Universal Record Statistics of Random Walks

GGI Workshop in Advances in Non-Equilibrium Statistical Mechanics

Grégory Schehr, LPTMS (Orsay)

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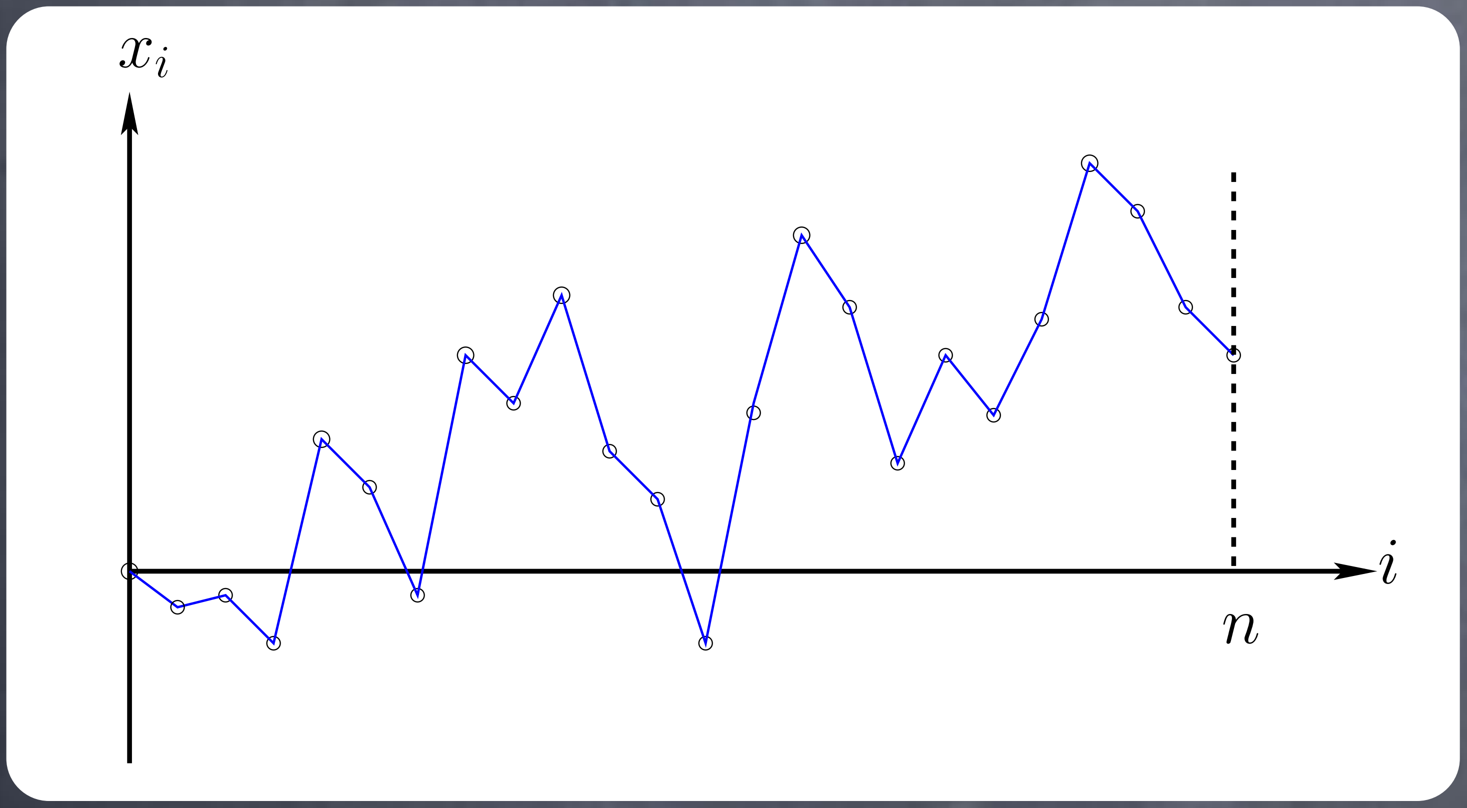
Collaborators:

- C. Godrèche (IPhT, Saclay)
- S. N. Majumdar (LPTMS, Orsay)
- G. Wergen (Uni. of Cologne)

Statement of the problem

$x_1, x_2, \cdots, x_n : n$ random variables (e.g. time series)



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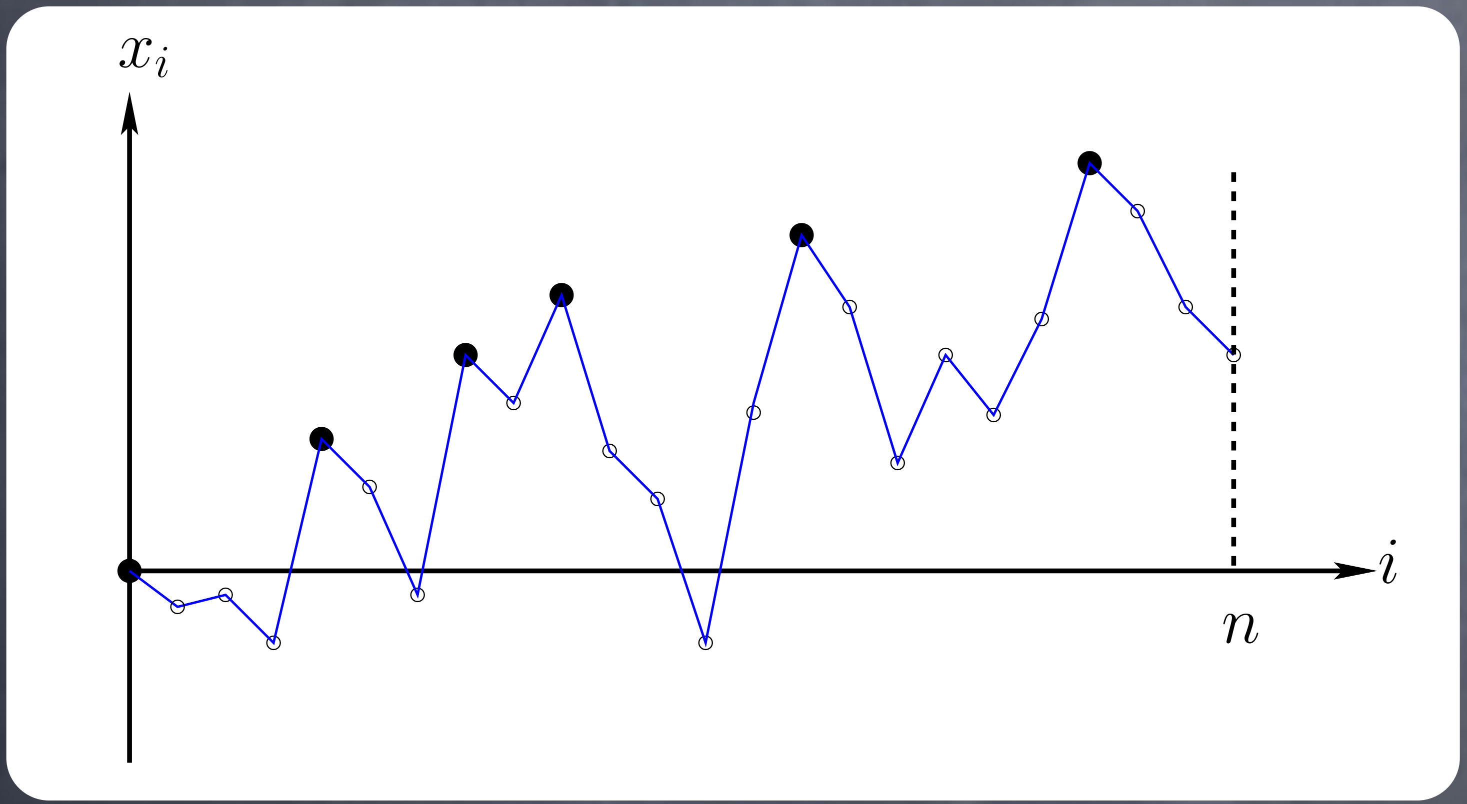

x_k is a record iff

$$x_k \geq \max(x_1, x_2, \cdots, x_{k-1})$$

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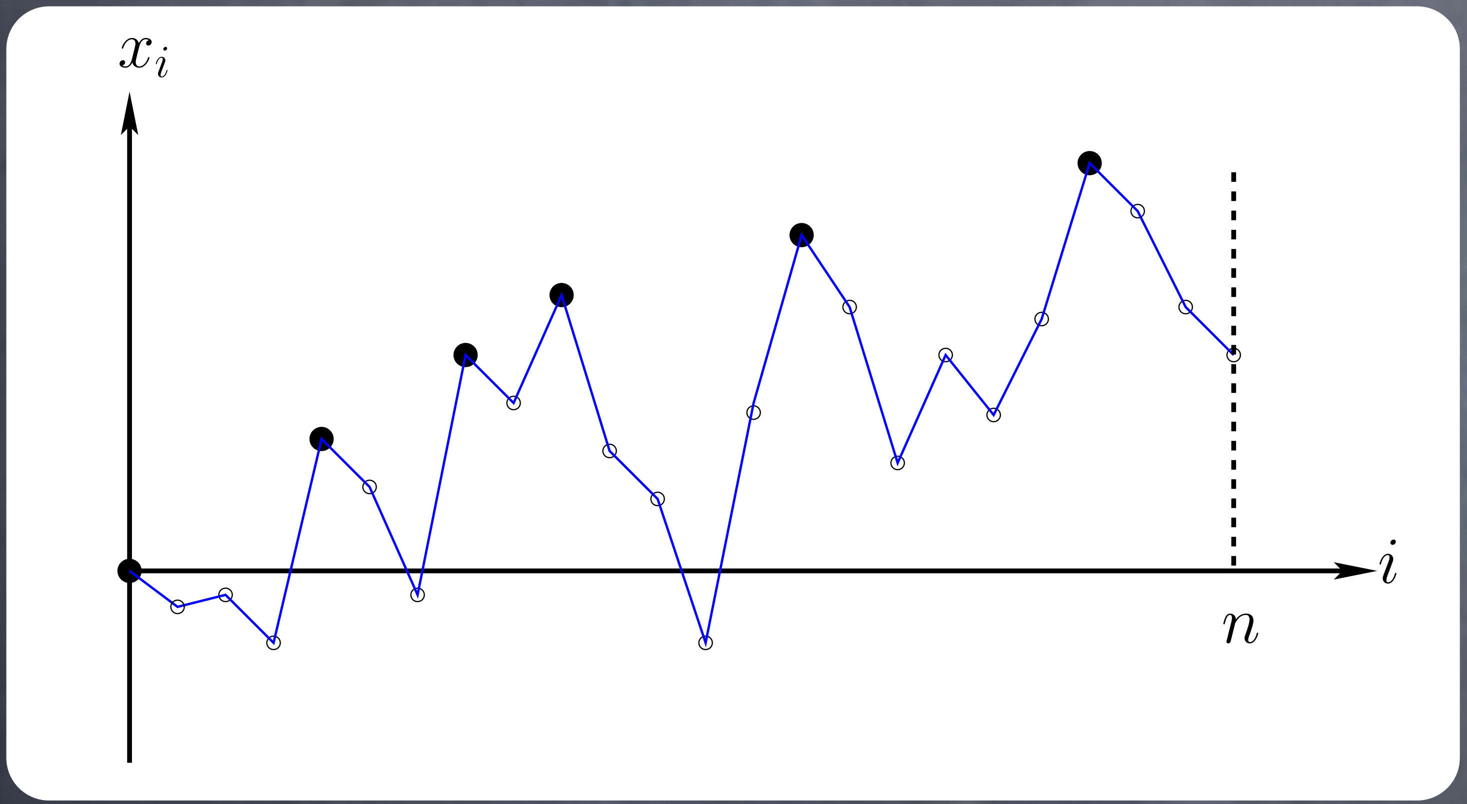

