Spatial extent of an outbreak in animal epidemics

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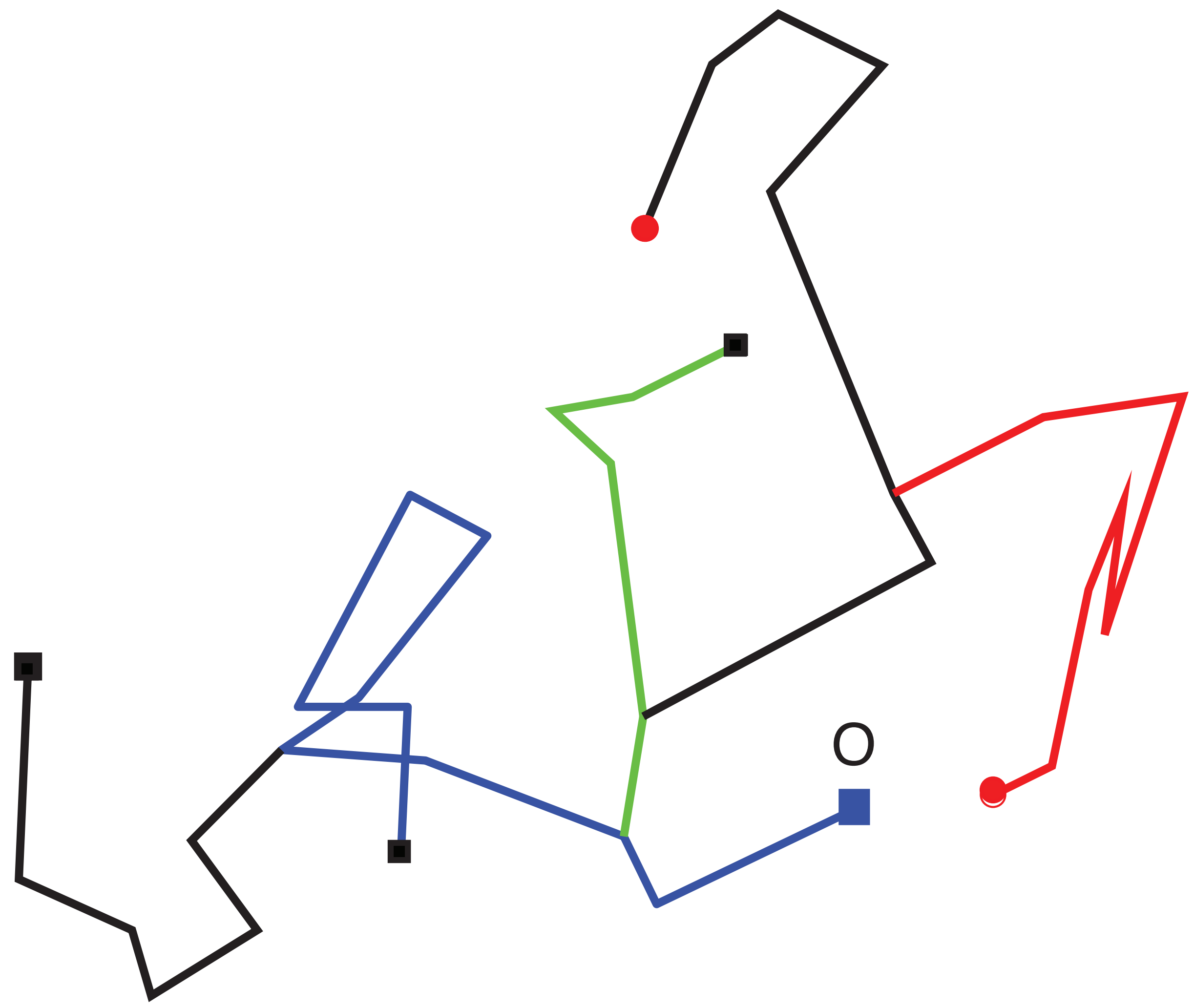

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SIR model for epidemics

Three species : susceptibles (S), infected (I), recovered (R)

$$\frac{dS}{dt} = -\beta\, I\, S$$
$$\frac{dI}{dt} = \beta\, I\, S - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

- mean field fully connected model
- β rate of infection transmission
- γ rate at which an infected recovers

$$\boxed{I(t) + S(t) + R(t) = N}$$

N being the total population

Outbreak of an epidemic

Initial condition : $I(0) = 1,\ S(0) = N - 1 \approx N,\ R(0) = 0$

Outbreak regime

$$\frac{dS}{dt} = -\beta\, I\, S$$
$$\frac{dI}{dt} = \beta\, I\, S - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

$t \approx 0,\ S \approx N$

$$\frac{dI}{dt} \simeq (\beta\, N - \gamma)\, I$$

Reproduction rate: $\boxed{R_0 = \frac{\beta N}{\gamma}}$

Deterministic and stochastic models

SIR is a deterministic model. In the outbreak fluctuations are important

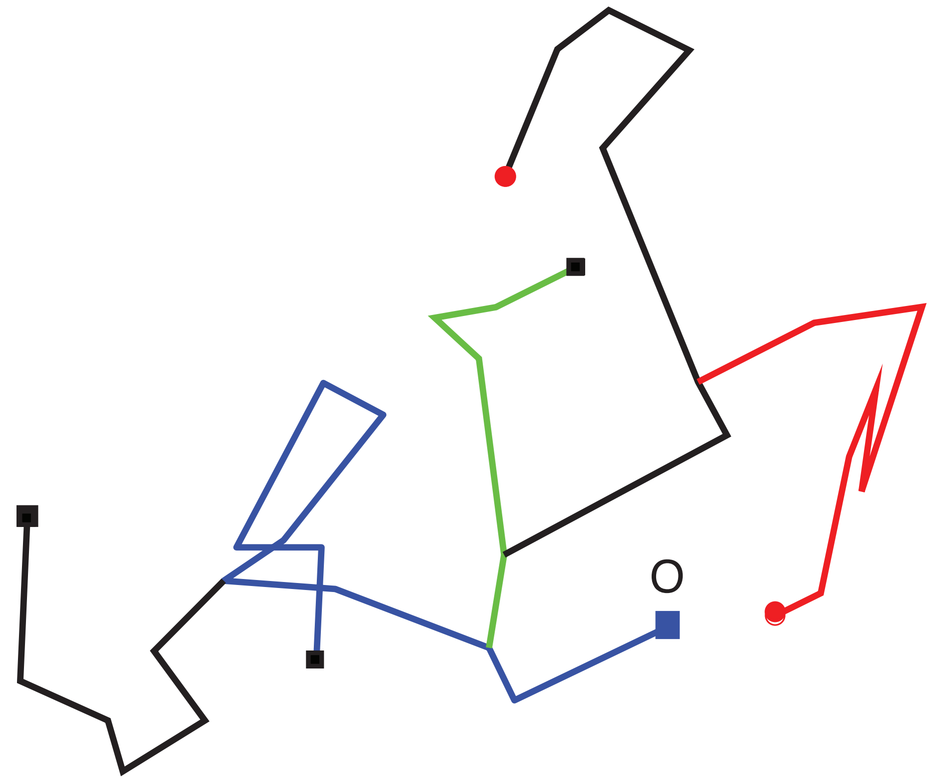
- Stochastic process: Galton-Watson (mean field)
- each infected individual transmits the disease at rate $N\beta$
- each infected individual recovers at rate γ

Reproduction rate: $\boxed{R_0 = \frac{\beta N}{\gamma}}$

- $R_0 < 1$ epidemics extinction
- $R_0 > 1$ epidemics invasion
- $R_0 = 1$ critical case

How far in space can an epidemic spread?

Problem 1: How to model the space?

