# Fronts and random matrix spectra in the XX chain


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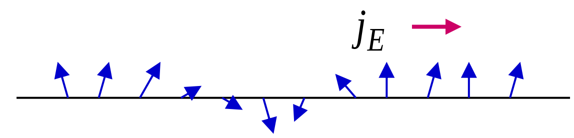
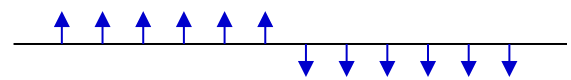
An excursion from nonequilibrium steady states
to extreme order statistics.




## Questions:

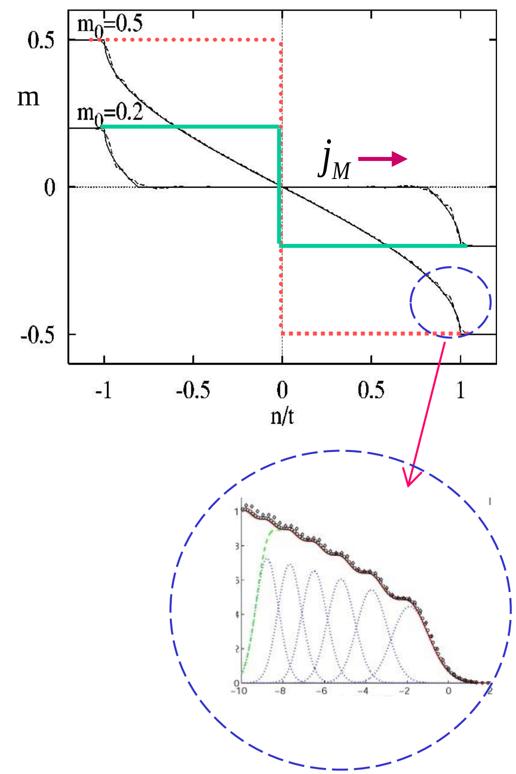


How to construct nonequilibrium steady states
for quantum systems?

General and distinct features of the steady states?

How do quantum systems relax to the steady state?

How to understand quantum fronts?



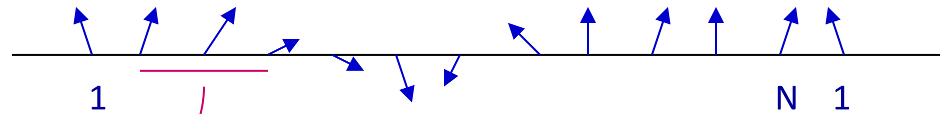

## Latest:

The front can be described in terms of the statistics
of the edge spectrum of random matrices (GUE).

# Transverse Ising model with energy flux

PRL **78**, 167 (1997)
T. Antal, L. Sasvari, Z. R.

$$\hat{H}_I = -\sum_{n=1}^{N} S_n^x S_{n+1}^x - \frac{h}{2}\sum_{n=1}^{N} S_n^z$$




local energy

$$\varepsilon_n = -\tfrac{1}{2} S_n^x (S_{n-1}^x + S_{n+1}^x) - h S_n^z$$

Find the ground state of

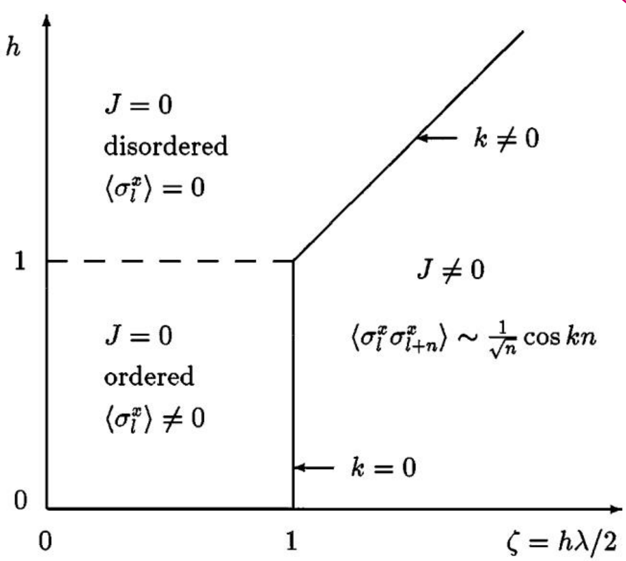
$$\hat{H} = \hat{H}_I + \lambda \hat{J}_E$$

$$\dot{\varepsilon}_n = i[\hat{H}, \varepsilon_n] = \underline{j_{E,n-1\to n}} - j_{E,n\to n+1}$$

local energy flux

Energy flux

$$\hat{J}_E = \frac{h}{4}\sum_{n=1}^{N} (S_n^x S_{n+1}^y - S_n^y S_{n+1}^x)$$

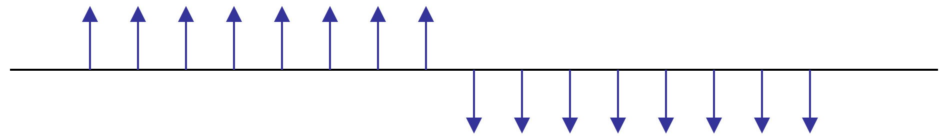


# Evolution from natural initial states

PRE **59**, 4912 (1999)
T. Antal, A. Rakos, Z.R., G.M. Schütz

$$\hat{H}_{xx} = -\sum_{n=1}^{N-1} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y)$$

initial state



**Questions:** Are there steady states in the $t \to \infty$ ?

Can they be described by $H_\lambda$ ?