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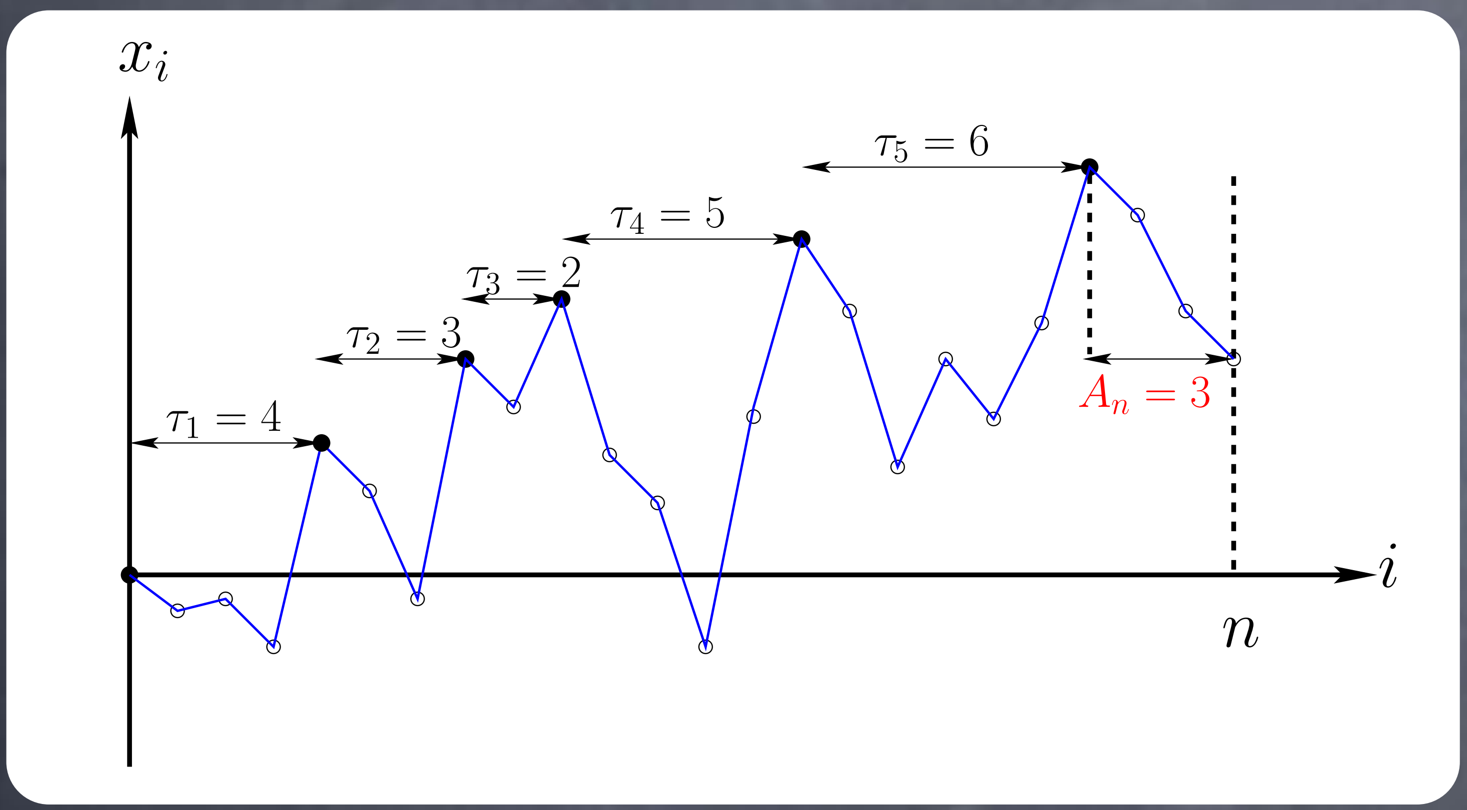

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Questions:
- Statistics of the number of records R_n ?
- Statistics of the ages of records $\tau_1, \tau_2, \cdots, A_n$?

Some recent applications of records in physics

- Domain wall dynamics   Alessandro et al. '90
- Evolutionary biology   Jain & Krug '05
- Global warming   Redner & Petersen '06, Wergen & Krug '10
- Spin-glasses   Sibani '07
- Random walks   Majumdar & Ziff '08, Wergen, Majumdar, G. S. '12
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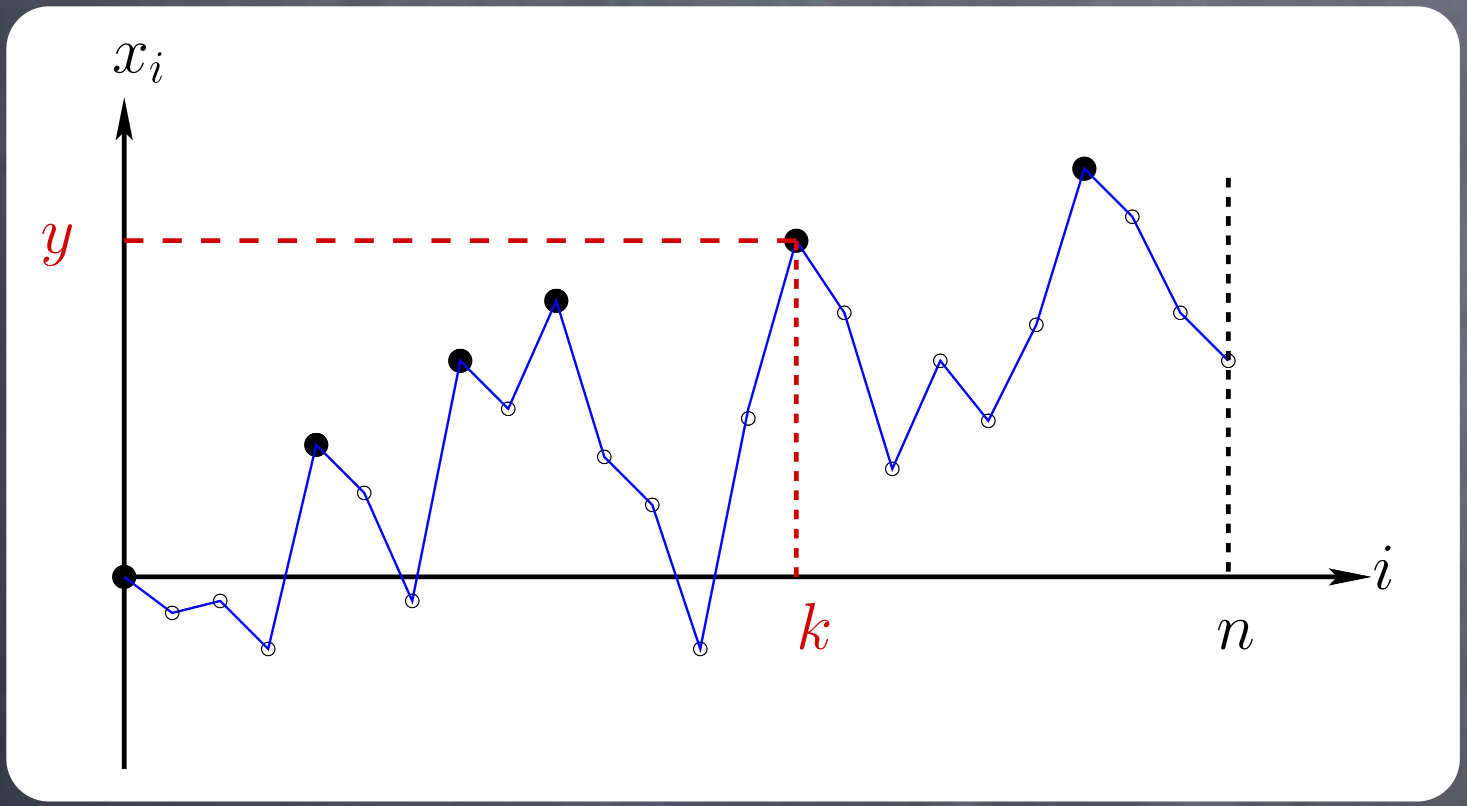
Record statistics of i.i.d. random variables

$x_1, x_2, \cdots, x_n : n$ i.i.d. random variables with PDF $p(x)$



Nber of records R_n

$$R_n = \sum_{k=1}^{n} \sigma_k$$

$$\sigma_k = \begin{cases} 1\,, & \text{if } x_k \text{ is a record} \\ 0\,, & \text{if } x_k \text{ is NOT a record} \end{cases}$$

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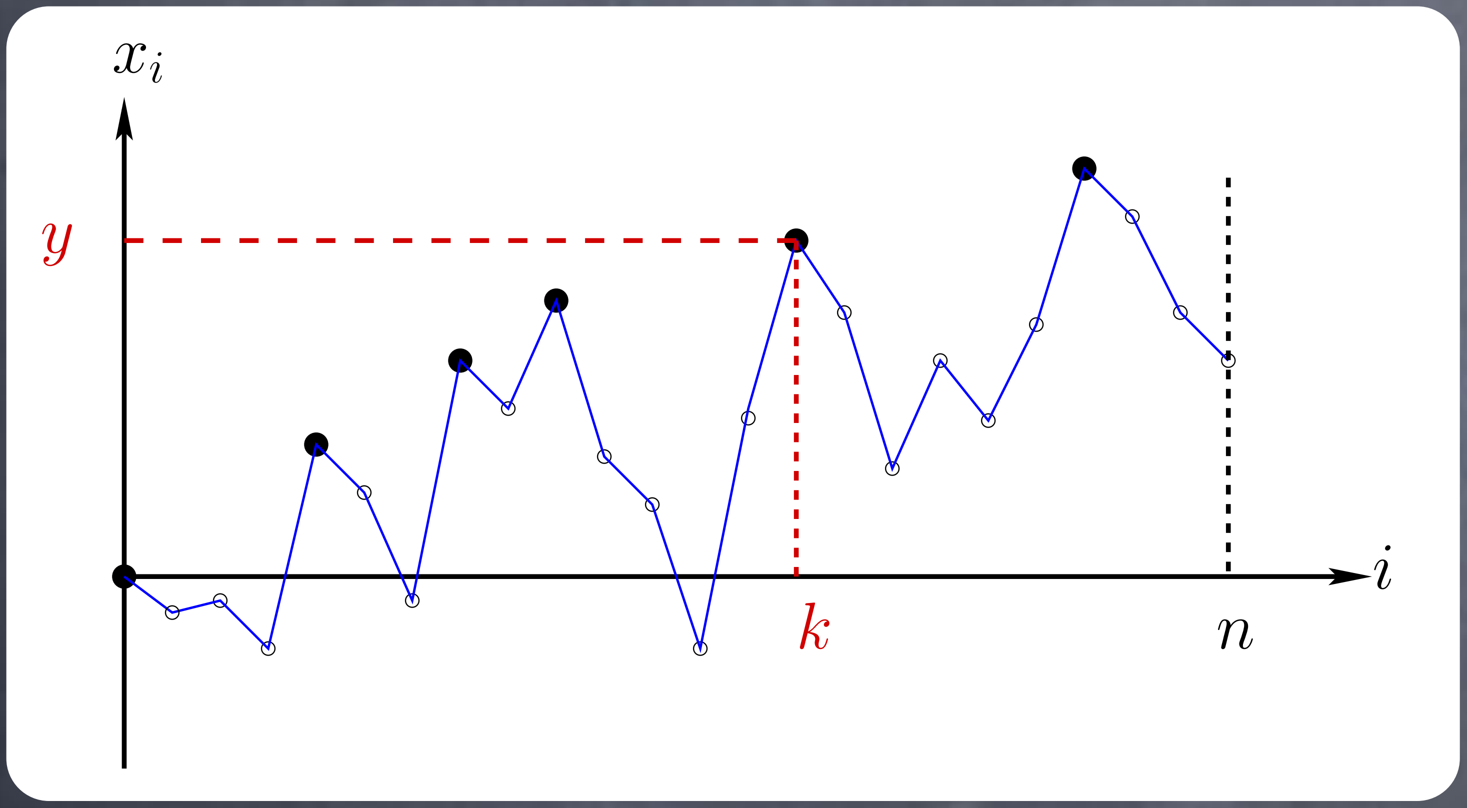

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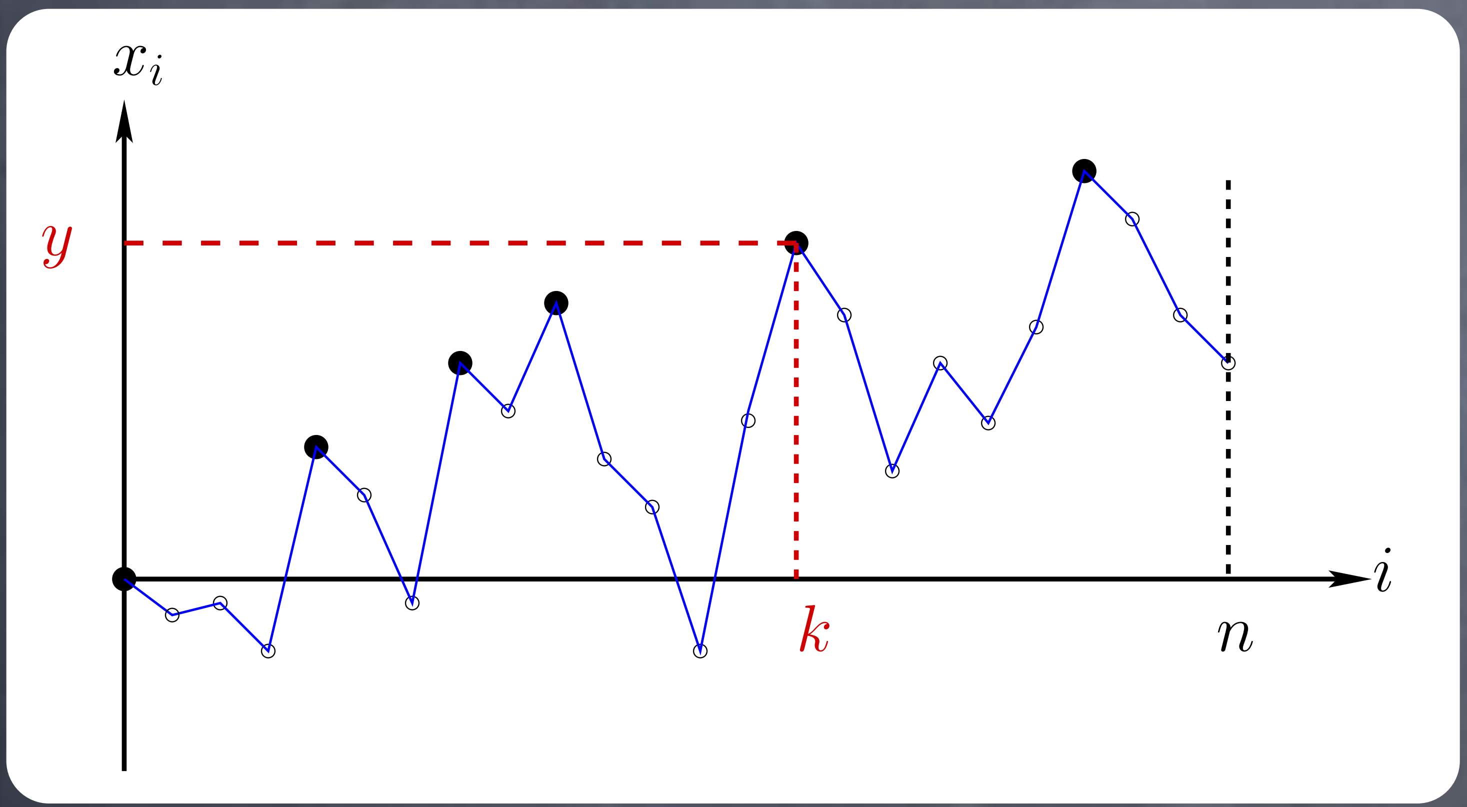