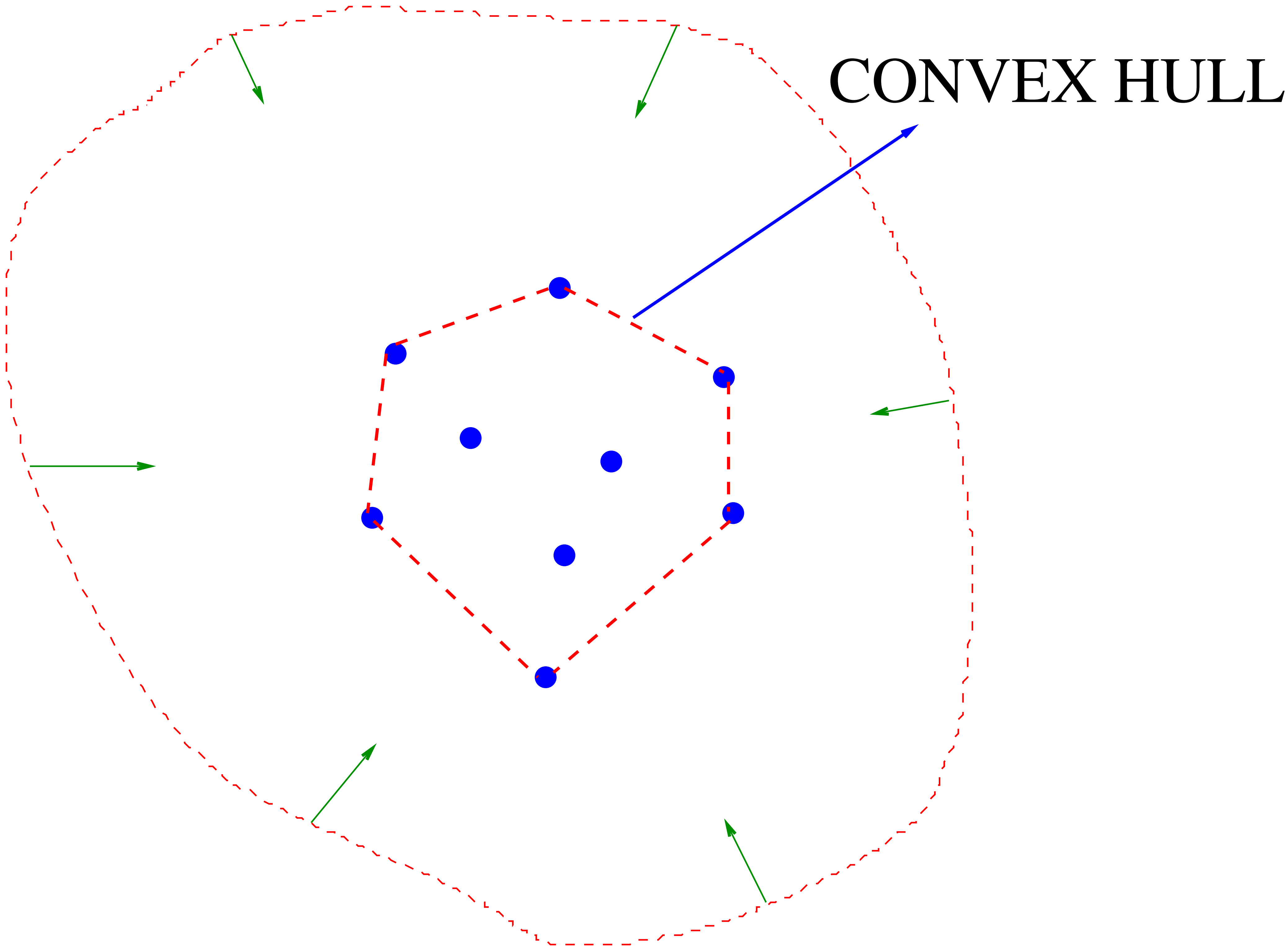


The good candidate: Brownian process with branching and death

In dt, each infected can:

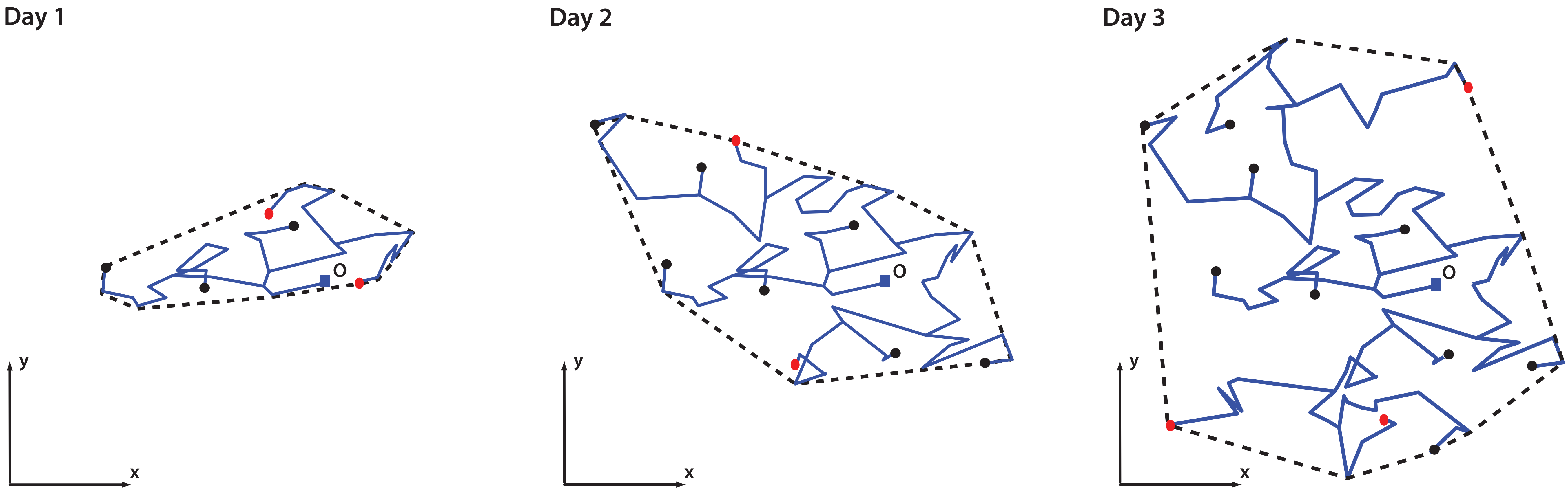
- recovers with probability $\gamma\, dt$
- infects with probability $\beta N\, dt = \gamma R_0 dt$
- otherwiese, it diffuses (D diffusion const.)

Problem 2: How to quantify the area that needs to be quarantined?