$$\hat{H}_\lambda = \hat{H}_{xx} - \lambda\, j_M$$

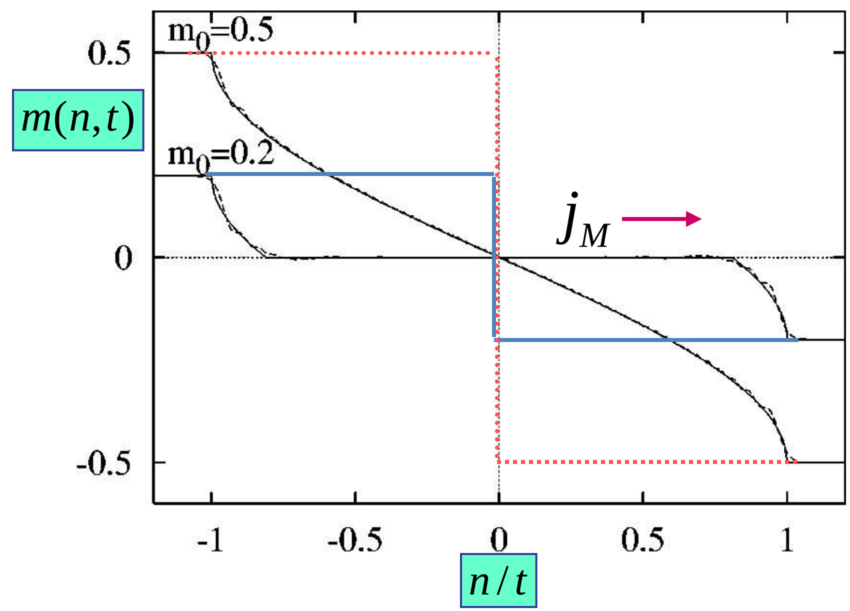
Time evolution (exact):

$$m_0 = 1/2$$

$$\langle S_n^z(t) \rangle = m(n,t) = -\tfrac{1}{2} \sum_{\ell=1-n}^{\ell=1+n} \mathrm{J}_\ell^2(t)$$

Scaling limit $\quad t \to \infty \quad \mathrm{n/t} = v$

$$m(n,t) \to -\tfrac{1}{\pi} \arcsin(\tfrac{n}{t})$$




- $J_M$ states are OK.

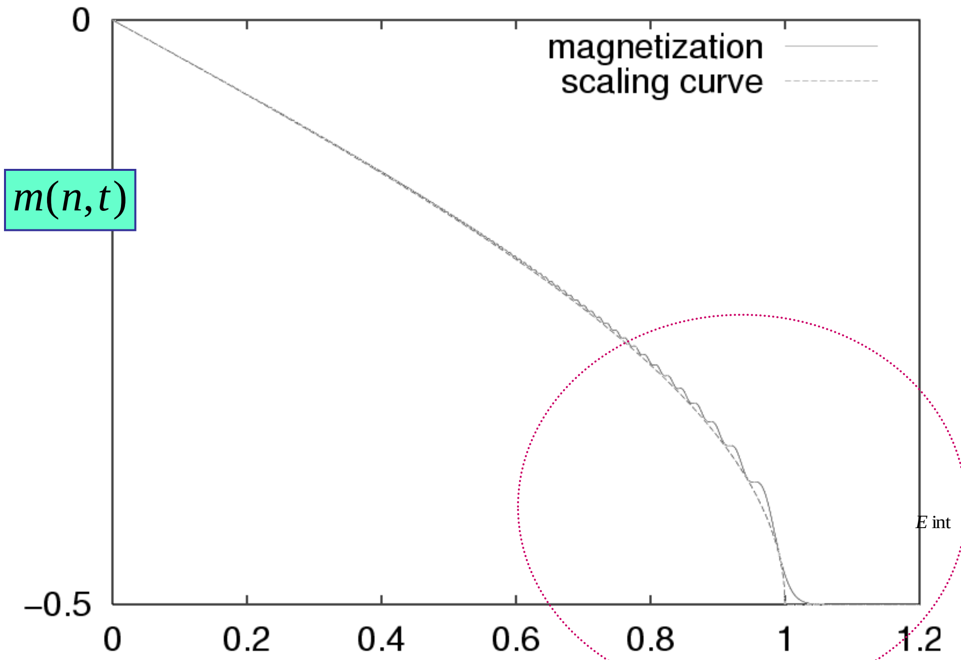
Problems in case of energy flux:

Y.Ogata,
PRE **66**, 016135 (2002)

# Scaling structure of the front

PRE **69** 066103 (2004)
V. Hunyadi, L. Sasvari, Z.R.