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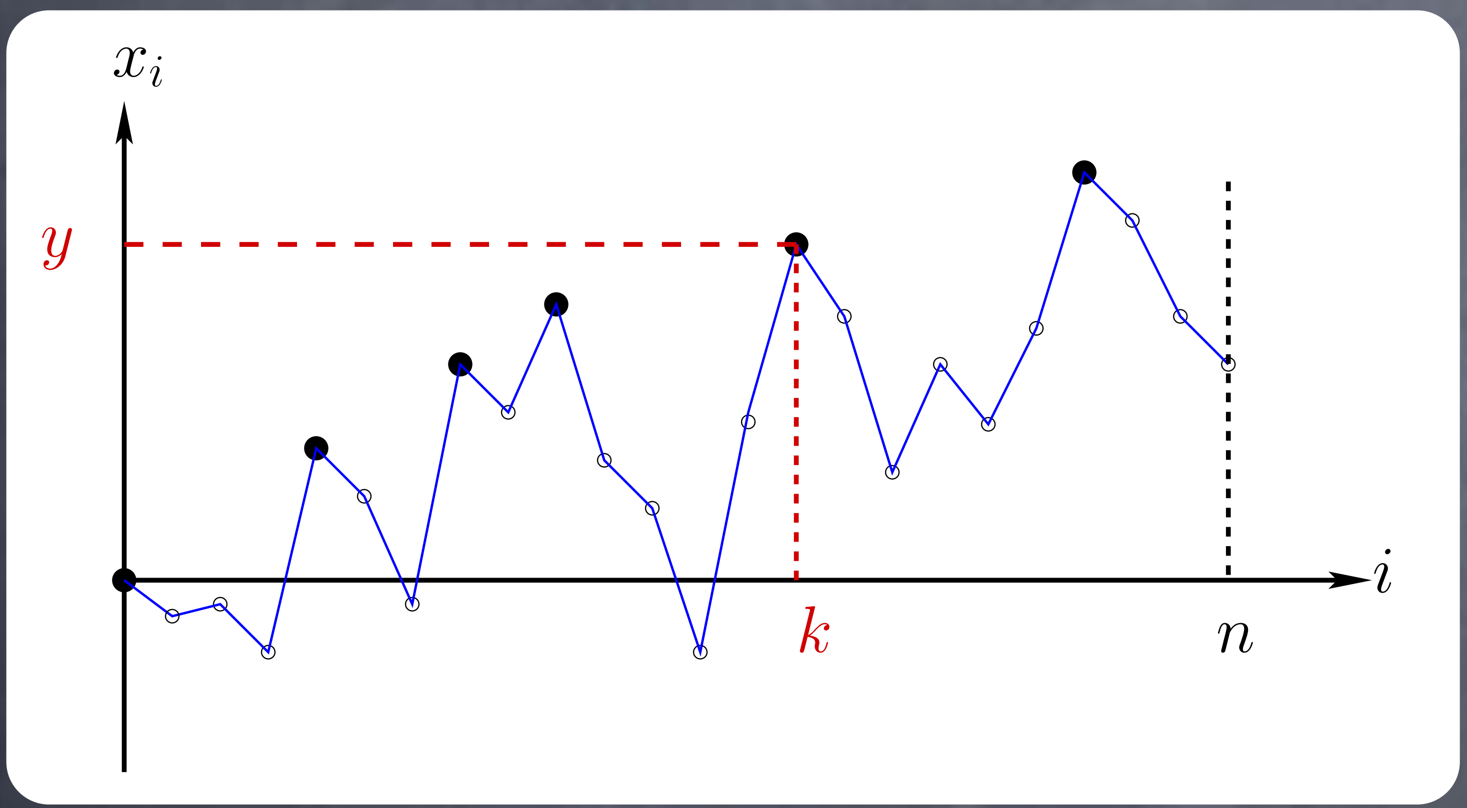

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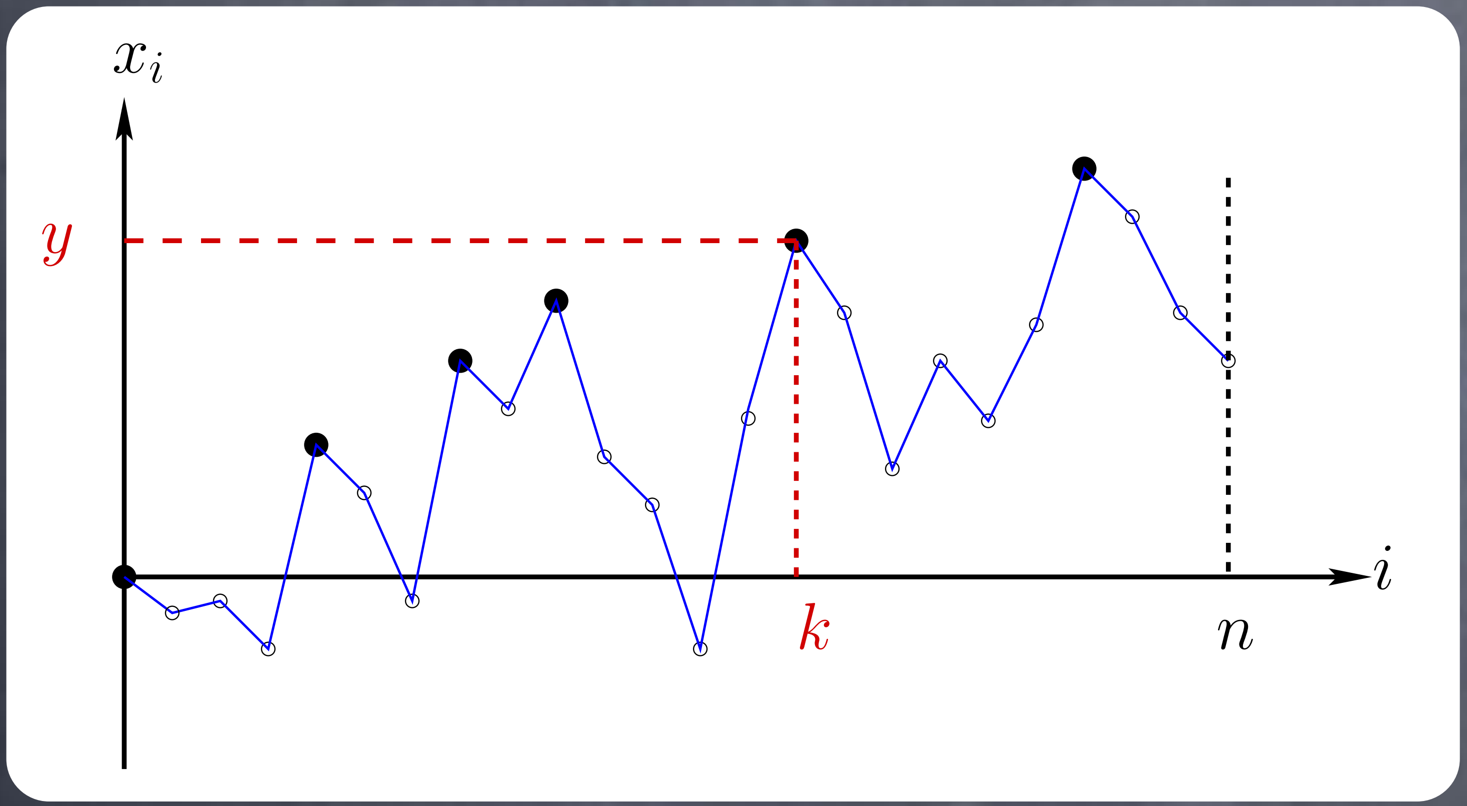

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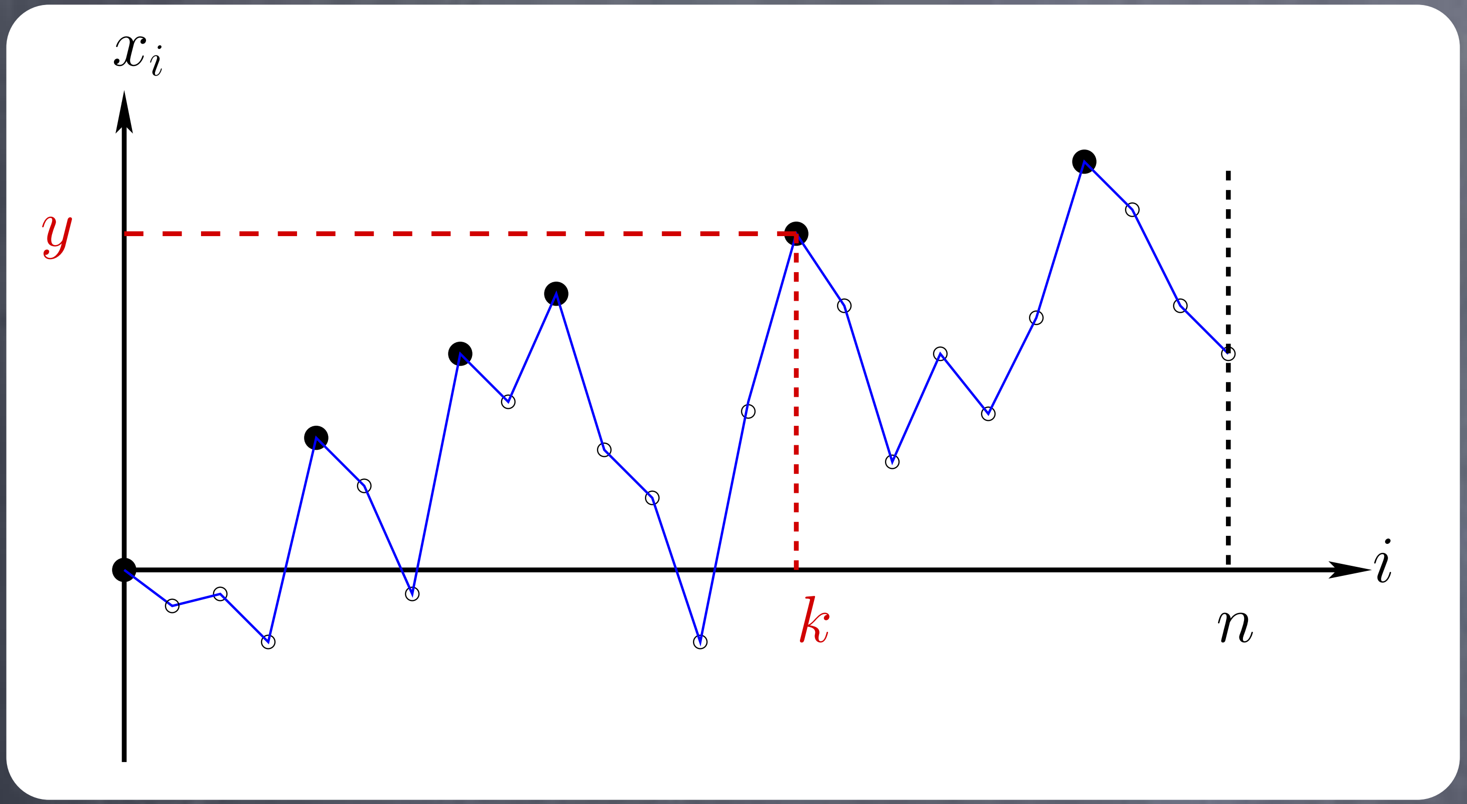