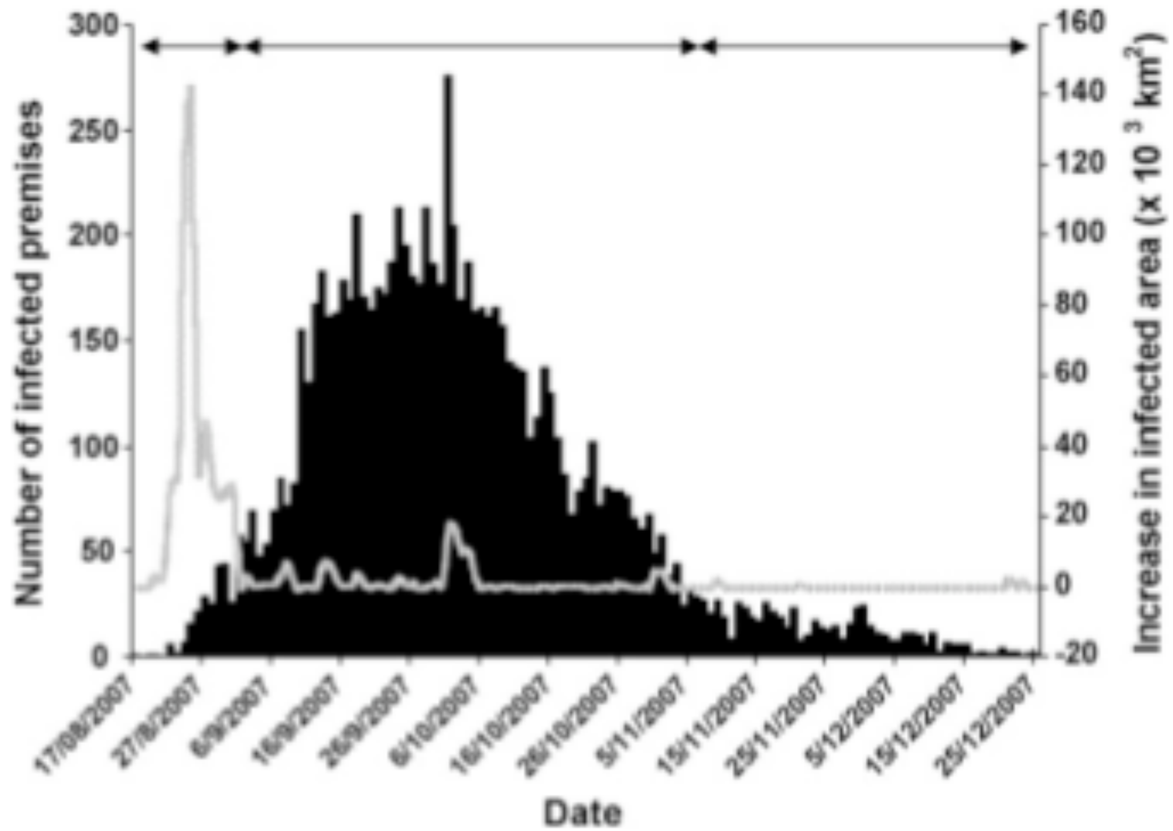


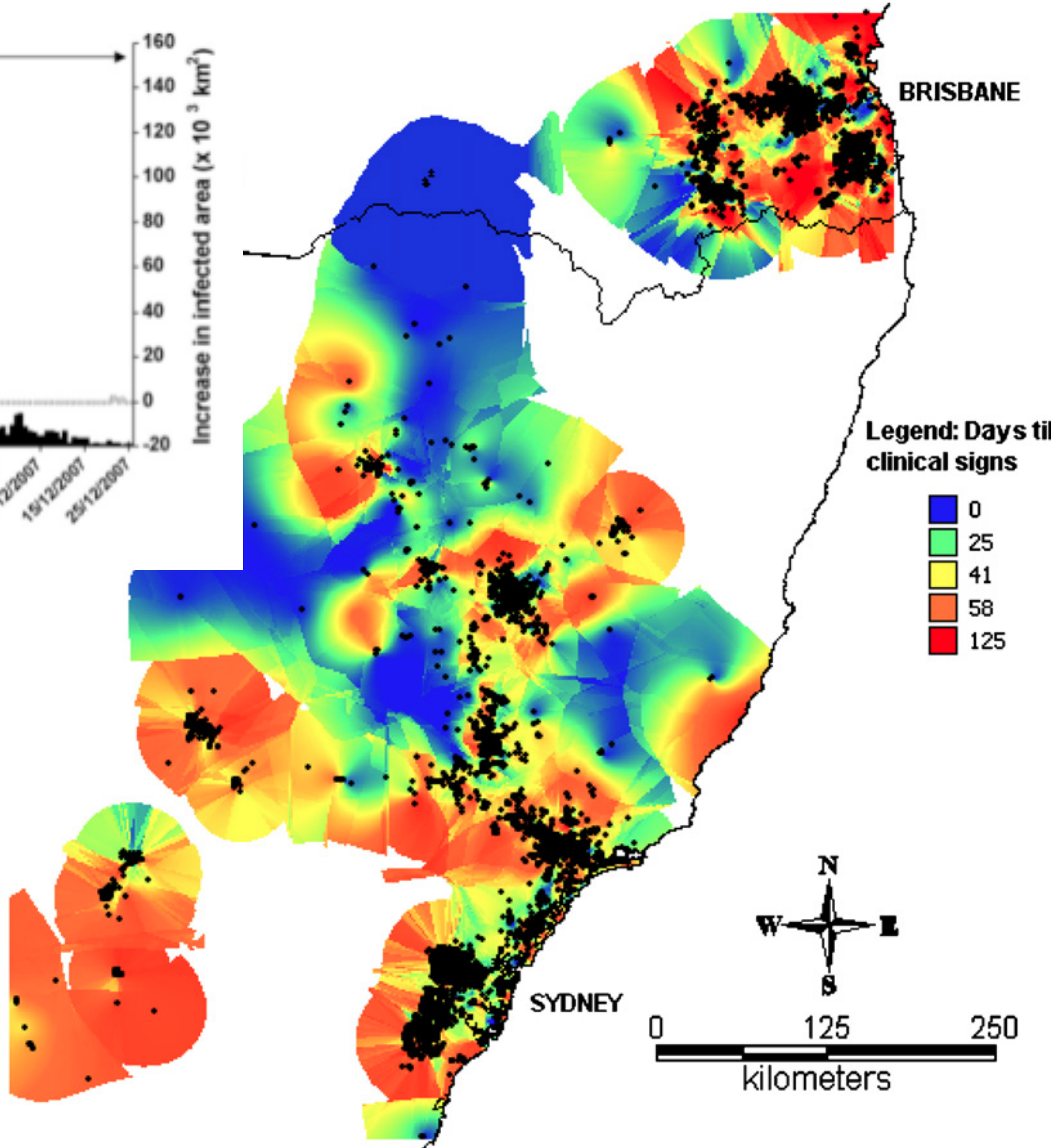
Algorithms: Graham Scan (Nlog(N))

Monitoring the outbreak

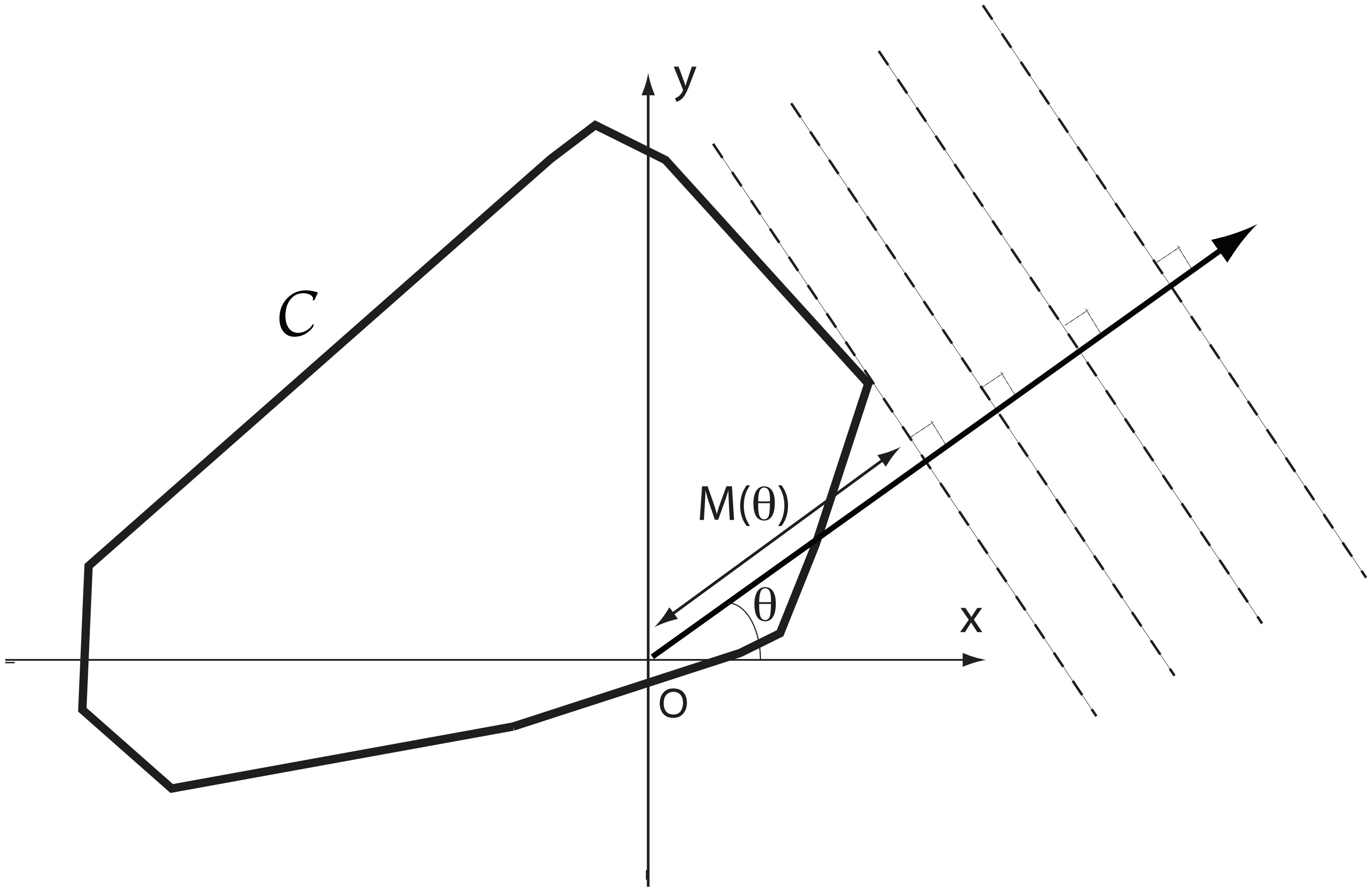


Real applications





How to compute the convex hull of Branching processes?



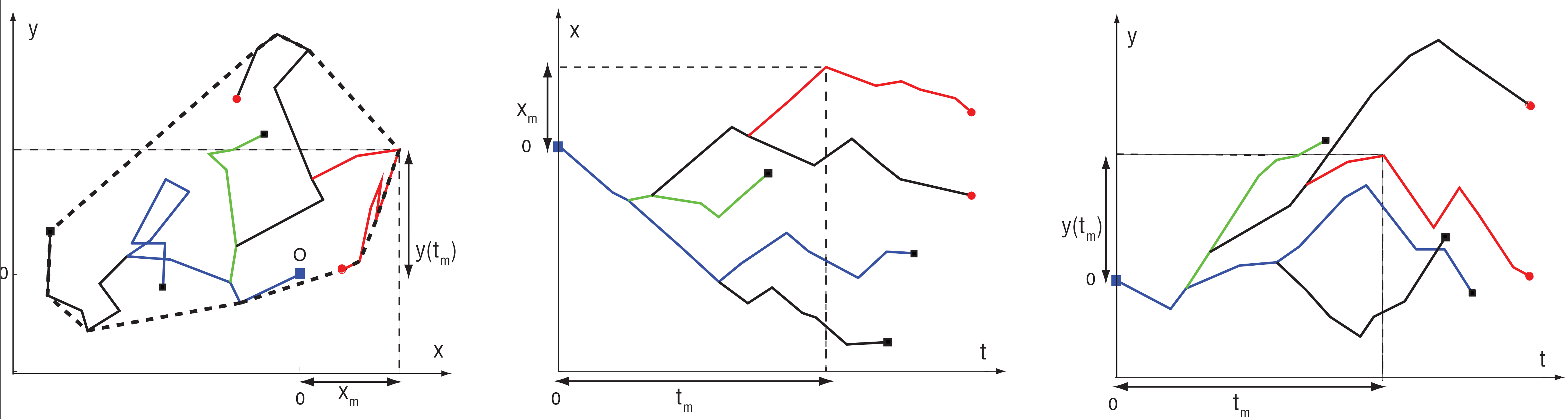
Cauchy formulas

$$L = \int_0^{2\pi} M(\theta)\, d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} \left[M^2(\theta) - (M'(\theta))^2\right] d\theta$$

