrida, who carefully studied it. The system may stay in 2^N states and each state k has an energy E_k . The energies are i.i.d. random variables with variance $N^{1/2}$.

Let us call E^* the energy for which $2^N = \exp(1/NE^*)$

The computation of the probability distribution of the ground state energy can be done; for E near E^* we find a Gumbel distribution.

For $e < e^*$ the probability is given by

$$\exp(-NA(e^* - e)) ,$$

while for $e < e^* \equiv E^*/N$ the probability is. of order

$$\exp(-\exp(NA(e - e^*))) .$$

The very large deviation region is quite peculiar. The probability for deviations upward is very very small as can be seen from the Gumbel law.

The $p = 2$ Spherical model

Here the Hamiltonian is given by

$$H = - \sum_{i,k=1,N} J_{i,k} \sigma_i \sigma_k \ ,$$

where the J are Gaussian random independent variables with variance N^{-1} and the N variable σ satisfy the constraint

$$\sum_{i=1,N} \sigma_i^2 = N \ .$$

The total ground state energy is the largest normalised eigenvalue of the random $N \times N$ matrix and its probability distribution is well know.

An explicit rigorous computation (Tracy and Widom) shows that

$$P_N(e) \propto \exp(-AN|e+1|^{3/2}) \qquad e < -1$$

$$P_N(e) \propto \exp(-BN^2|e+1|^3) \qquad e > -1$$

Each time we move an eigenvalue of a quantity of order 1, the probability decreases by a factor $\exp(-CN)$.

In order to lower the ground state we have to move one eigenvalue, in order to increase the ground state we have to move an number of eigenstates that is proportional to N, hence the N^2 factor.

The eigenvalue density is positive for $e > -1$, so that in the most likely case the total number of eigenvalues less the e is proportional to N.

The SK model

Here the Hamiltonian is given by

$$H = - \sum_{i,k=1,N} J_{i,k} \sigma_i \sigma_k$$

where the J are random independent variables (as before with variance N^{-1} and the N variable σ are equal to ± 1.

Here no explicit computation can be done and we have to use the replica method.

For $n > 0$ the appropriate computation has been proposed by Kondor and the relevant formula has been proved rigorously to be valid by Talagrand: it is a (simpler) variation of the computation of the ground state.

One formally arrives to the formula

$$f(n) = \min F_n(Q),$$

where Q is a matrix in replica space.

This expression can be translated in the standard world and we get
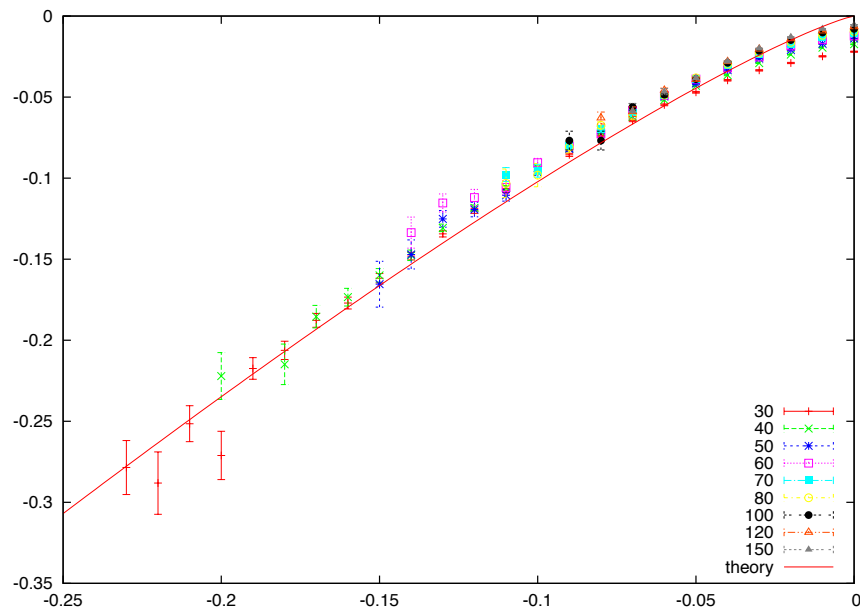
$$f(n) = \max F_n[q],$$

where $q(x)$ is a function in the interval $[0 - 1]$.