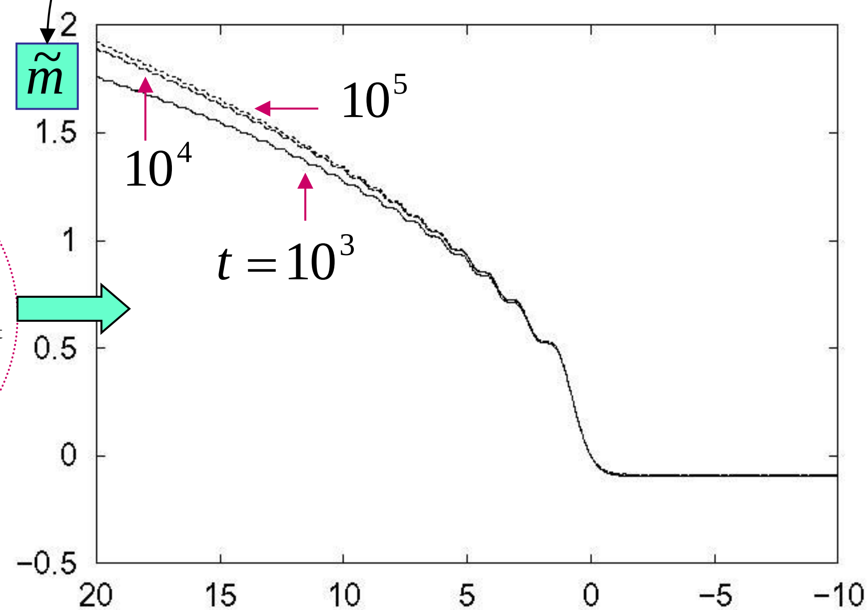
$m(n,t)$




$n/t$

$$\tilde{m} = t^{1/3}[m(n,t) - m(t,t)]$$

$\tilde{m}$




$$z = (t-n)/t^{1/3}$$

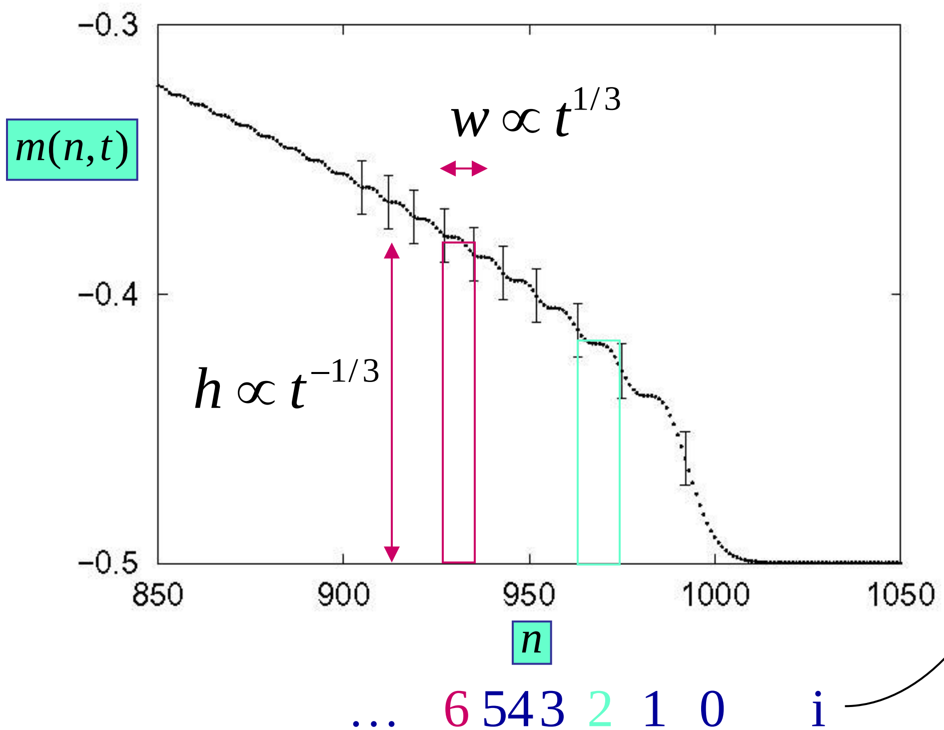
Scaling in the front:

$$m(n,t) - m(t,t) = \frac{1}{t^{1/3}}\Phi\left(\frac{t-n}{t^{1/3}}\right)$$

integral of the Airy function

- Front shape is formed early
  Width of the front $\propto t^{1/3}$
  Height of the front $\propto 1/t^{1/3}$