Support Function

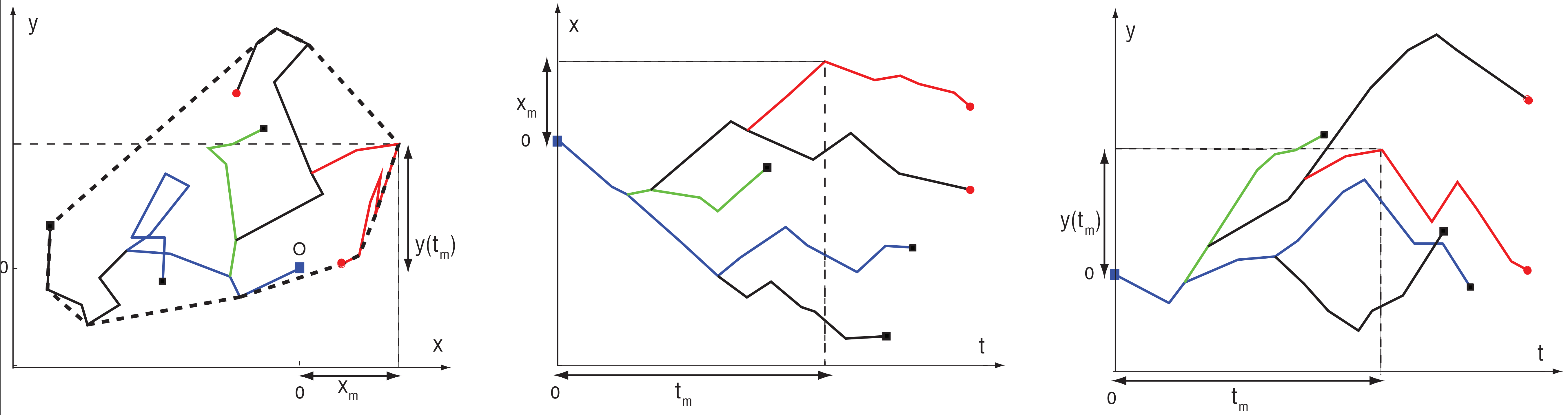
$$M(\theta) = \max_{0 \le \tau \le t} \left[x_\tau \cos\theta + y_\tau \sin\theta \right]$$



$$M(0) = x_{\tau = t_m} = x_m(t)$$

- $x_m(t)$ x-maximum up to time t
- t_m time *location* of the maximum

$$M'(\theta = 0) = -x_{t_m} \sin\theta + y_{t_m} \cos\theta|_{\theta=0} = y_{t_m}$$



$$\langle L(t)\rangle = 2\pi \langle x_m(t)\rangle$$
$$\langle A(t)\rangle = \pi\left[\langle x_m^2(t)\rangle - \langle y^2(t_m)\rangle\right]$$

consider a 1d branching process evolving in $(0, t)$

- x_m is the global maximum
- t_m is the location of the maximum
- $\langle y^2(t_m)\rangle = \ldots = 2D\, t_m$

Backward Fokker Planck equation

$$Q_t(x_m) = \text{Proba}[\text{global max up to } t < x_m]$$

$$Q_{t+dt}(x_m) = \gamma dt + R_0 \gamma dt Q_t^2(x_m) + [1-\gamma(R_0+1)]dt\langle Q_t(x_m - \Delta x)\rangle$$

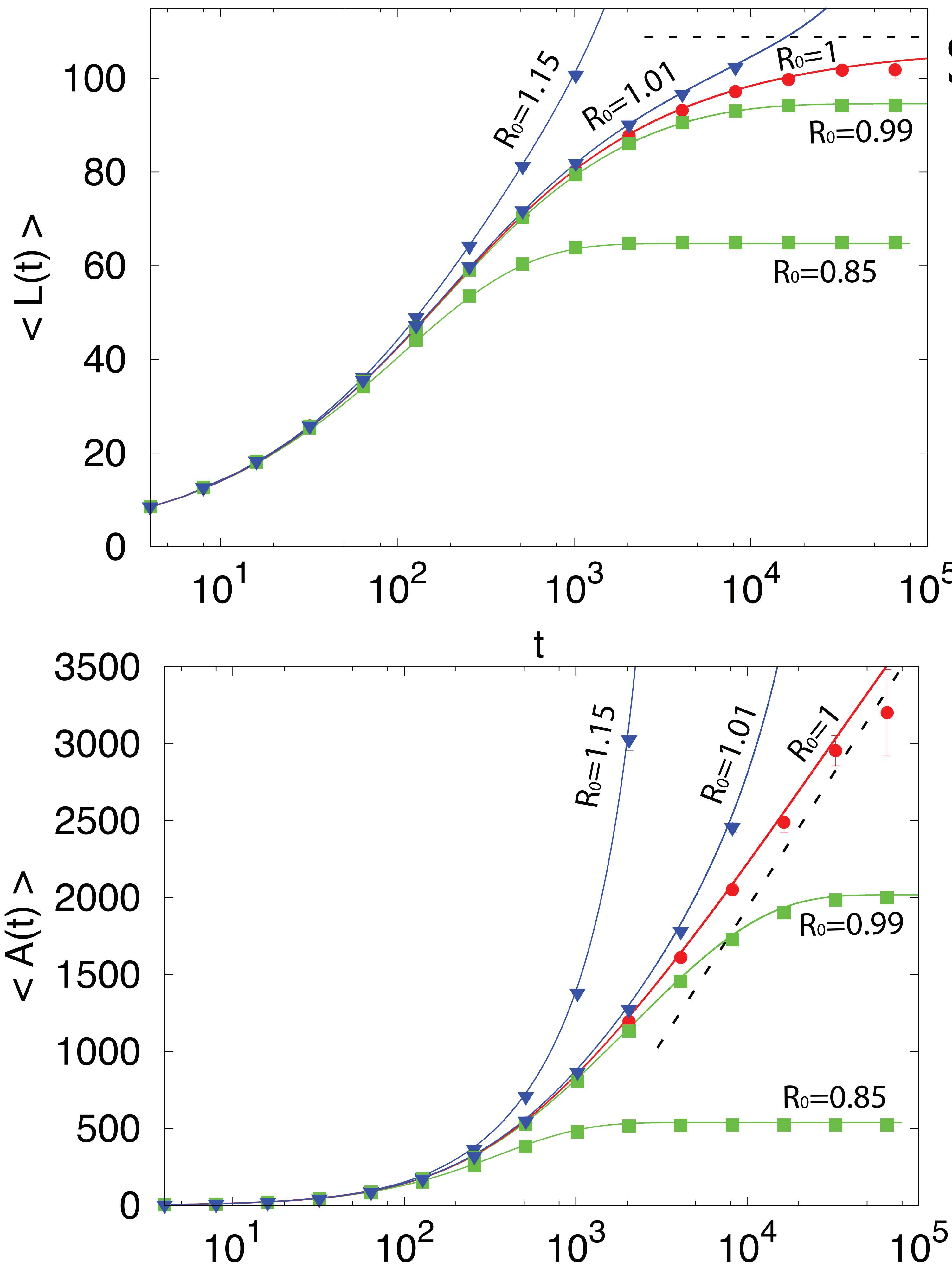
- $\langle Q_t(x_m - \Delta x)\rangle = Q_t(x_m) - \langle \Delta x\rangle Q_t'(x_m) + \langle \frac{\Delta x^2}{2}\rangle Q_t''(x_m) + \ldots$
- $\langle \Delta x\rangle = 0$
- $\langle \Delta x^2\rangle = 2Ddt$

$$\langle Q_t(x_m - \Delta x)\rangle = Q_t(x_m) + Ddt\partial_x^2 Q_t(x_m) + \ldots$$

$$\boxed{\frac{\partial}{\partial t}Q = D\frac{\partial^2}{\partial x_m^2}Q - \gamma(R_0+1)Q + \gamma R_0 Q^2 + \gamma}$$