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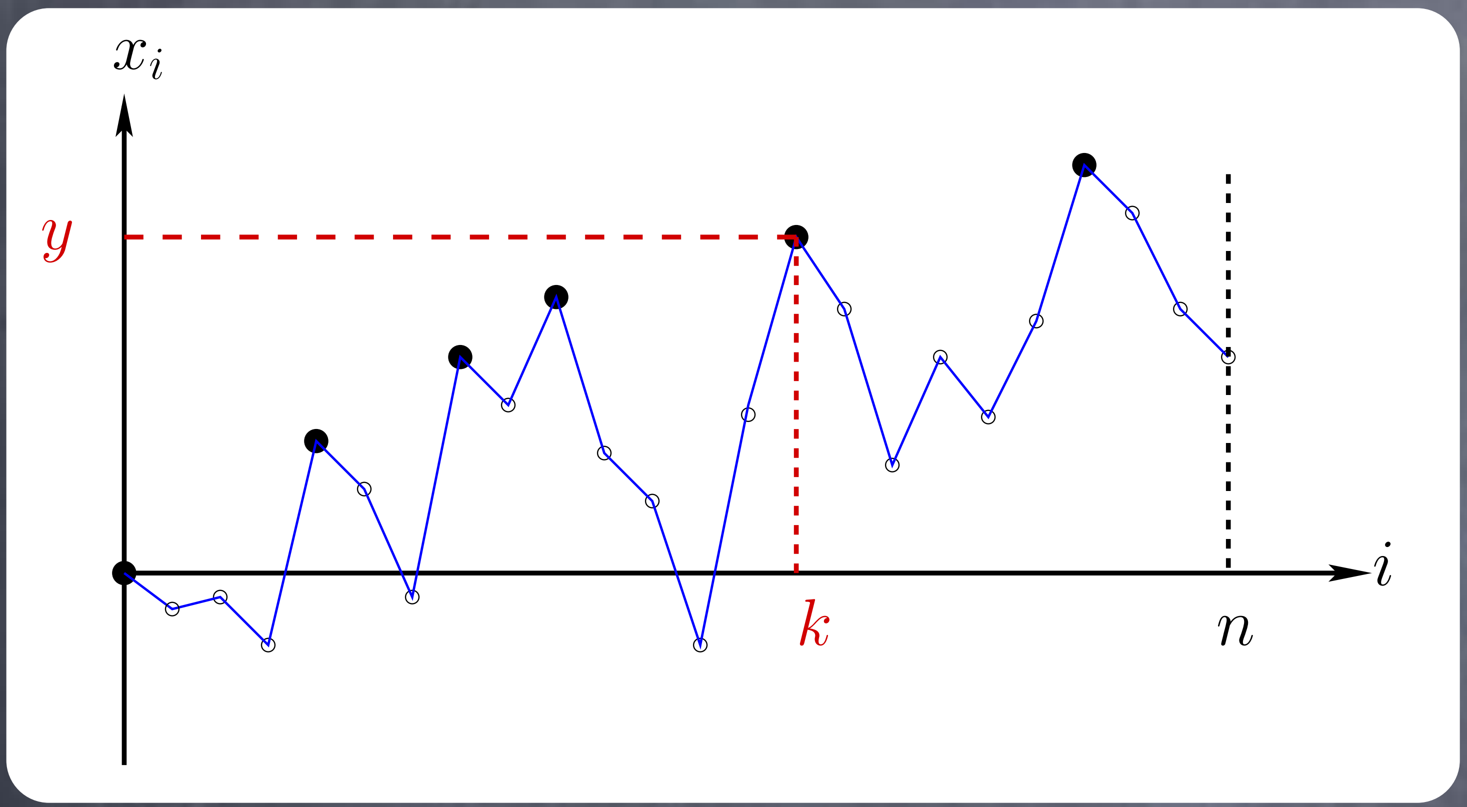

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$$\langle R_n \rangle = \sum_{k=1}^{n} \frac{1}{k} \sim \log n$$

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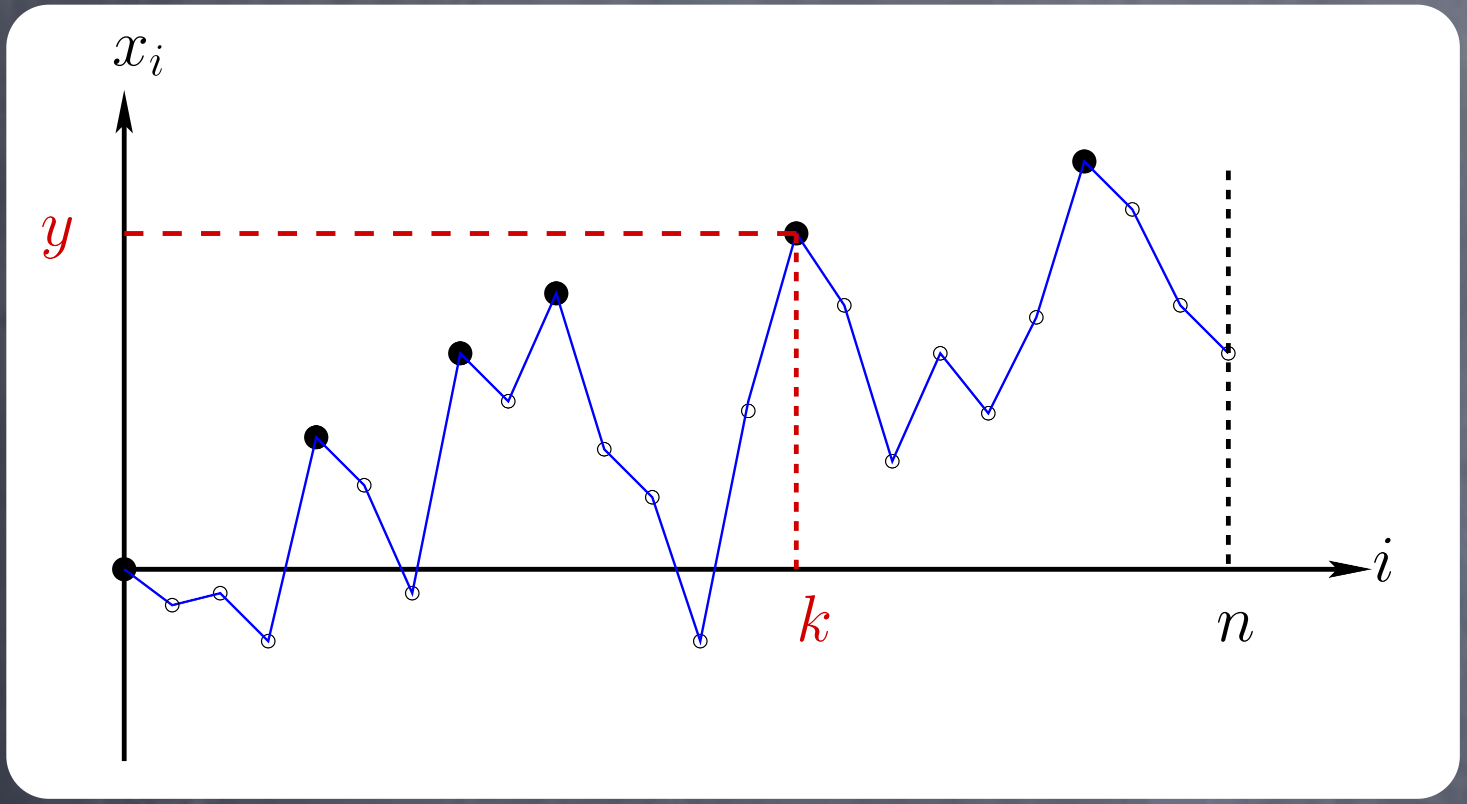

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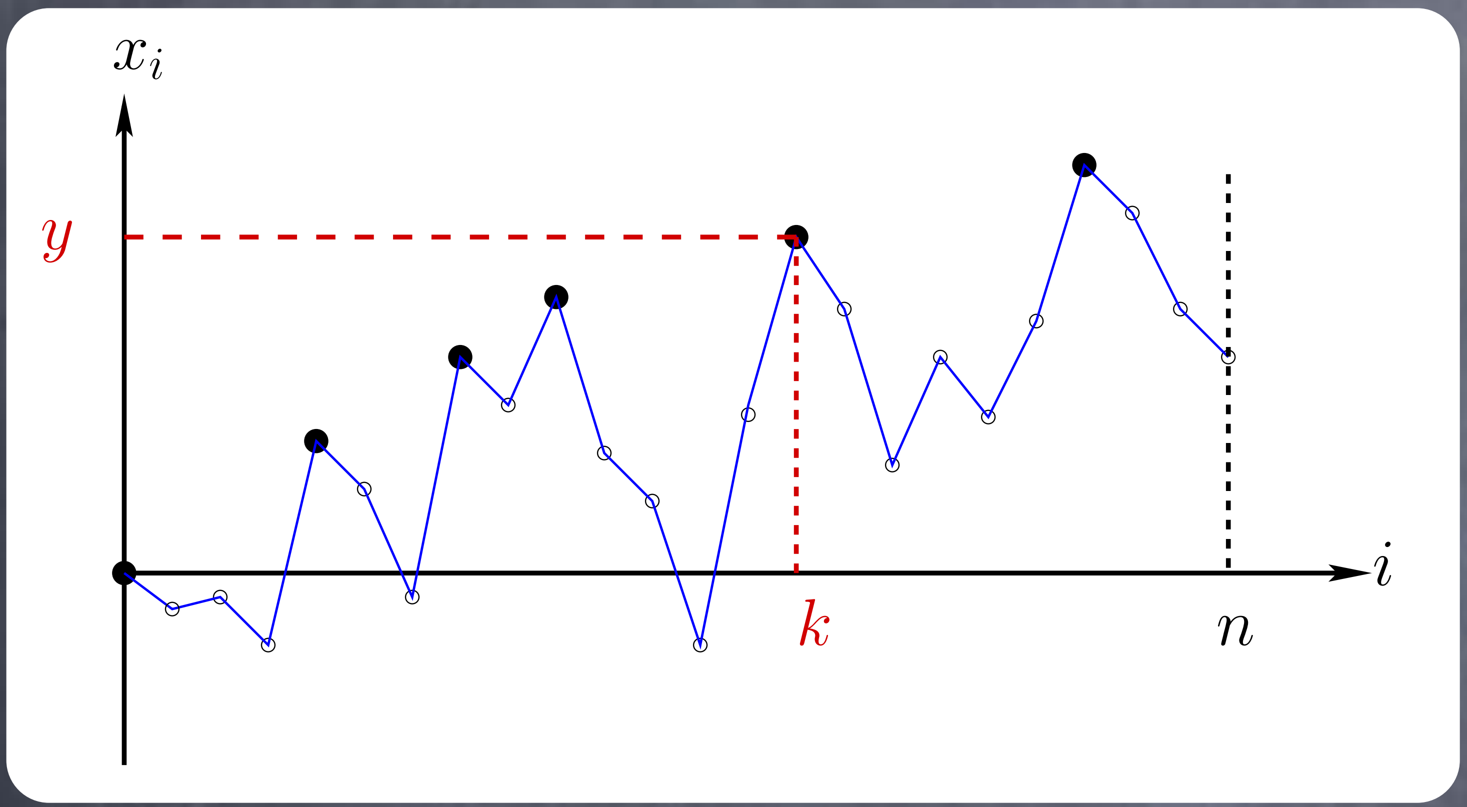