# Quasi-classical picture of quantum transport?



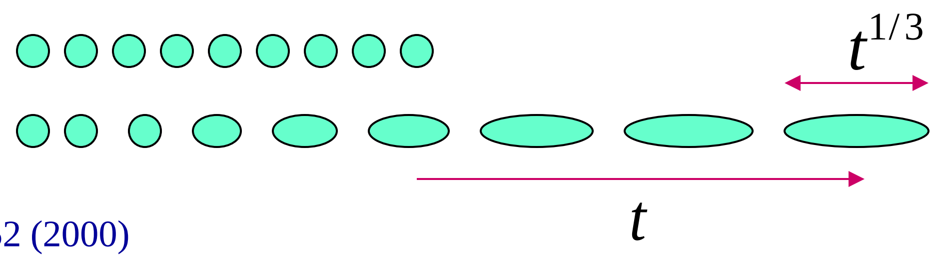

Area under the steps is constant

$$\boxed{hw \cong 1}$$

| i | $t = 10^3$ | $10^5$ |
|---|---|---|
| 1 | 1.018 | 1.038 |
| 2 | 0.971 | 1.009 |
| 3 | 1.037 | 0.993 |
| . | . | . |

- The steps carry one $\uparrow$ spin
- Quasi-classical picture:

Subdiffusive spreading

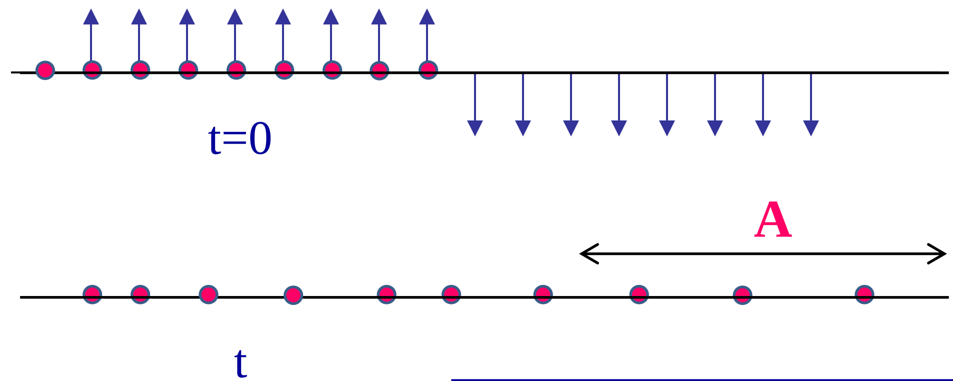



V. Popkov and M, Salerno, PRB 62, 352 (2000)

# Fermionic description: Full counting statistics

PRL **110**, 060602 (2013)
V. Eisler, Z.R.

$$\hat{H} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}(c_n^+ c_{n+1} + c_n c_{n+1}^+)$$




$$\hat{N}_A(t) = \sum_{n\in A} c_n^+(t) c_n(t)$$

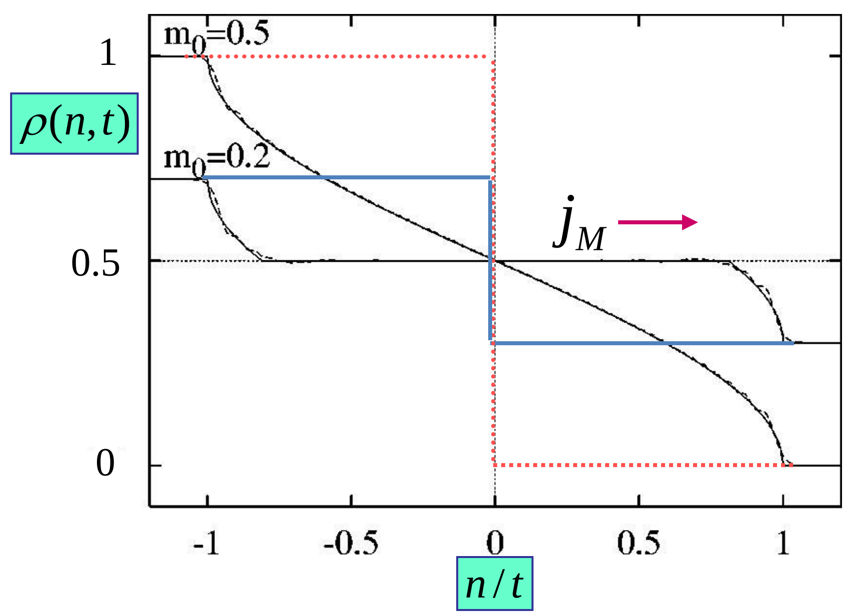
**Questions:**

Number of particles in **A**: $\langle \hat{N}_A \rangle$ ?