- initial condition $Q_{t=0}(x_m) = \theta(x_m)$
- boundary condition $Q_t(x_m < 0) = 0$
- boundary condition $Q_t(x_m \to \infty) = 1$

$$\langle L(t)\rangle = 2\pi\int_0^\infty [1 - Q_t(x_m)]dx_m.$$



Solid lines: our predictions

Blue lines: super-critical

Red lines: critical

Green lines: sub-critical

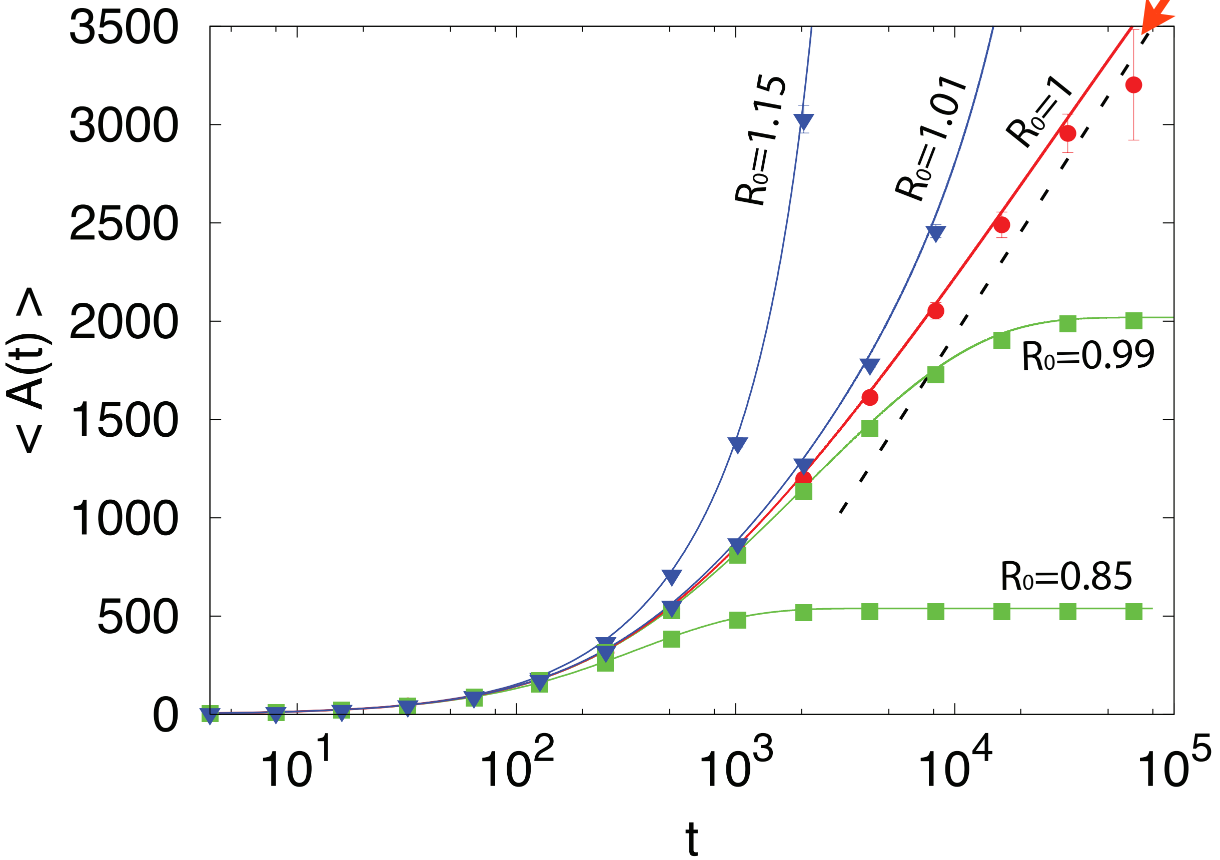
Symbols: Monte Carlo simulations

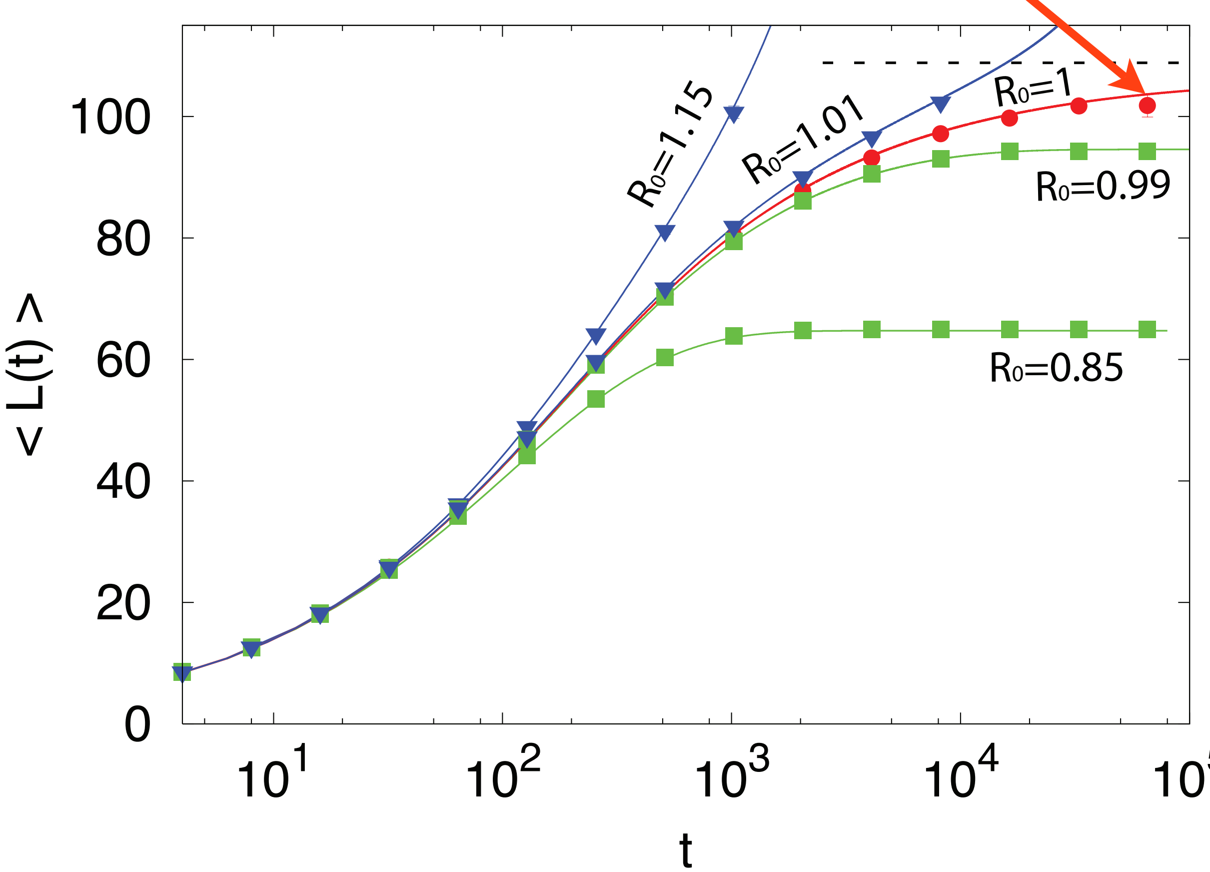
Dashed lines: analytical asymptotic results

The critical case

$$\langle L(t \to \infty) \rangle = 2\pi \sqrt{\frac{6D}{\gamma}} + \mathcal{O}(t^{-1/2})$$

$$\langle A(t \to \infty) \rangle = \frac{24\pi D}{5\gamma} \ln t + \mathcal{O}(1)$$