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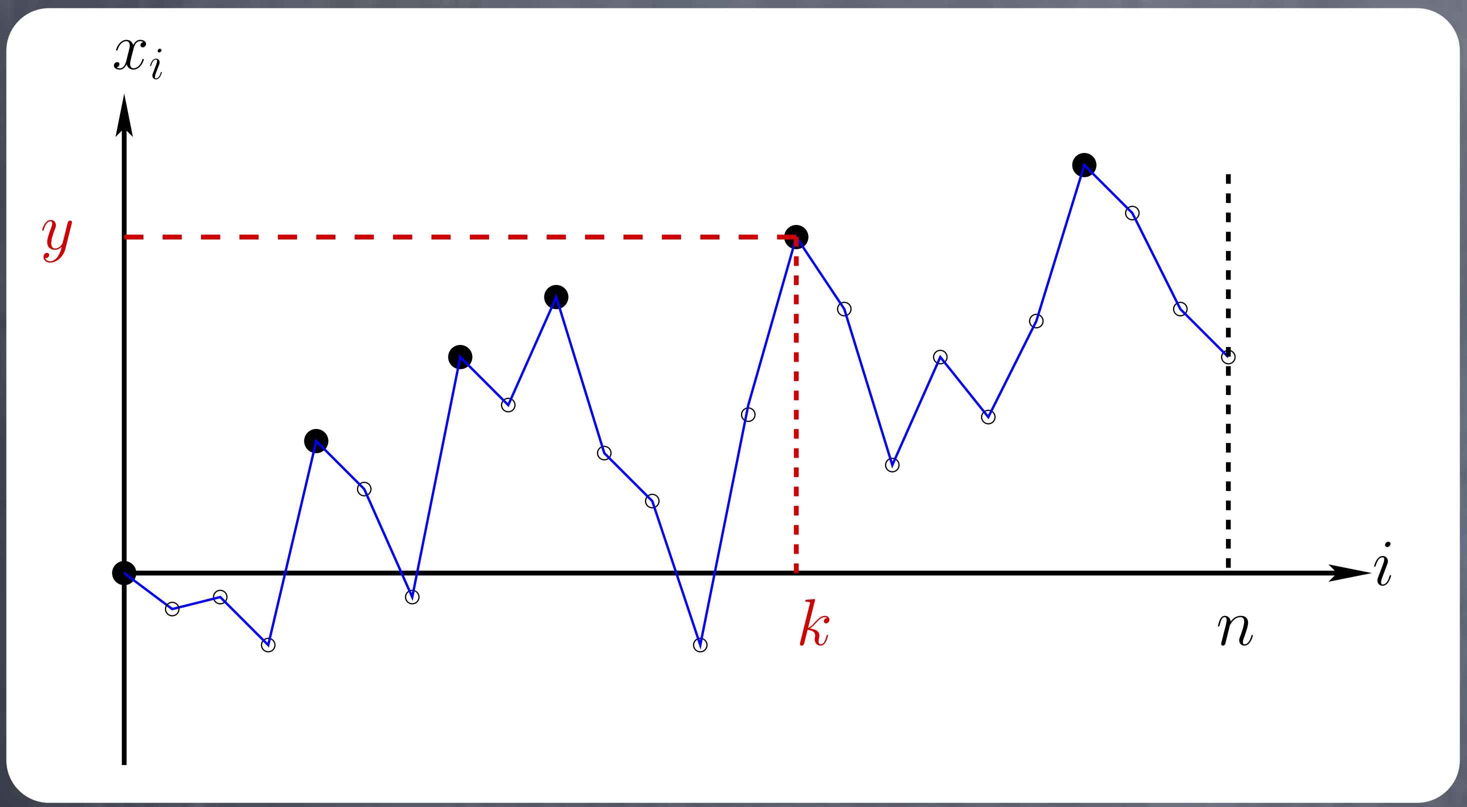

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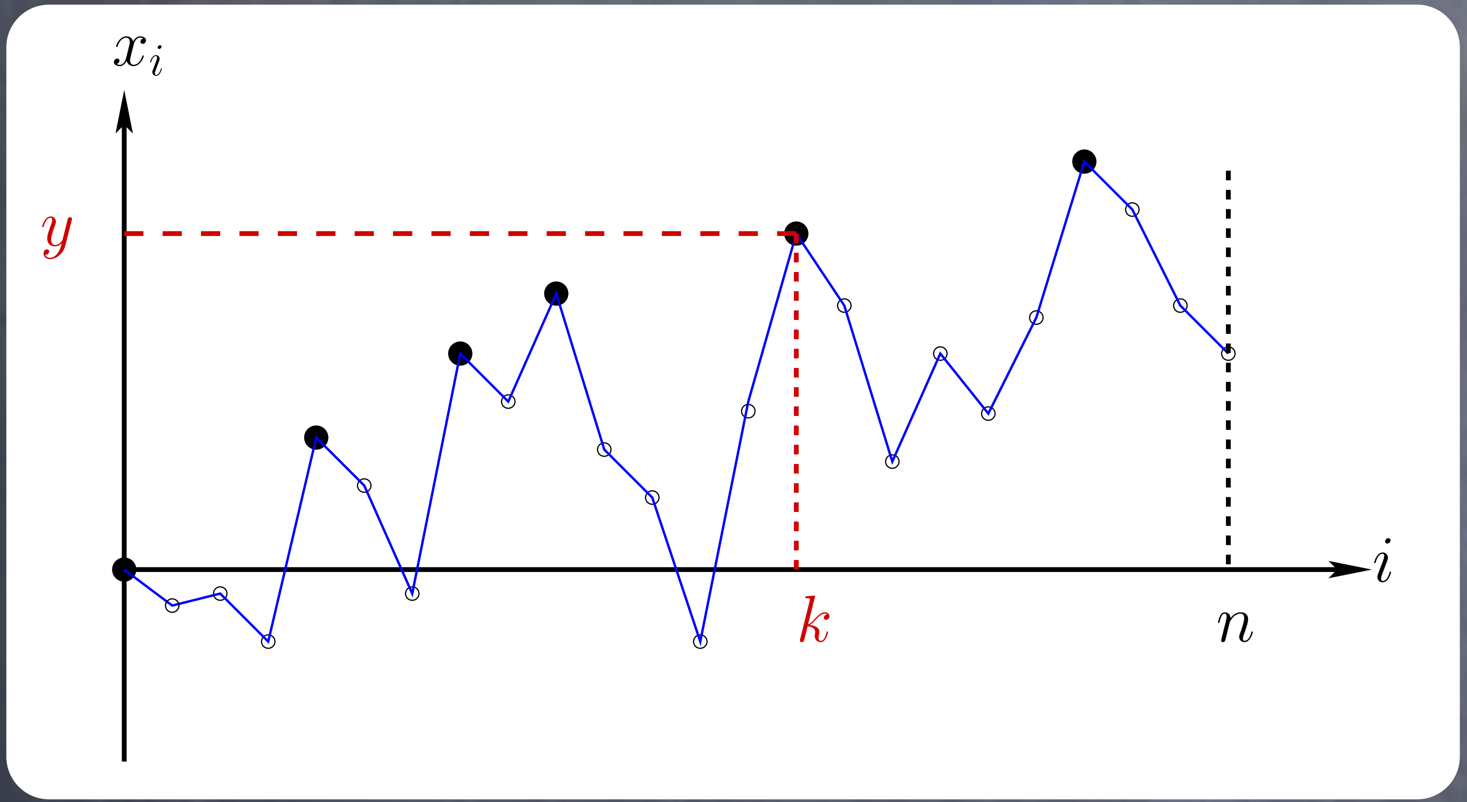

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Gaussian for large n

$$\sim \frac{1}{\sqrt{2\pi \log n}} \exp\left(-\frac{(M - \log n)^2}{2 \log n} \right)$$

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- Ordinary random walks $\sigma^2 = \int_{-\infty}^{\infty} \eta^2\, p(\eta)\, d\eta < \infty$

$$x_n \sim \sigma \sqrt{n}$$

- Lévy flights

$$p(\eta) \propto a^{\mu}\, |\eta|^{-1-\mu}\ , \ |\eta| \to \infty$$

$$0 < \mu < 2$$

$x_n \sim a\, n^{1/\mu}$ μ is the Lévy index

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Q: Dependence of records on the jump distribution ?

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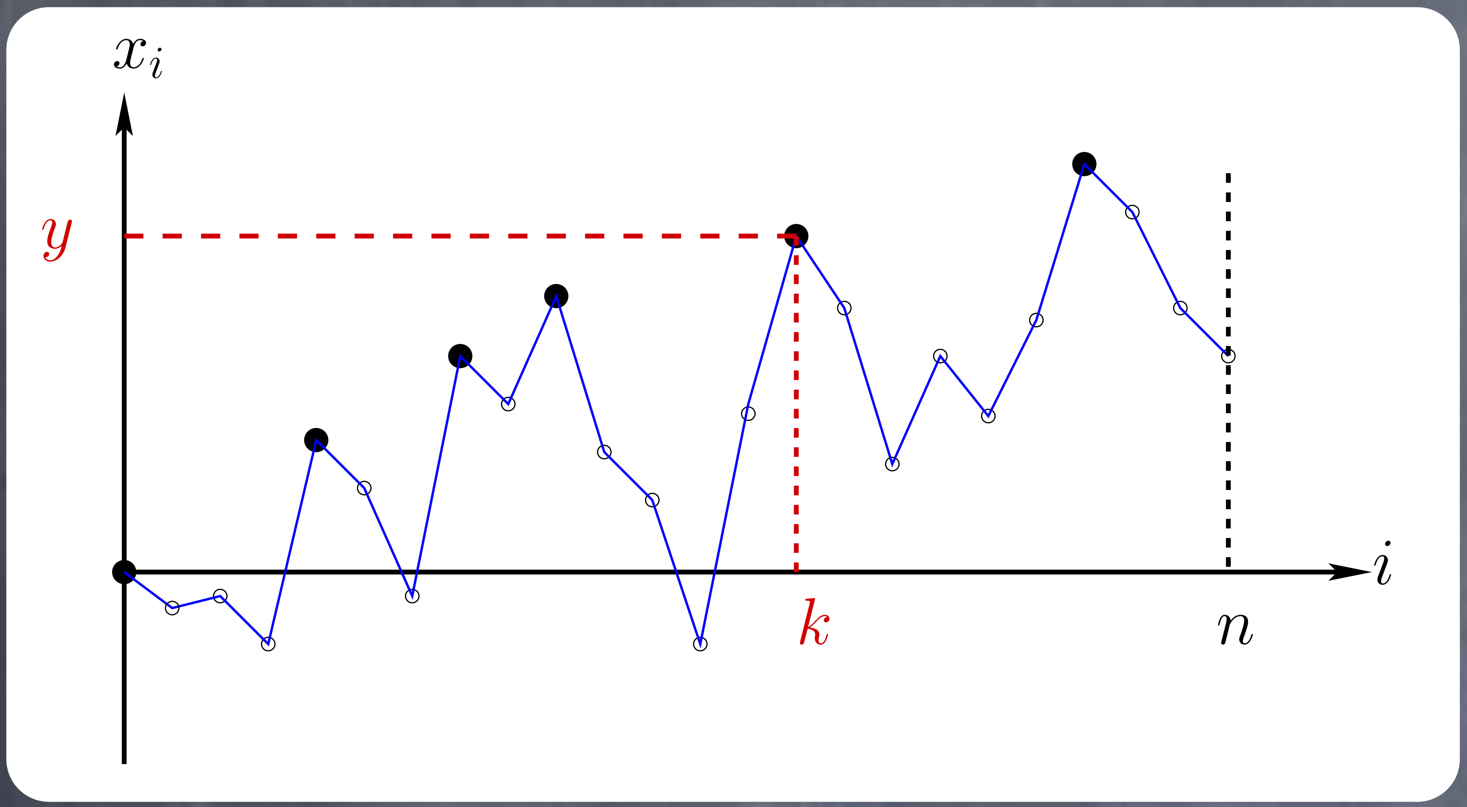

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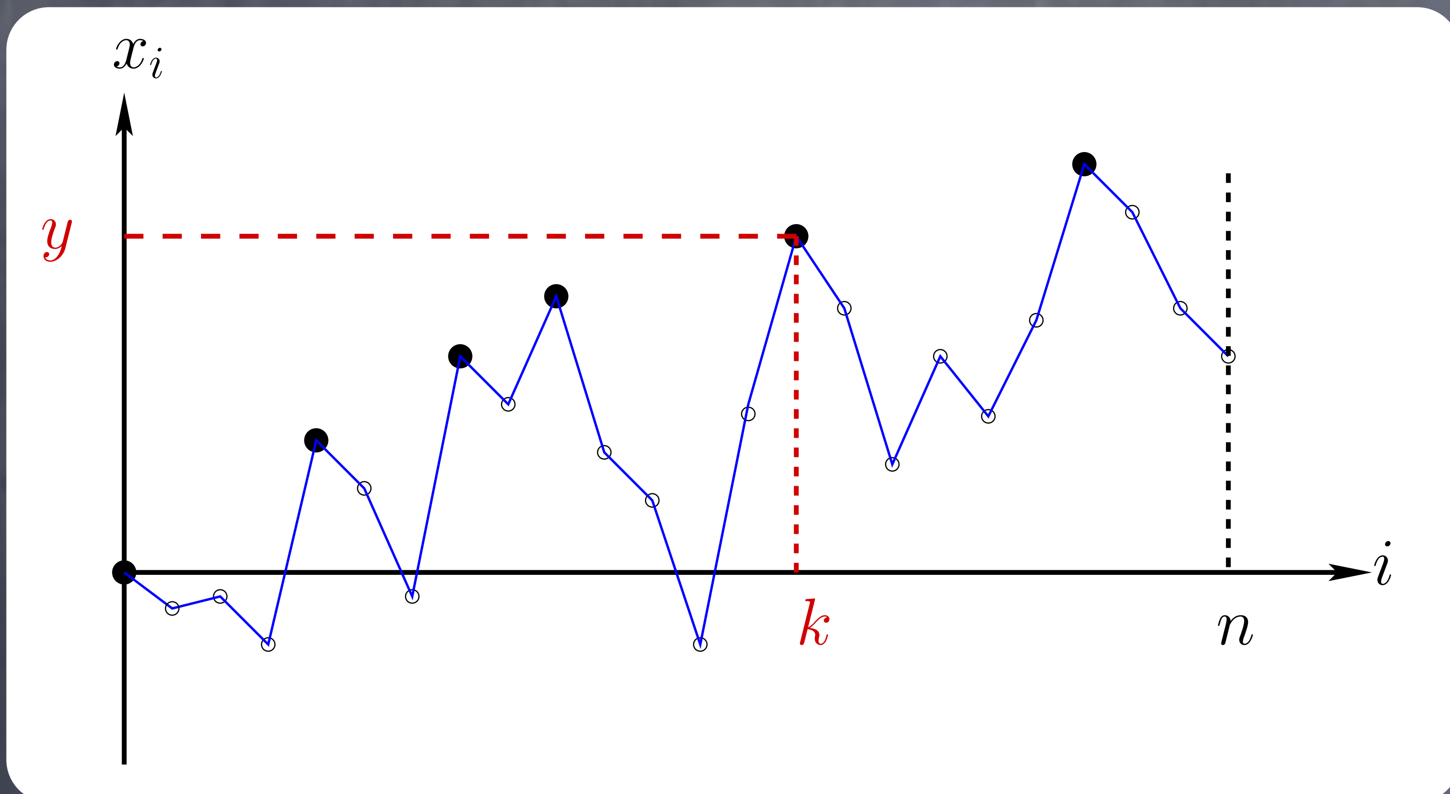