Fluctuations in **A**: $\langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2$ ?

$$c_n(t) = \sum_{j=-\infty}^{\infty} i^{j-n} J_{j-n}(t) c_j(0)$$

$$\rho(n,t) = \langle c_n^+(t) c_n(t) \rangle$$

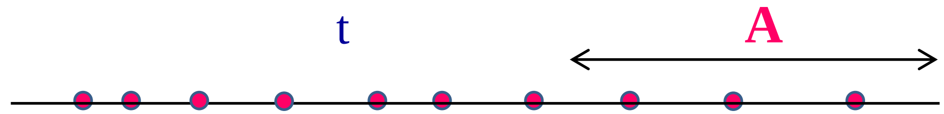



Generating function for
full counting statistics:

$$\chi(\lambda,t) = \left\langle \exp[i\lambda \hat{N}_A(t)] \right\rangle$$

# Generating function for full counting statistics (FCS)

$$\chi(\lambda,t) = \left\langle \exp[i\lambda \hat{\mathrm{N}}_{\mathrm{A}}(t)] \right\rangle$$




$$\hat{\mathrm{N}}_{\mathrm{A}}(t) = \sum_{n\in \mathrm{A}} c_n^{+}(t) c_n(t)$$

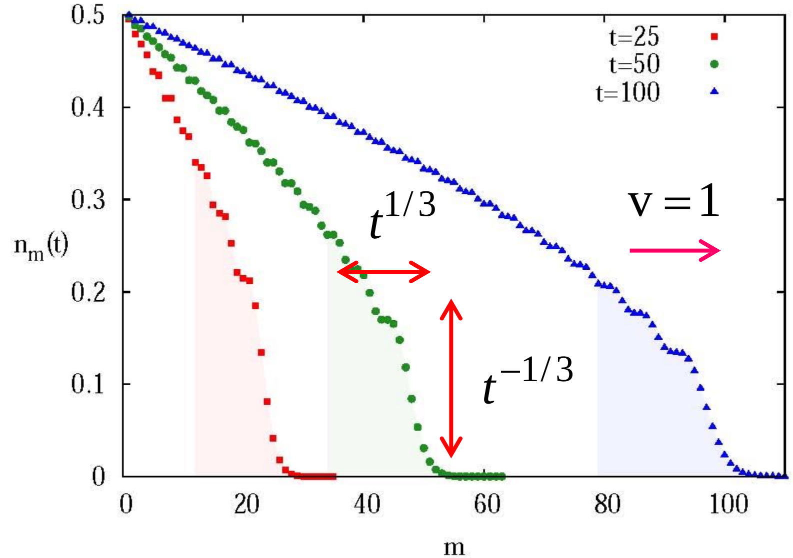
Expression in terms of a determinant

I. Klich in Quantum Noise in Mesoscopic Physics (2003)

K. Schönhammer, PRB **75,** 205329 (2007)

$$\boxed{\chi(\lambda,t) = \det[\,\hat{\mathrm{I}} + (e^{i\lambda} - 1)\hat{\mathrm{C}}\,]}$$

$$\hat{C}_{mn}(t) = \langle c_m(t) c_n(t) \rangle = -\frac{i^{n-m} t}{2(m-n)} \left[ J_{m-1}(t) J_n(t) - J_m(t) J_{n-1}(t) \right]$$




To get the scaling regime in the front:

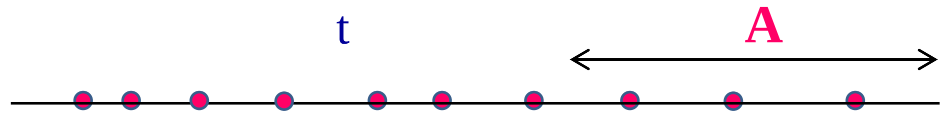
$$m = t + \left(\frac{t}{2}\right)^{1/3} x \qquad n = t + \left(\frac{t}{2}\right)^{1/3} y$$

$$\mathrm{J}_{m-1}(t) = \dot{\mathrm{J}}_{\mathrm{m}}(t) + \frac{m}{t} \mathrm{J}_m(t)$$

$$\mathrm{J}_{\mathrm{m}}(t) = -\left(\frac{2}{t}\right)^{1/3} \mathrm{Ai}(x) \qquad \dot{\mathrm{J}}_{\mathrm{m}}(t) = -\left(\frac{2}{t}\right)^{1/3} \mathrm{Ai}'(x)$$

# Full counting statistics in the front

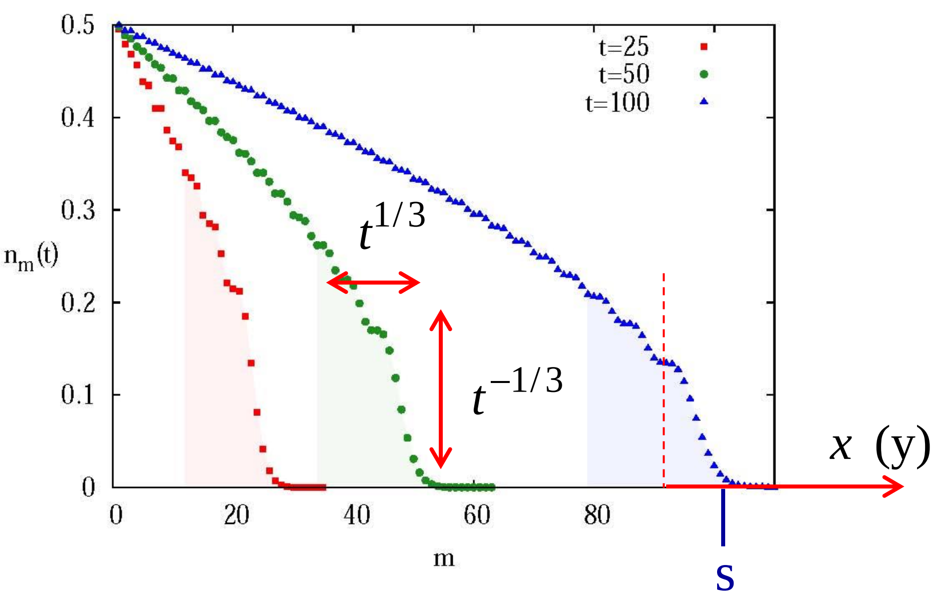
$$\chi(\lambda,t)=\left\langle \exp[i\lambda\,\hat{\mathrm{N}}_{\mathrm{A}}\,(t)]\right\rangle$$