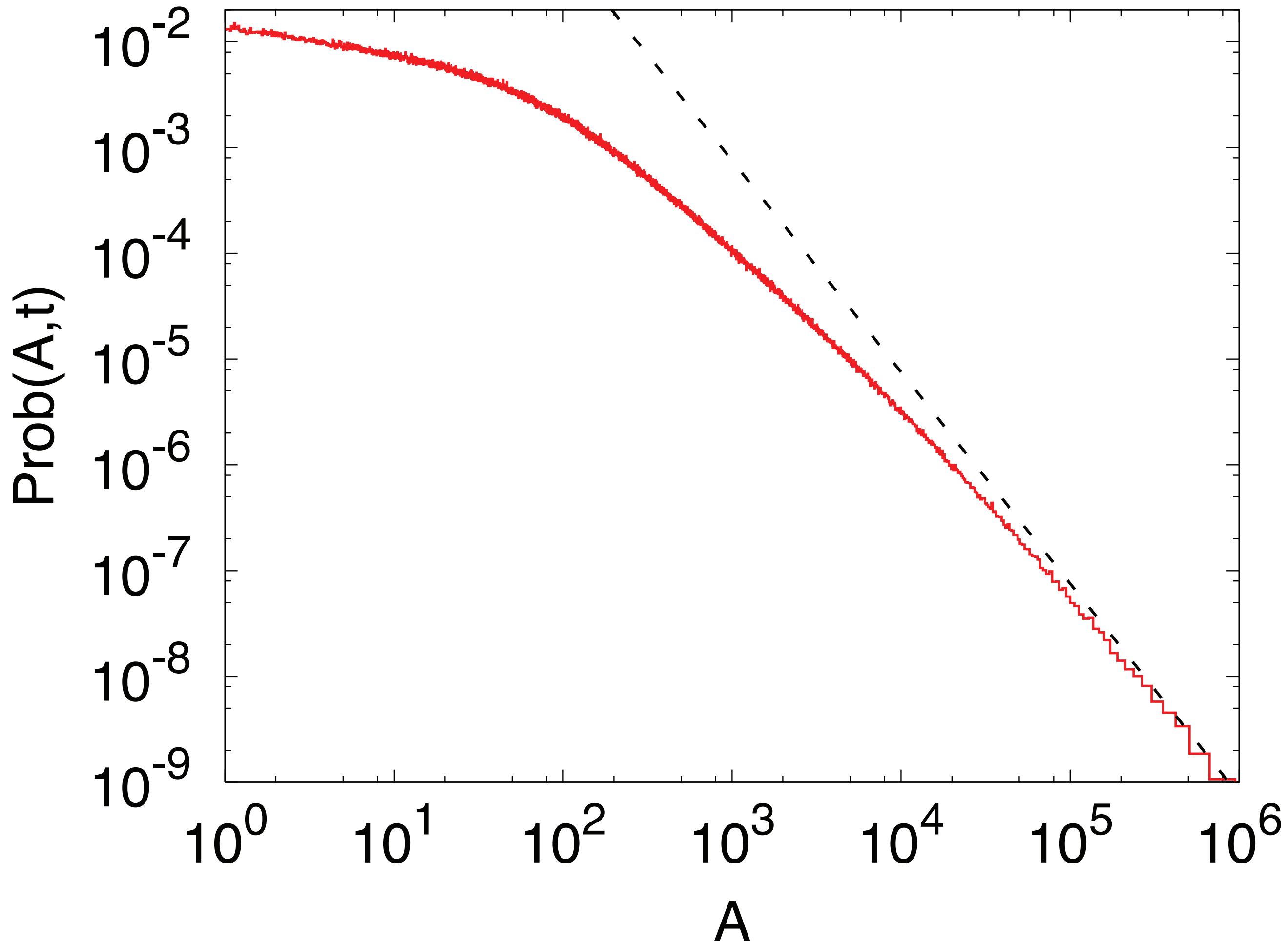


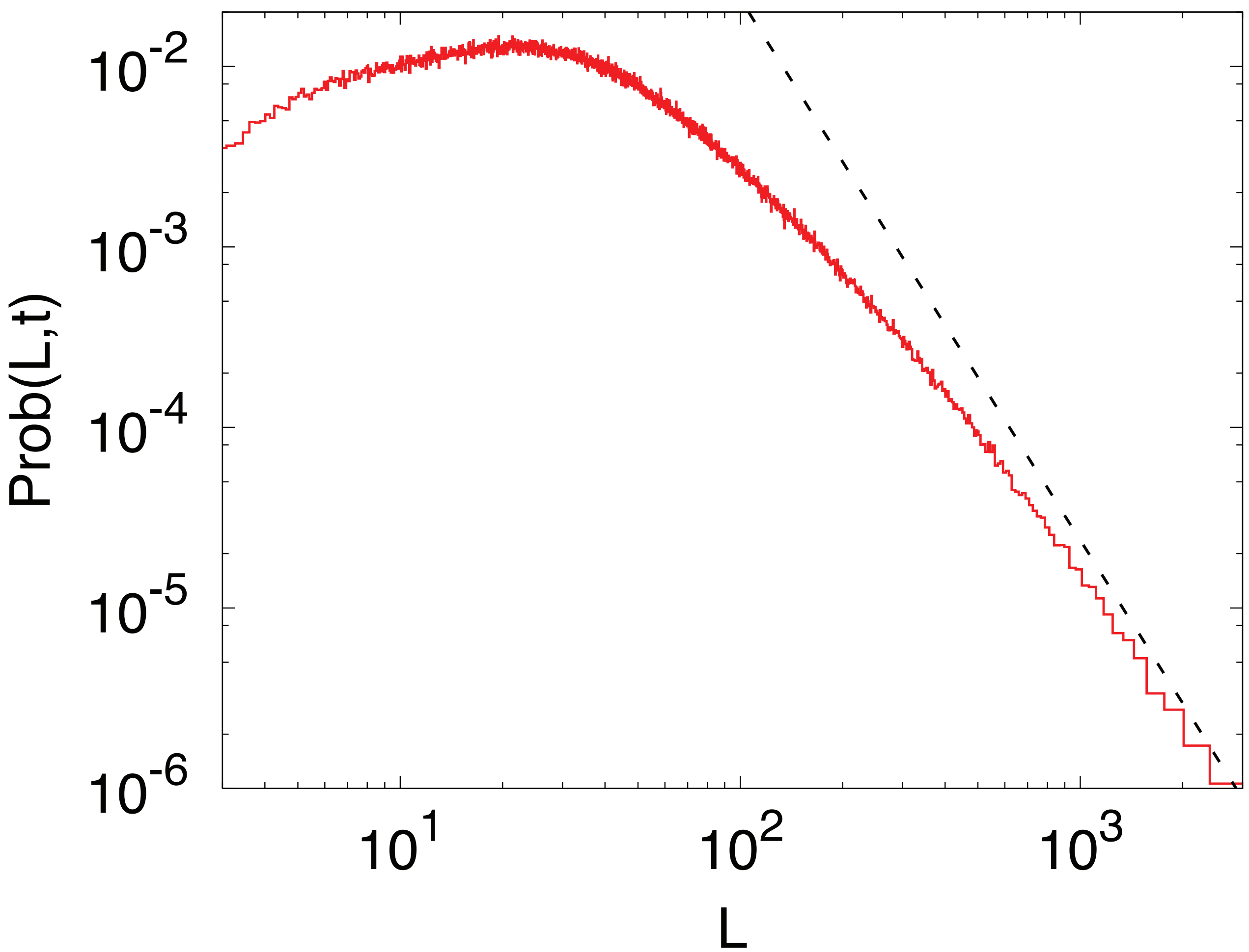


When $t \to \infty$ the perimeter remains finite, but the area diverges!

How it is possible ? ... Fluctuations

$$\mathrm{Prob}(A) \xrightarrow[A\to\infty]{t=\infty} \frac{24\pi D}{5\gamma} A^{-2} \qquad \mathrm{Prob}(L) \xrightarrow[L\to\infty]{t=\infty} L^{-3}$$