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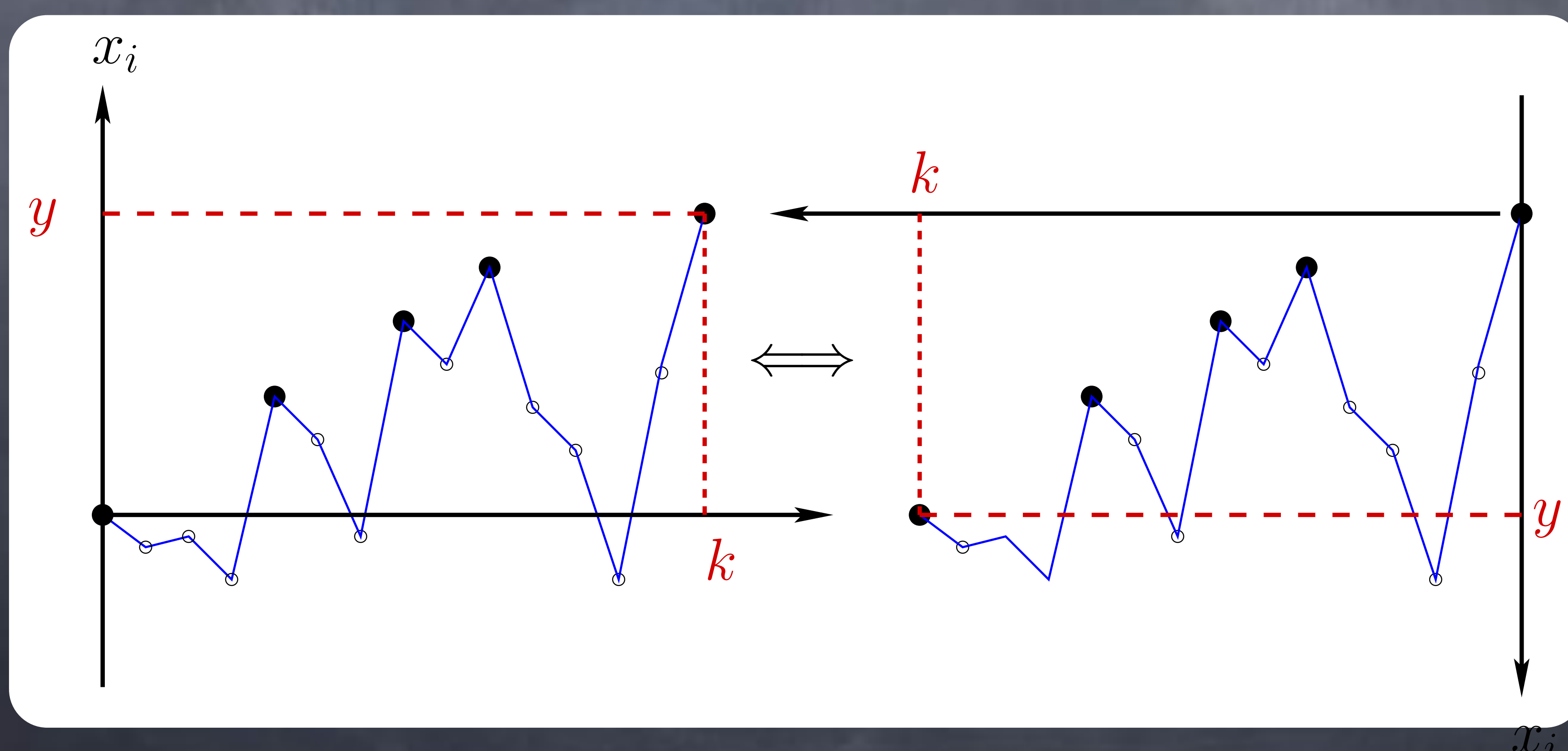

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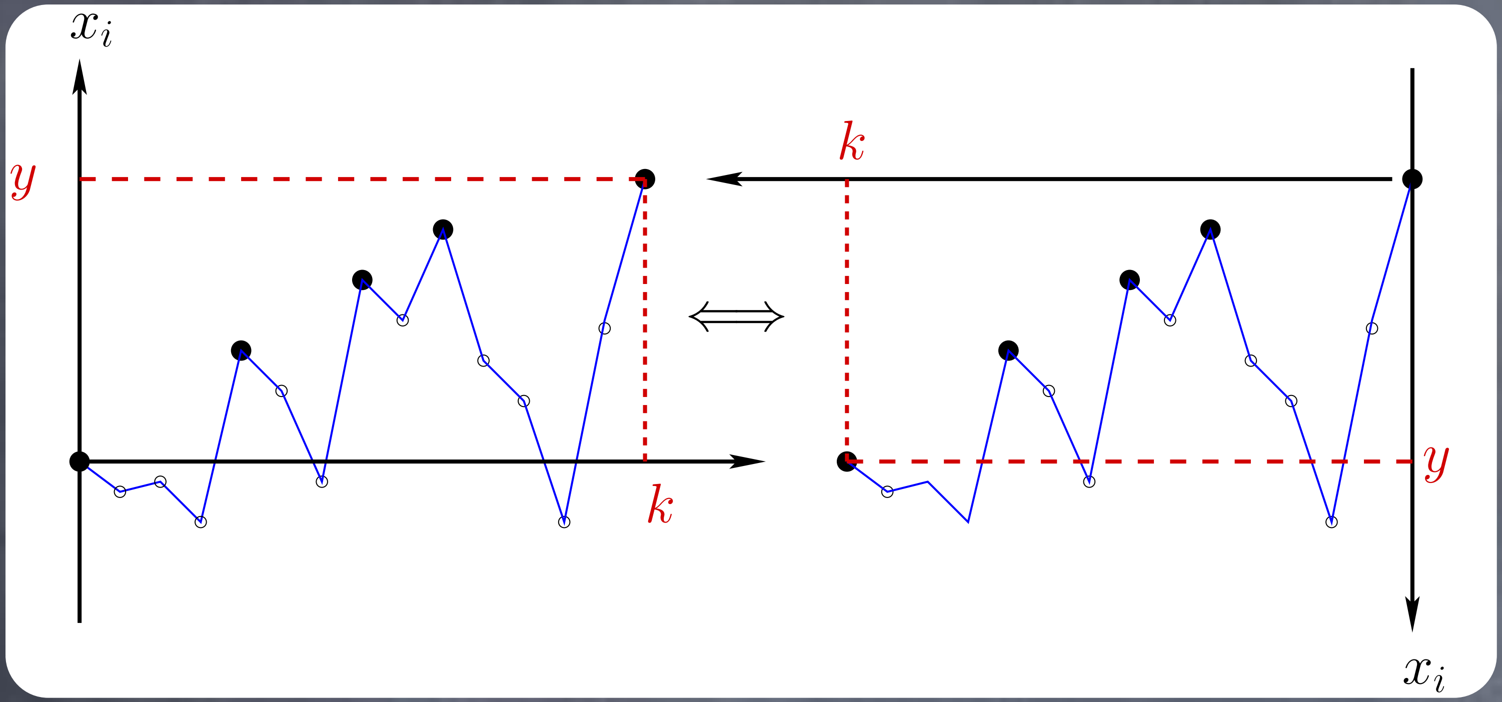

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$$\Longrightarrow r_k = q_-(k)$$

$=$ Proba. that the walker stays negative up to step k starting from the origin

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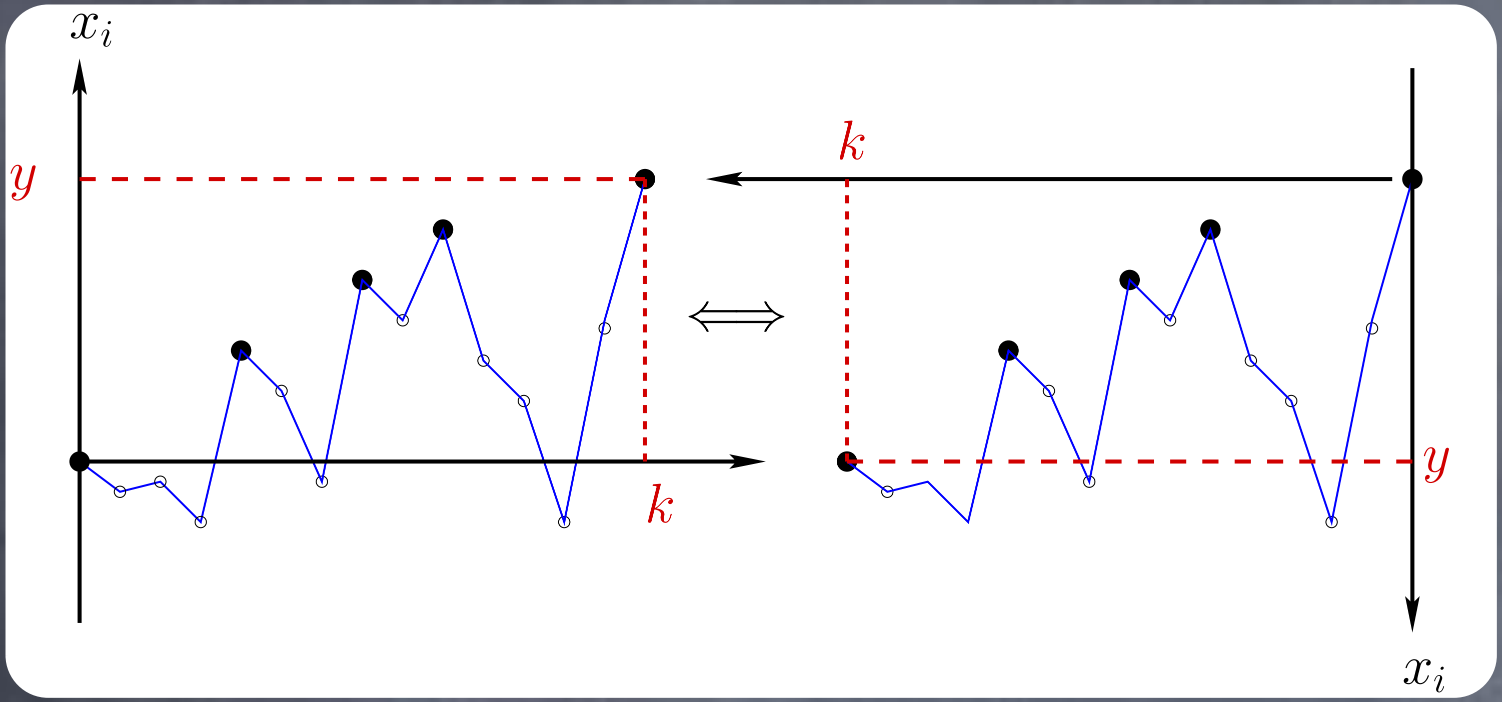

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For symmetric RW

$$\sum_{k=0}^{\infty} q_-(k) z^k = \frac{1}{\sqrt{1-z}} \Longrightarrow q_-(k) = \frac{1}{2^{2k}} \binom{2k}{k} \sim \frac{1}{\sqrt{\pi k}}$$

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Universal, i.e. independent of the jump distribution !

$$\langle R_n \rangle = \frac{2\,\Gamma(3/2+n)}{\sqrt{\pi}\, n!} \sim \frac{2}{\sqrt{\pi}} \sqrt{n}$$

Majumdar, Ziff `08

Record statistics of random walks with a drift

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RW with a drift $y_n = x_n + c\,n$

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(Generalized) Sparre Andersen theorem

$$\boxed{\sum_{k=0}^{\infty} q_-(k) z^k = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \Pr(y_k < 0) \right)}$$