$$\boxed{\hat{C}_{mn}(t)=\frac{2^{1/3}}{t^{1/3}}K(x,y)+\mathrm{O}\left(\frac{1}{t^{2/3}}\right)}$$

Airy kernel

$$\boxed{K(x,y)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}}$$



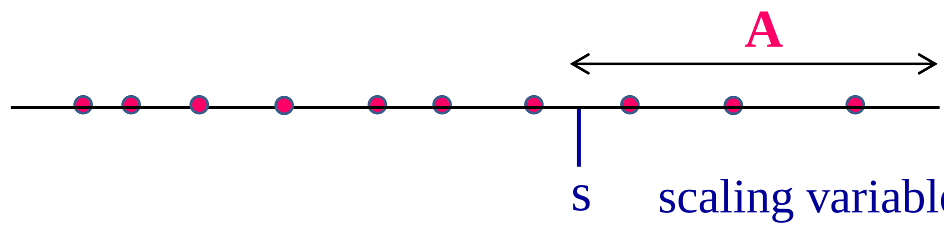

$$\boxed{\chi(\lambda,s)=\det[\,\mathrm{I}+(e^{i\lambda}-1)\mathrm{K}\,]}$$

**A** $\longrightarrow$ **A**(s,∞)

$$m=t+\left(\frac{t}{2}\right)^{1/3}x \quad \text{(y)}$$

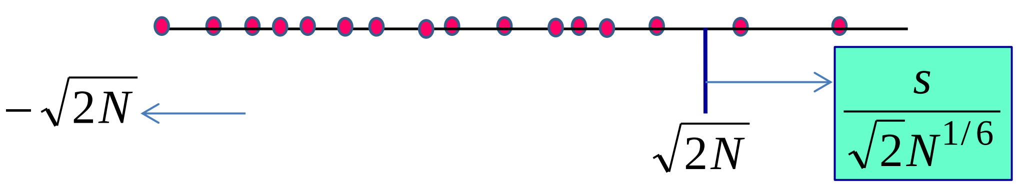
# Full counting statistics and random matrices

$$\chi(\lambda, s) = \det[\,\mathrm{I} + (e^{i\lambda} - 1)\mathrm{K}\,]$$




scaling variable for FCS

Comparison with $N$ x $N$ random matrices from GUE:




Probability that exactly $\boxed{n}$ eigenvalues lie in the interval $\boxed{(s,\infty)}$:

$$E(n,s) = \frac{(-1)^n}{n!}\frac{d^n}{dz^n}\det(1 - zK)\Big|_{z=1}$$

$$\tilde{\chi}(\lambda, s) = \sum_n e^{i\lambda n} E(n,s) =$$

$$\sum_n e^{i\lambda n}\frac{(-1)^n}{n!}\frac{d^n}{dz^n}\det(1 - zK)\Big|_{z=1} = \sum_n \exp\left(-e^{i\lambda}\frac{d}{dz}\right)\det(1 - zK)\Big|_{z=1} = \det[\mathrm{I} + (e^{i\lambda} - 1)K]$$