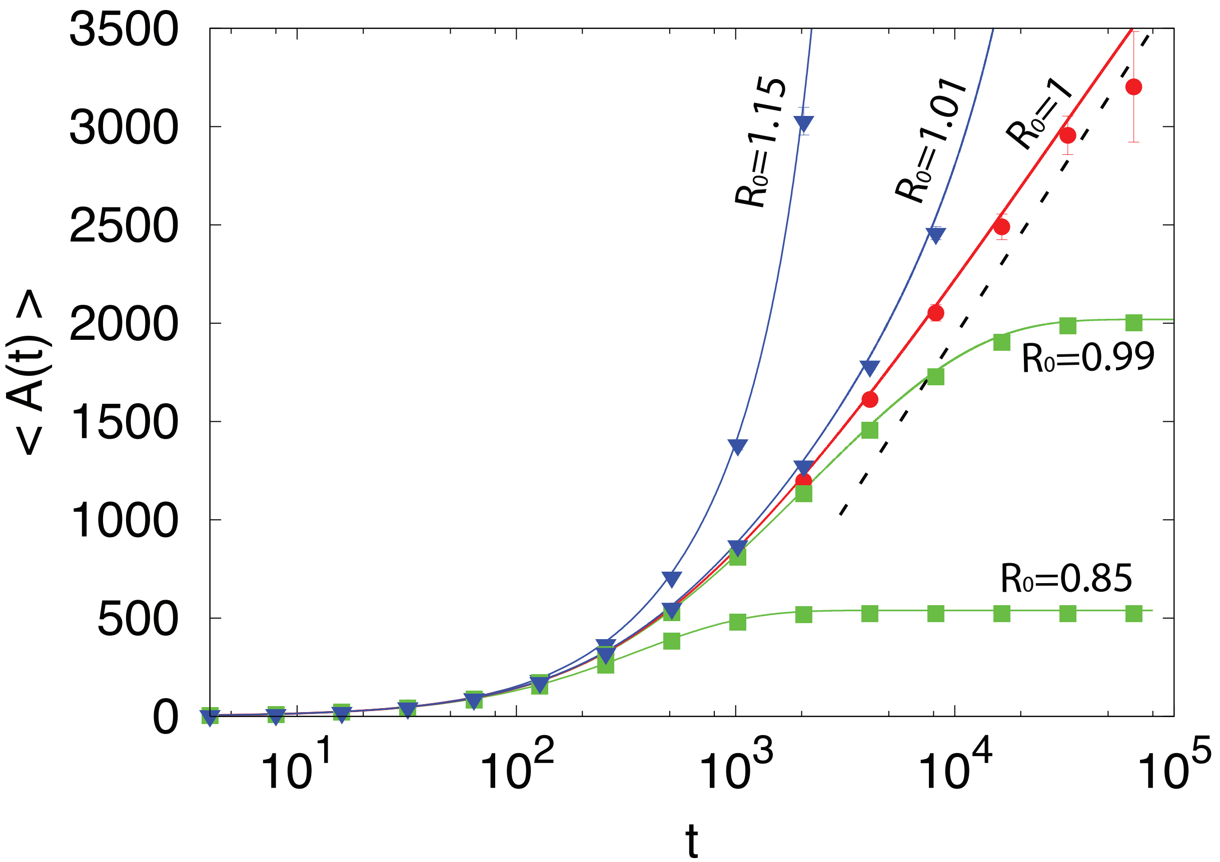
Out of criticality

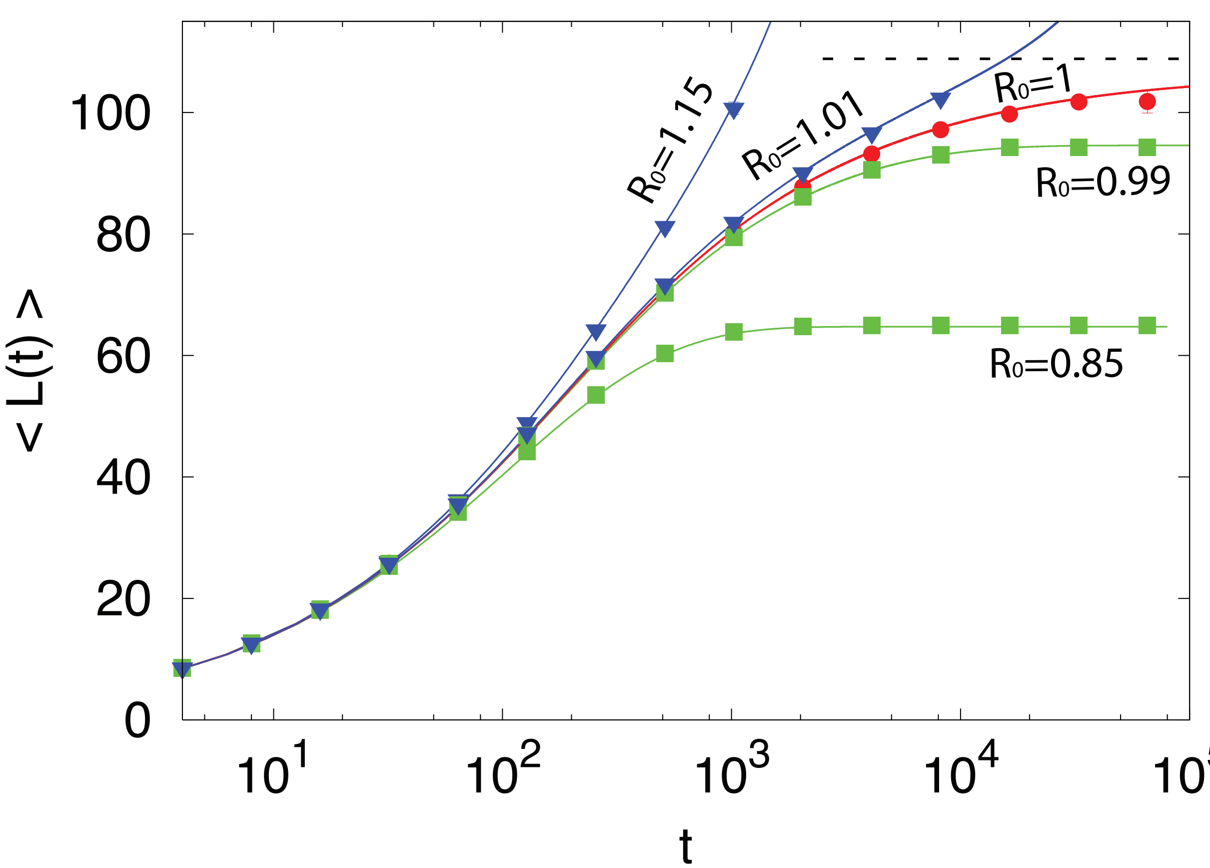
When $R_0 \neq 1$, characteristic time $t^* \sim |R_0 - 1|^{-1}$.

For times $t < t^*$ the epidemic behaves as in the critical regime.

In the *subcritical* regime, for $t > t^*$ the epidemic goes to extinction.

In the *supercritical* regime, with probability $1 - 1/R_0$ epidemic explodes.