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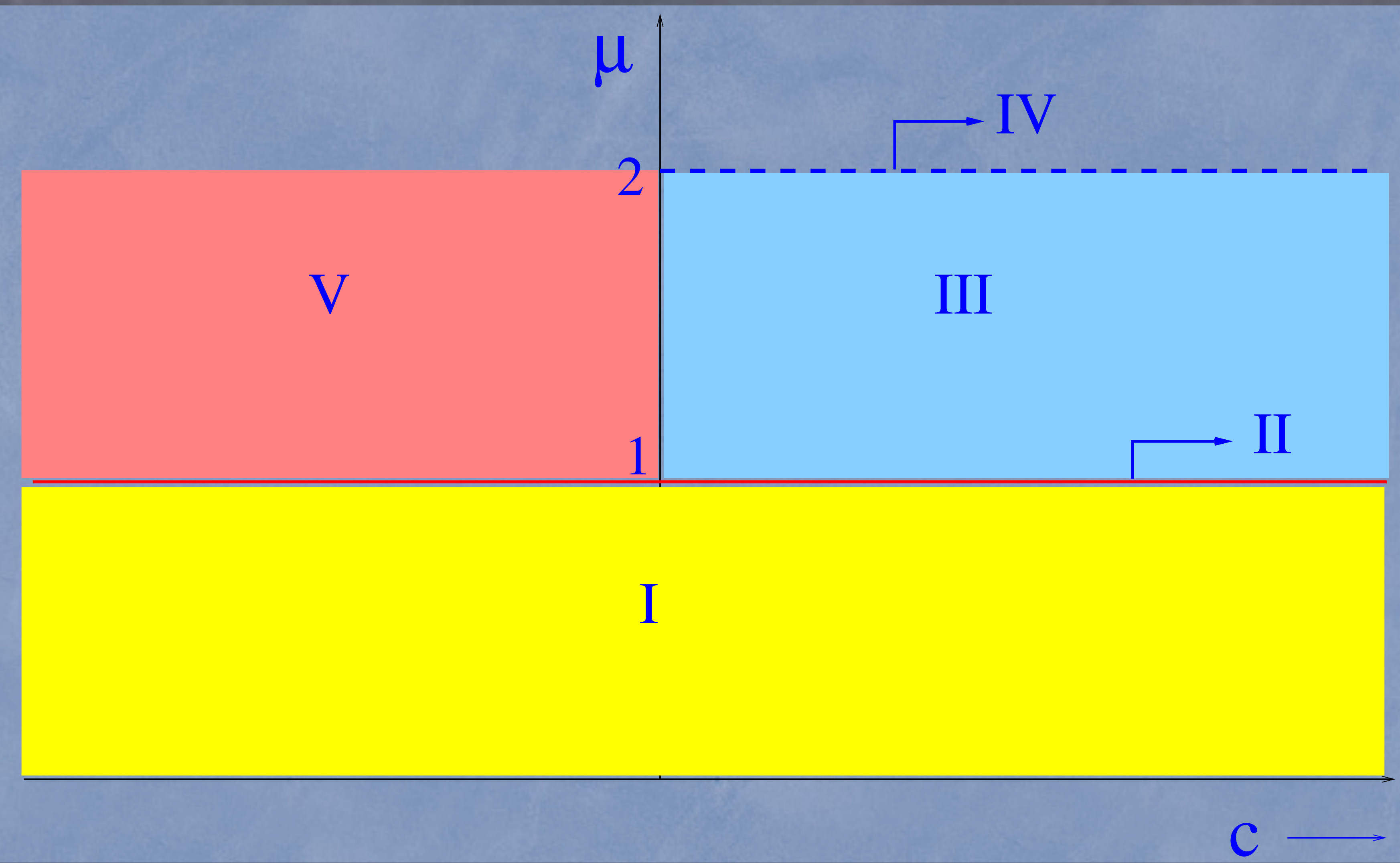
RW with a drift

Majumdar, G. S., Wergen `12



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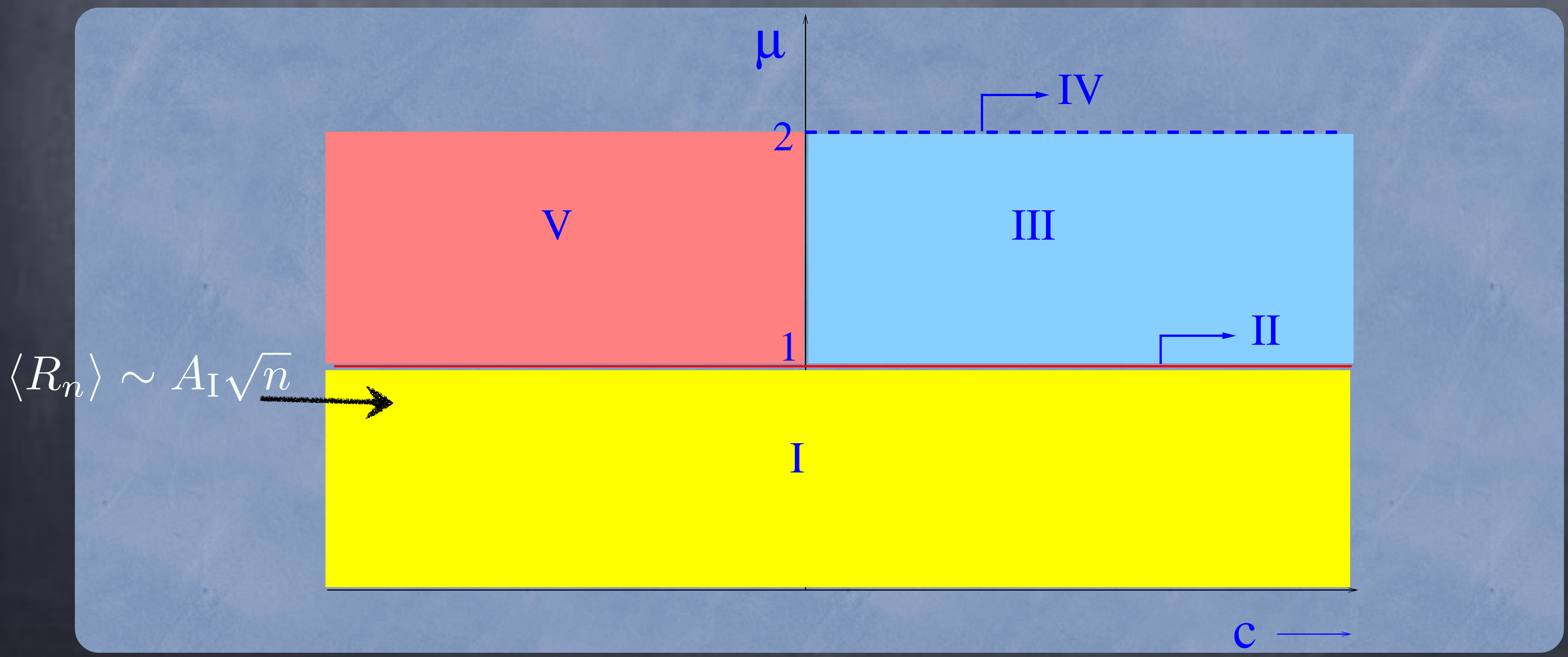

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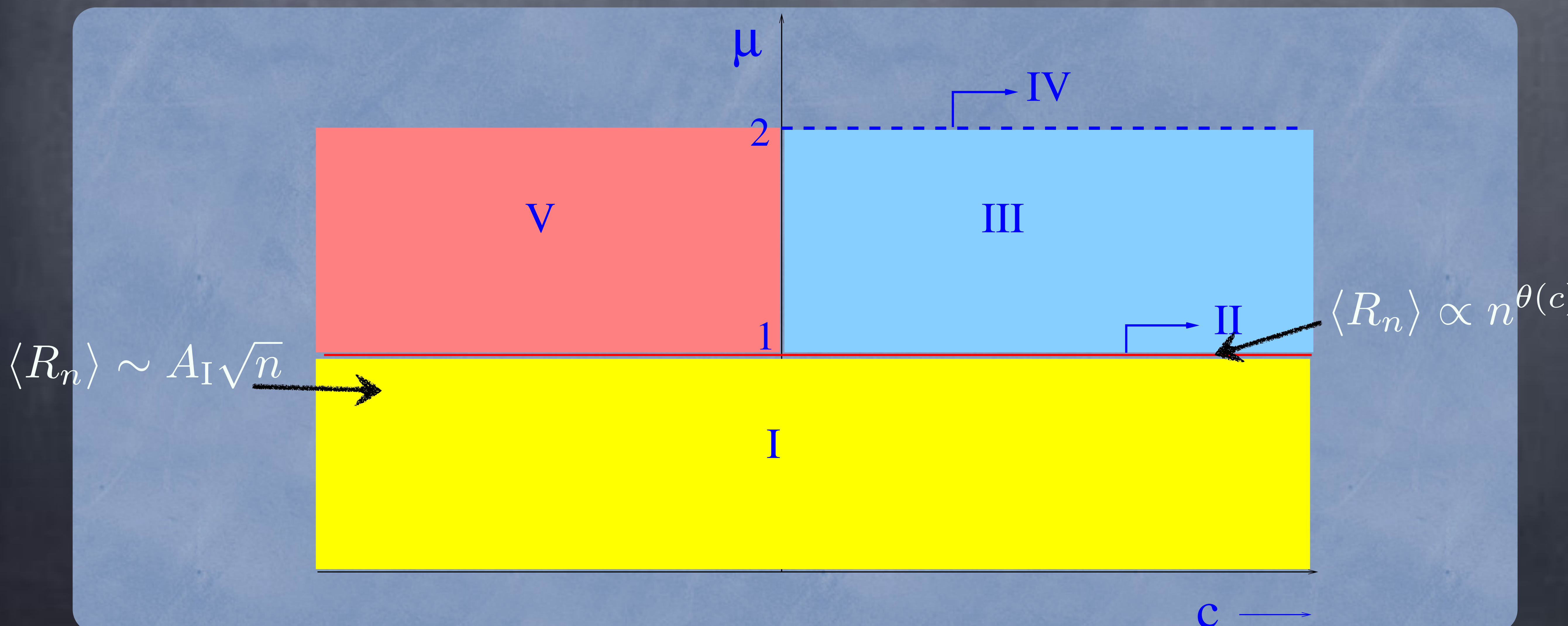

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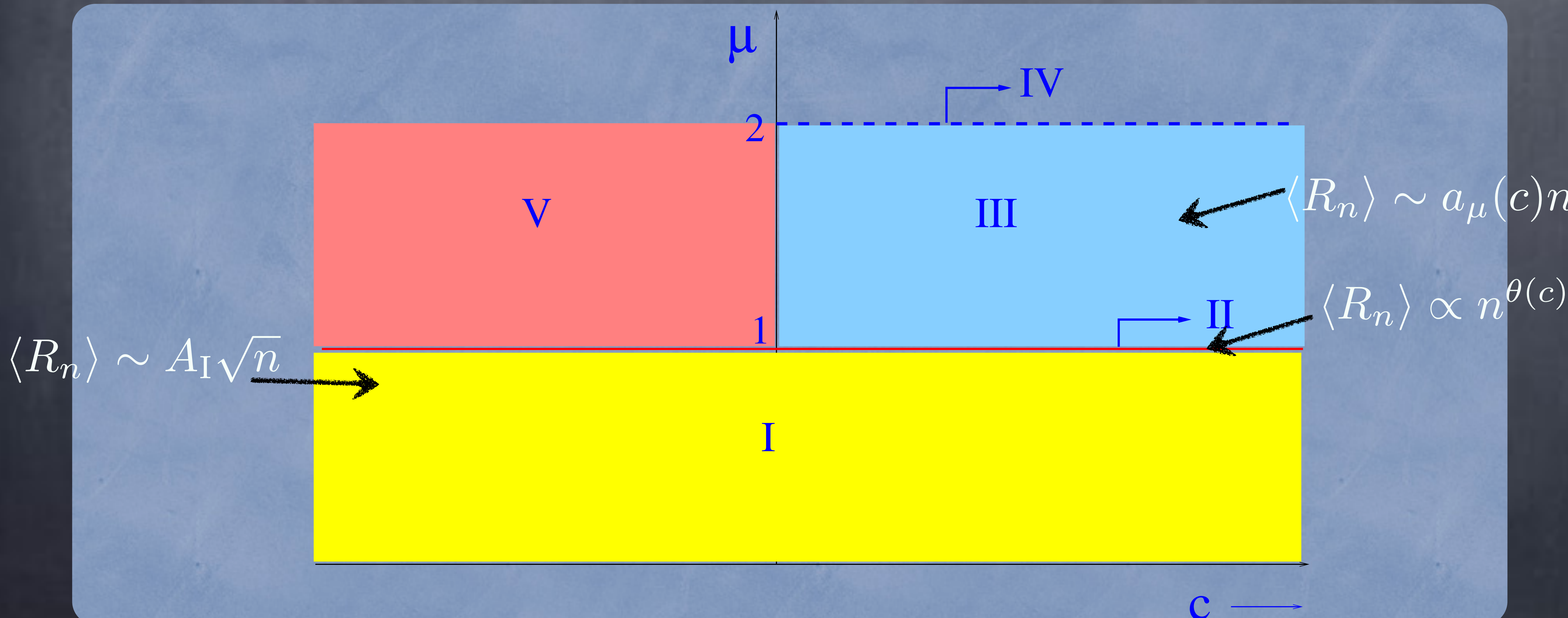