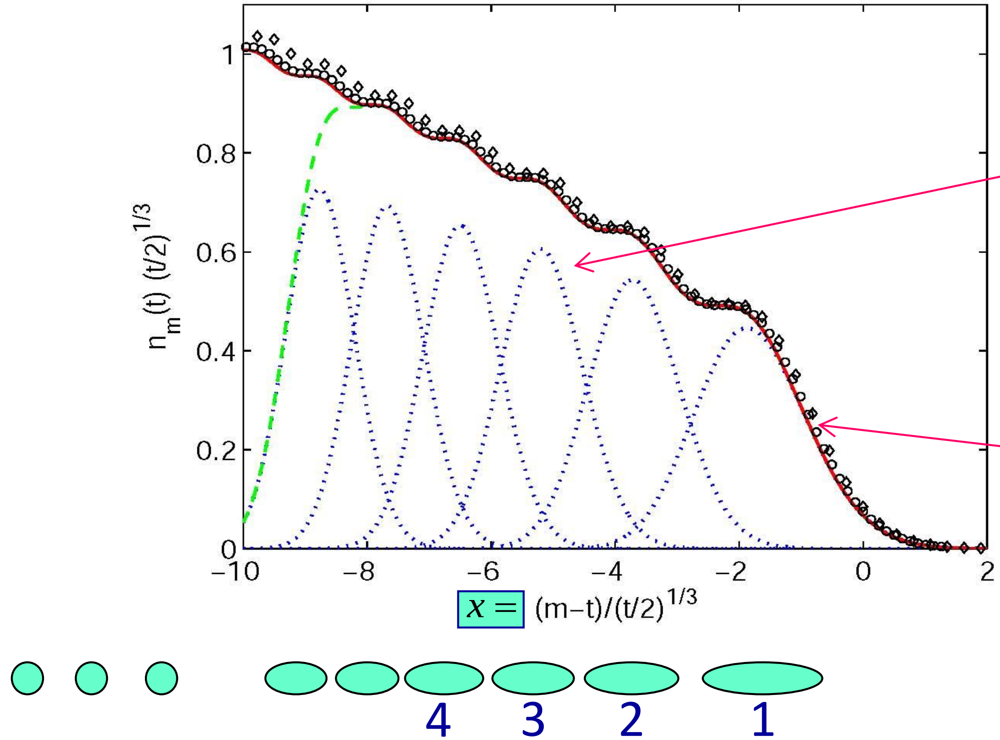
$$= \chi(\lambda, s)$$

# Using the random matrix results

Probability density of the $\boxed{n}$-th largest eigenvalue: $$F(n,x)=\sum_{k=0}^{n-1}\frac{dE(k,x)}{dx}$$

Density of eigenvalues = sum of single eigenvalue densities:

$$\boxed{\rho(x)}=-\sum_{k=0}^{\infty}k\frac{dE(k,x)}{dx}=\boxed{\sum_{k=0}^{\infty}F(k,x)}=K(x,x)=[\mathrm{Ai}'(x)]^2-x\mathrm{Ai}^2(\mathrm{x})$$




Extreme order statistics:

$F(3,x)$

Probability distribution of the 3rd particle

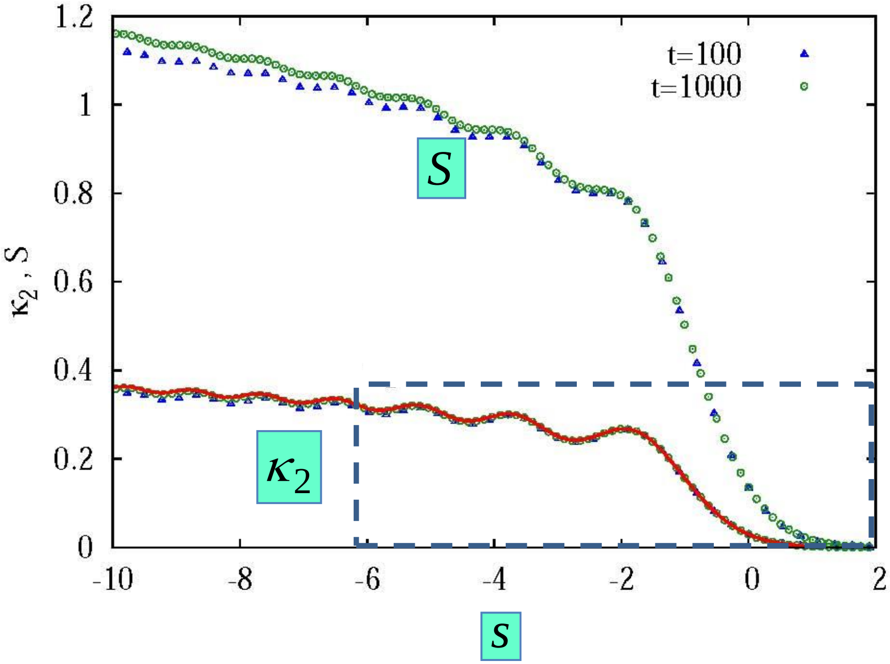
$\rho(x)$

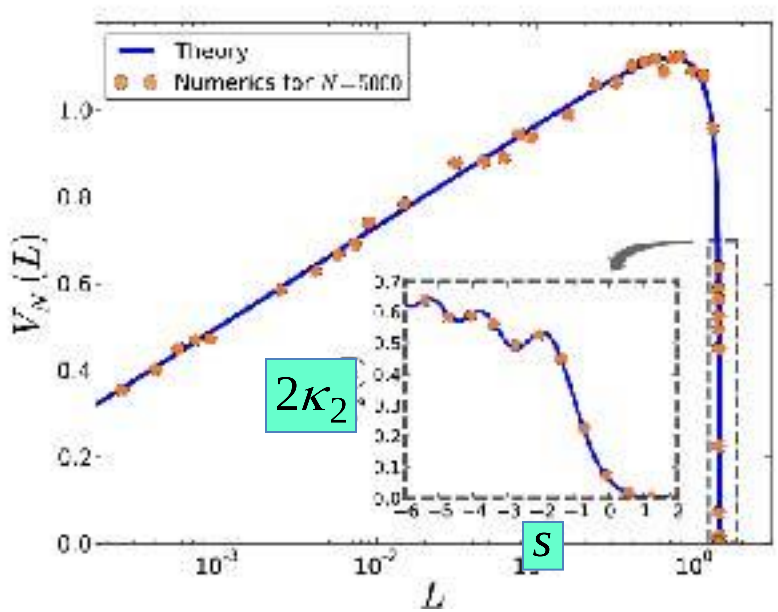
Total particle density

# Using the random matrix results

Fluctuations in particle number: $\kappa_2 = \langle N_A^2 \rangle - \langle N_A \rangle^2 = \mathrm{Tr}K(1-K)$