With Tommaso Rizzo we have done the appropriate computations. In the zero temperature limit for n positive and we find

$$f(n) = a + bn^5 + O(n^7)$$

In this way we can compute the probability for all negative value of $\Delta e = e - e^*$ and in the region of small Δe we find

$$P_N(\Delta e) \propto \exp(-1.61N|\Delta e|^{6/5})$$

Very large deviations: a tentative analysis

The function $f(n, N)$ is computed as

$$f(n, N) = \min_q F[q]$$

where $F[q]$ is a functional that depends on the a function $q(x)$, defined in the interval $n - 1$.

This formula is obtained using a saddle point method in computing n integrals and neglecting the corrections to the saddle point method.

The result obtained for $n < 0$ is $f(n, N) = f(0, N)$ as in the REM.

When $n = -\alpha N$ fluctuation cannot be neglected. The results are compatible with

$$f(n, N) = f(0, N) + N A \alpha^{8/5} .$$

In the same way we have

$$P_N(\Delta e) \propto \exp(-C N^2 |\Delta e|^{8/3}) \qquad C \approx 0.64 .$$

Large and Very large deviations together:

$$P_N(\Delta e) \propto \exp(-G(\delta) \quad \delta = N^{1/\omega}(E - E_0) \quad \omega = \frac{6}{5}$$

$$G(\delta) \approx G_-(-\delta)^\omega \quad \text{for} \quad \delta \to -\infty$$

$$G(\delta) \approx G_+(-\delta)^{2\omega} \quad \text{for} \quad \delta \to +\infty$$

One scaling functions for large and very large deviations.

This behaviour is an agreement with what happens in the spherical model.

Bethe Lattices: Large deviations

General speaking, the topology of the lattice and the couplings at fixed topology are both fluctuating one can distinguish the effects of the former from the effect of the latter.

There are two cases and in the large deviation region one can make the following conjecture for which there is a partial evidence:

- Locally homogenous problems: all points are indistinguishble with a local analysis, the length of loops of to infinity with N. (e.g. spin glasses on a random regular graph, i.e a fixed coordination lattice with $J = \pm 1$).

  They have a similar behaviour to infinite range models in the large deviation regions.

- Locally inhomogeneous problems. It is a simpler, more or less trivial problem:

$$P_N(\Delta e) = \exp(-AN(\Delta e)^2 + \ldots) .$$

Bethe Lattices: Very large deviations

The very large deviation region for locally homogeneous is more complex.

For example in a spin glass with fixed coordination number z, the total number of topologies is not higher that

$$((z-2)N/2)! \approx \exp(z-2)N \ln(N)$$

and the number of different coupling is $2^{(Nz/2)}$.

There is no room for a probability that is of order $\exp(-AN^2)$.

On the other ends highly frustrated regular lattice have an higher ground state energy. There must by a large deviation function

The number of lattices with M closed loops of length 3 should behave as

$$\exp(-H(\alpha)N\log(N)) \qquad \alpha = \frac{M}{N}$$

We can convincingly argue that in the very large deviation region

$$P_N(\Delta e) \propto \exp(\mathcal{L}(\Delta e)N\log(N))$$

but we have no idea on how to compute $\mathcal{L}(\Delta e)$.

In the case of spin glasses (and similar models) large deviations are well understood

For certain class of models we have a very large deviations regime for positive deviations of the free energy

In the SK model some computations are possible

$$P_N(\Delta e) \propto \exp(\mathcal{L}(\Delta e)N^2)\,.$$

In the Random Regular graph, no computations have been done. We expect that

$$P_N(\Delta e) \propto \exp(\mathcal{L}(\Delta e)N\log(N))\,.$$

This is a completely unstudied problem that is related to the problem of very large deviations in the topology of random regular lattice, a problem that has not been studied, as far as I know.

Some references

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