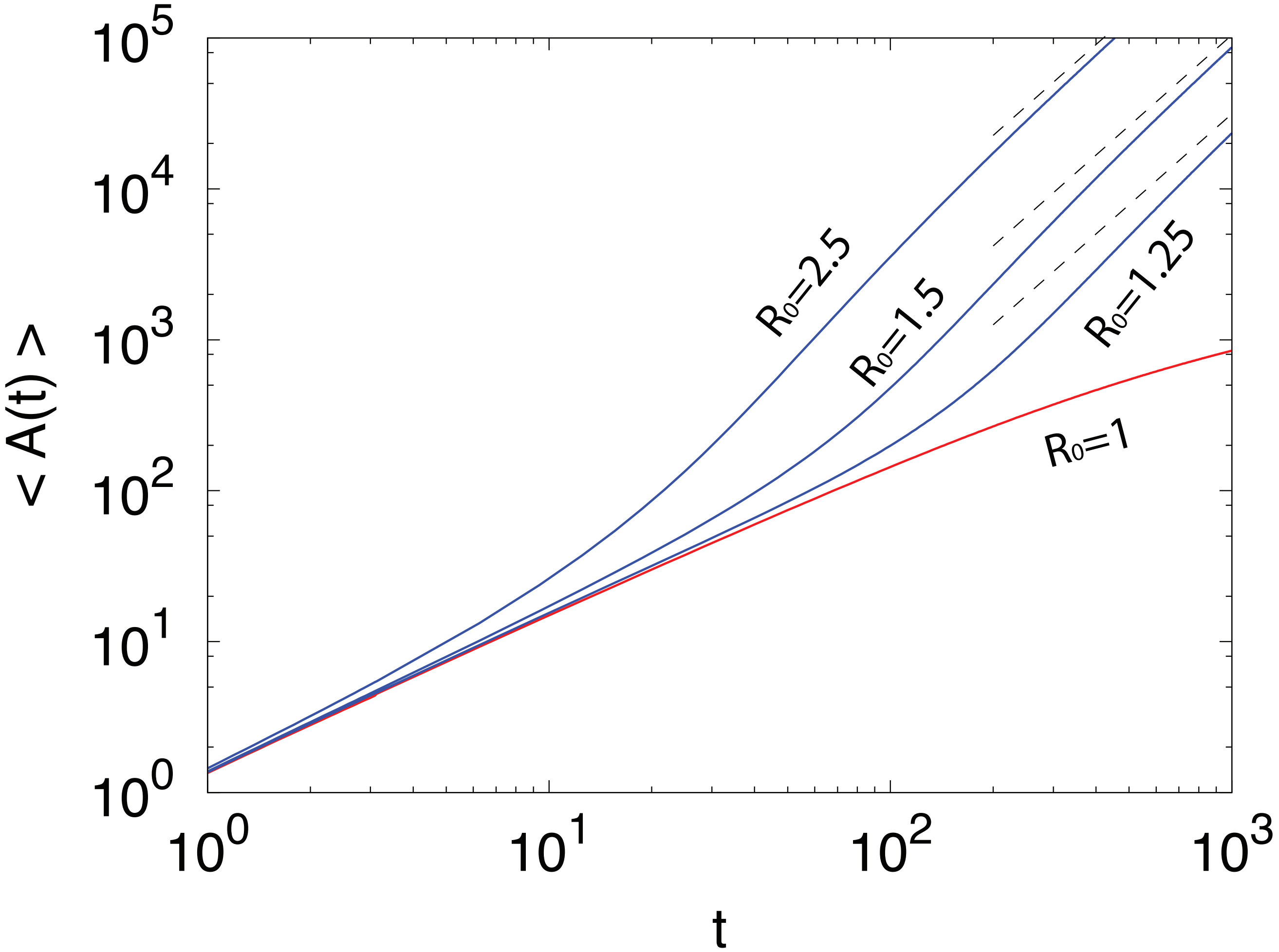


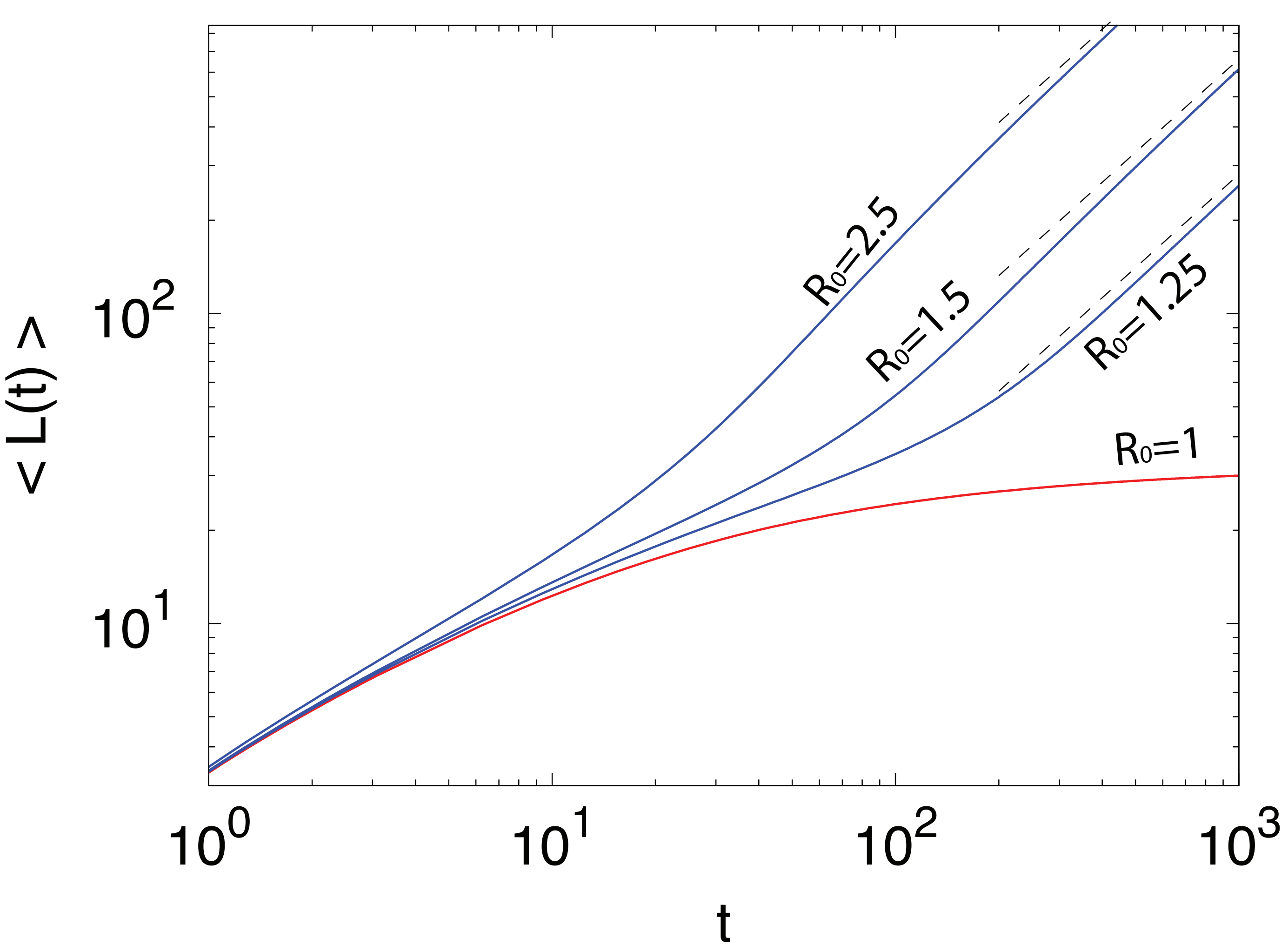
Supercritical

$$\langle L(t \gg t^*) \rangle = 4\pi \left(1 - \frac{1}{R_0}\right) \sqrt{D\,\gamma\,(R_0 - 1)}\, t$$

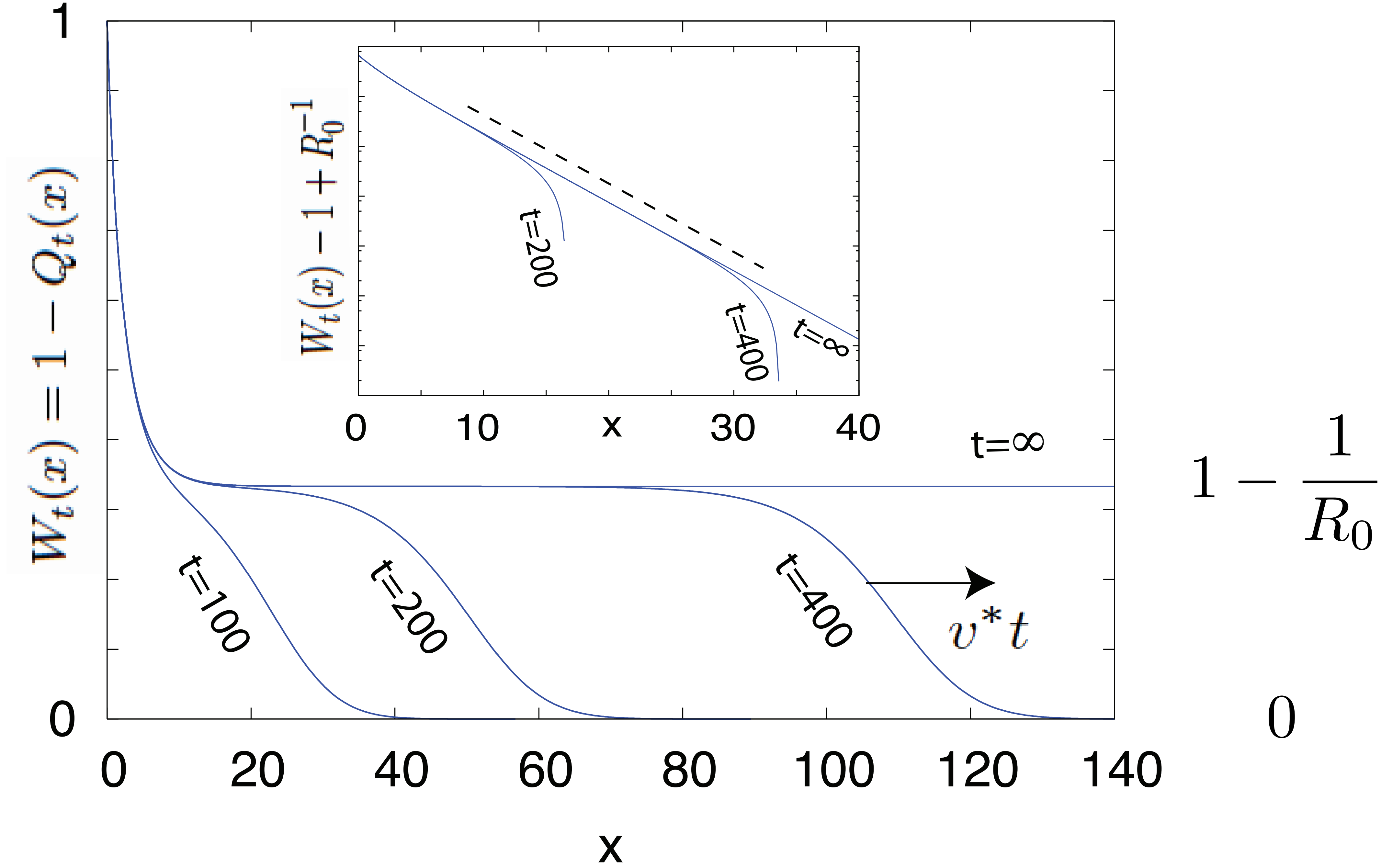
$$\langle A(t \gg t^*) \rangle = 4\pi \left(1 - \frac{1}{R_0}\right) D\,\gamma\,(R_0 - 1)\, t^2$$

$$t^* \sim |R_0 - 1|^{-1}$$





$$\boxed{\frac{\partial}{\partial t} W = D \frac{\partial^2}{\partial x_m^2} W + \gamma (R_0 - 1) W - \gamma R_0 W^2}$$



Traveling front solution

Conclusions:

- Branching Brownian motion with death as a model for the spatial extent of animal epidemics
- Using Cauchy Formulas we can map the convex hull problem in the extreme statistic of the 1-dimensional process
- Backward F-P equations for the extreme distributions
- Critical case has very large fluctuations
- Super Critical case: traveling front solution

How far in space can an epidemic spread?

Problem 1: How to model the space?

The population is uniformly distributed

At time $t = 0$ an infected individual appears

... and moves in space

Brownian motion is the paradigm of animal migration

while human beings take the plane (even when they are sick)

Similar calculations allows to express the mean area as:

$$\langle A(t)\rangle = \pi \int_0^\infty dx_m \left[2x_m(1 - Q_t(x_m)) - T_t(x_m)\right]$$

Where the evolution of $T_t(x_m)$ is governed by:

$$\boxed{\frac{\partial}{\partial t}T_t + \partial_x Q_t(x_m) = \left[D\frac{\partial^2}{\partial x_m^2} + 2\gamma R_0 Q_t - \gamma\left(R_0 + 1\right)\right] T_t,}$$

Both PDE can be integrated numerically and solved in some asymptotic limit

Dimensional reduction

$$L = \int_0^{2\pi} M(\theta)\, d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} \left[M^2(\theta) - (M'(\theta))^2\right] d\theta$$

If the process is rotationally invariant any average is independent of θ

$$\langle L(t)\rangle = 2\pi\langle M(0)\rangle$$
$$\langle A(t)\rangle = \pi\left[\langle M^2(0)\rangle - \langle M'(0)^2\rangle\right]$$

$$\langle L(t)\rangle = 2\pi\langle x_m(t)\rangle$$
$$\langle A(t)\rangle = \pi\left[\langle x_m^2(t)\rangle - \langle y^2(t_m)\rangle\right]$$

This relation is valid ONLY in average