Entanglement between $(-\infty, s)$ and $(s, \infty)$: $S = \ldots = \mathrm{Tr}[K \ln K - (1-K)\ln(1-K)]$



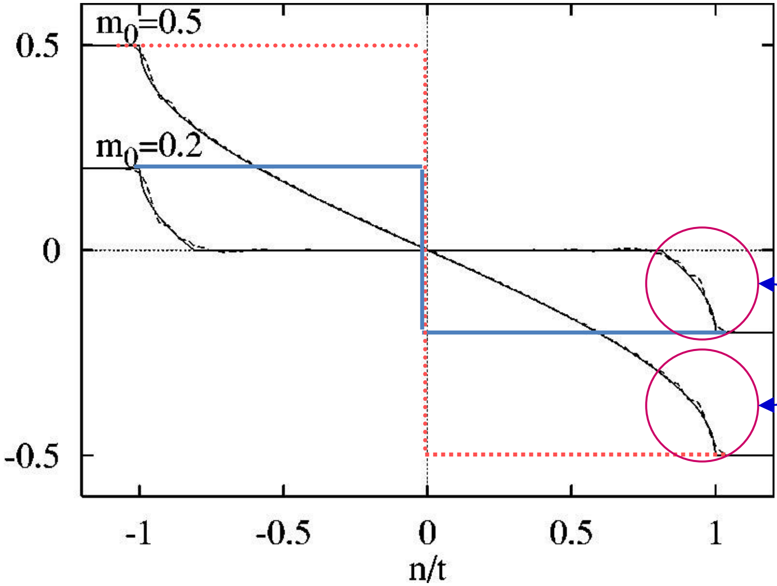

R. Marino, S. N. Majumdar, G. Schehr, and P. Vivo, arXiv: 1404.0575

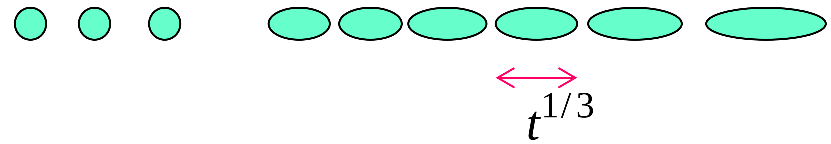
$P_{N,L}(N_L)$

Probability that a $N \times N$ Gaussian random matrix has $N_L$ eigenvalues in $[-L, L]$.

Gap scaling (A. Perret and G. Schehr, arXiv: 1312.2966).

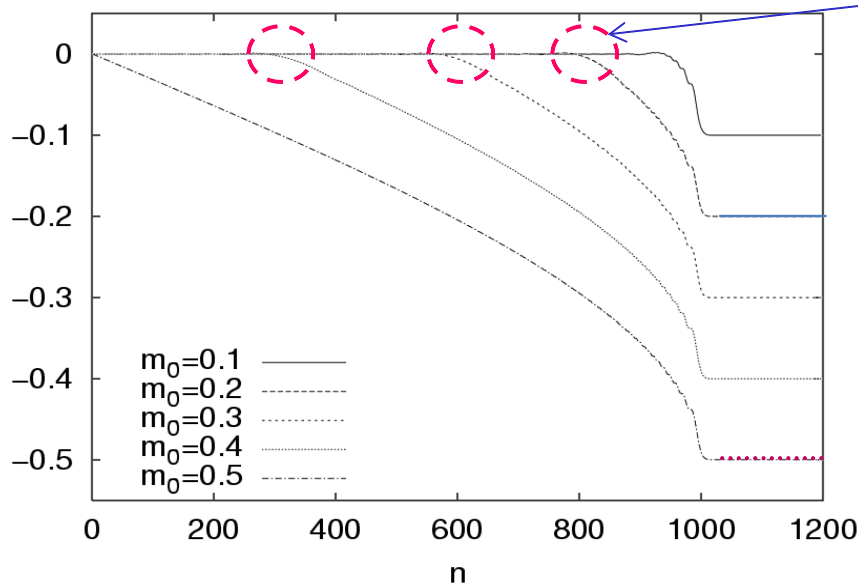
# Applications, problems, conclusions




subdiffusive spreading

Practically identical profiles,
compare with other spin chains.




Phase transition between the edge
and bulk eigenvalue spectrum?

$m_0$ determines the number of
steps arriving at time t.

Control over moving spins?