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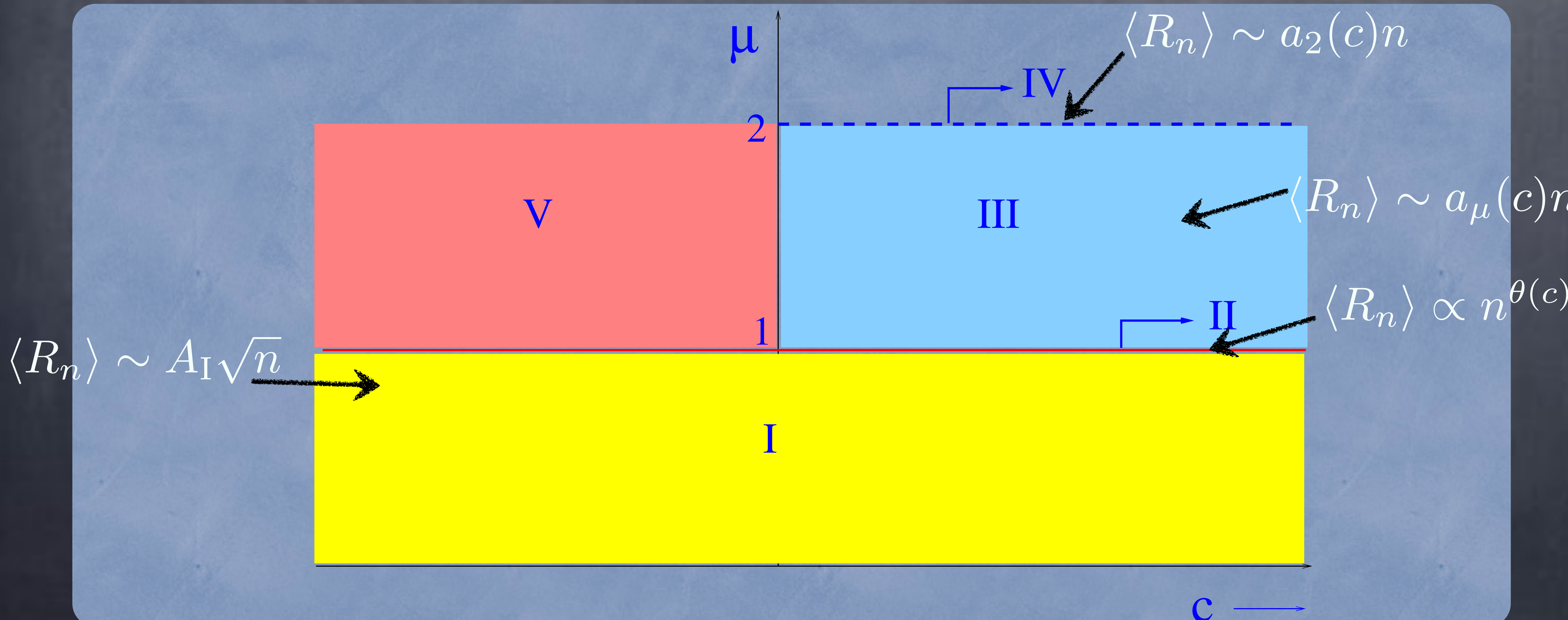

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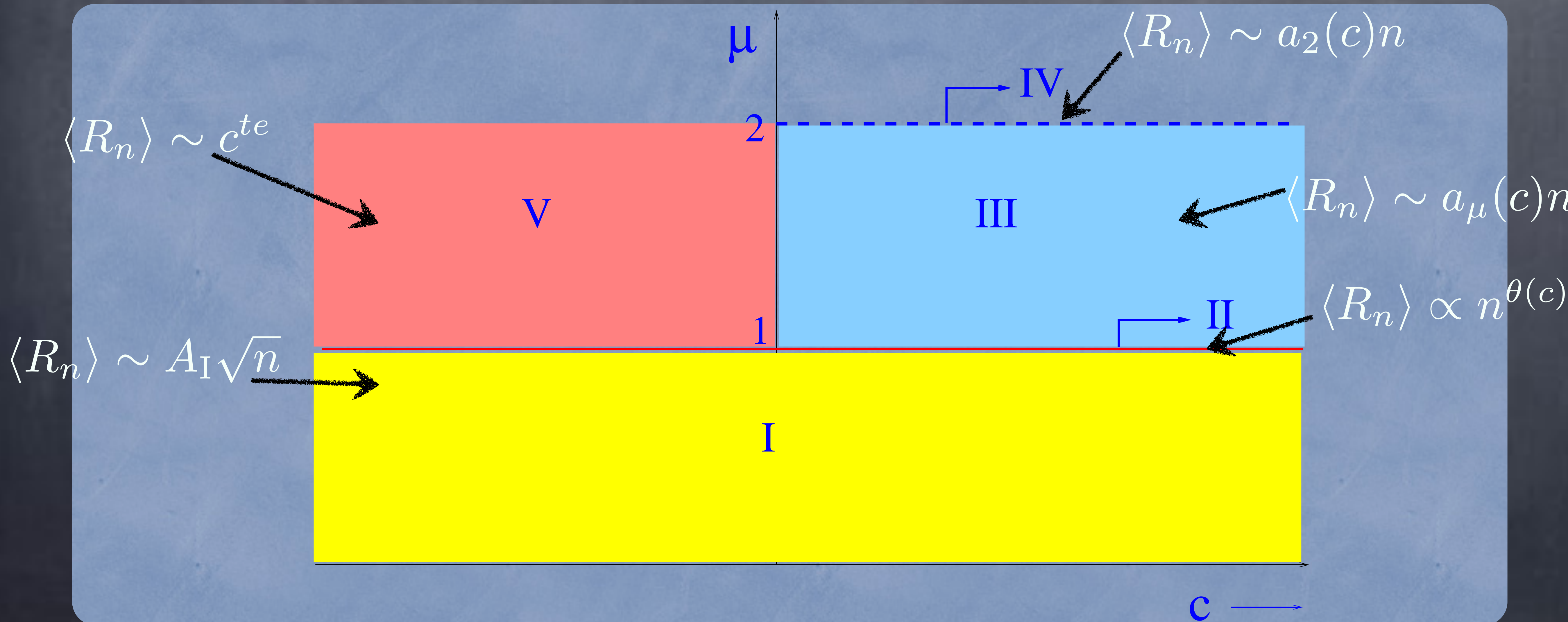

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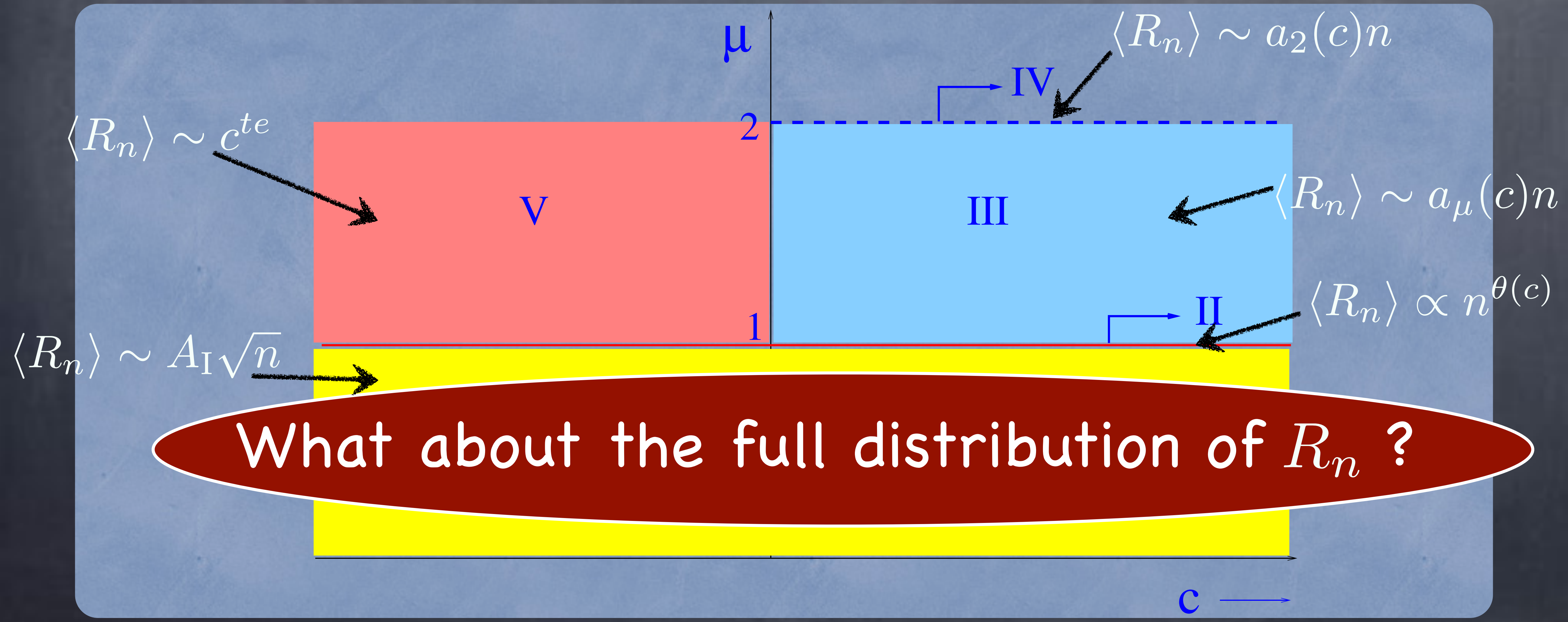
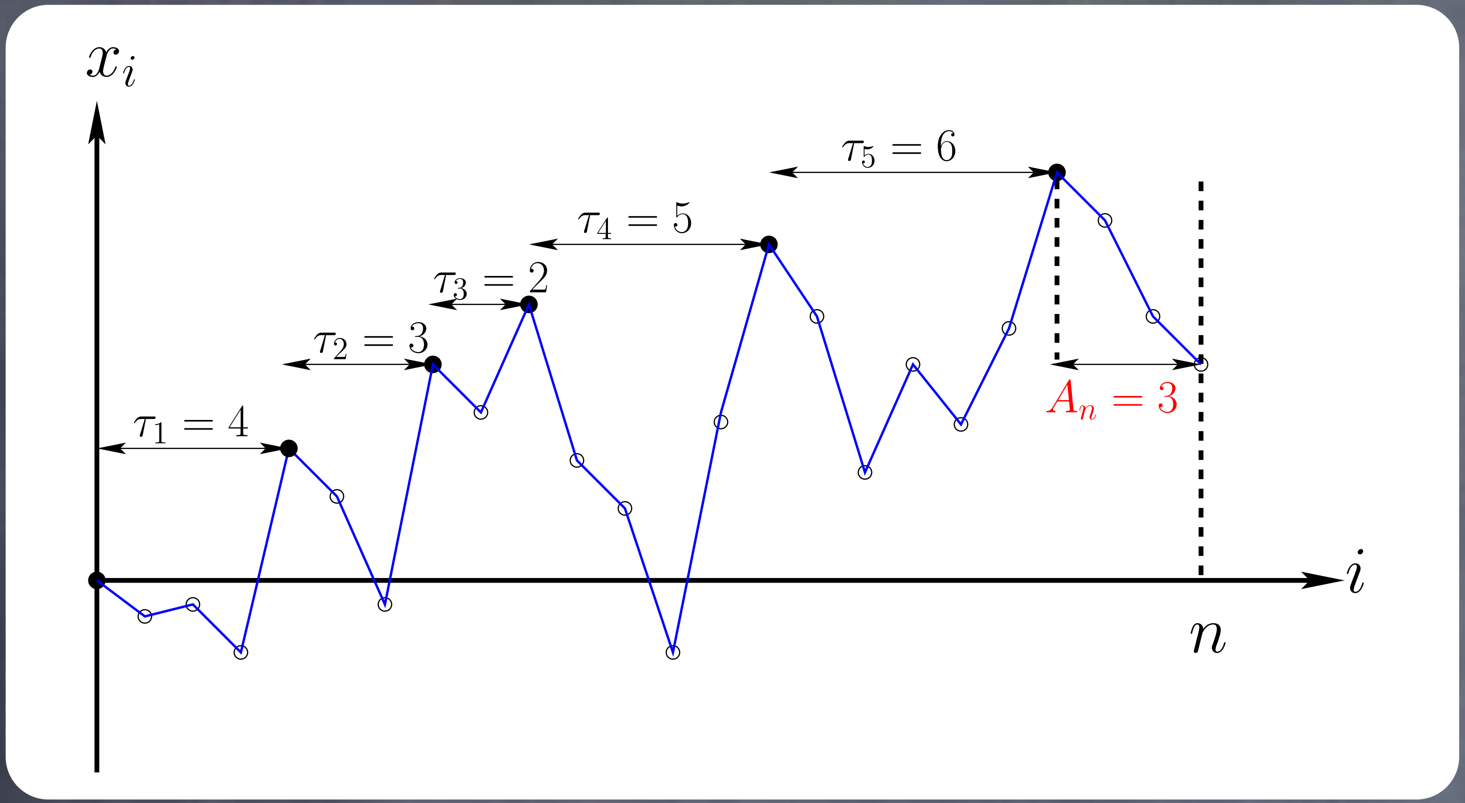


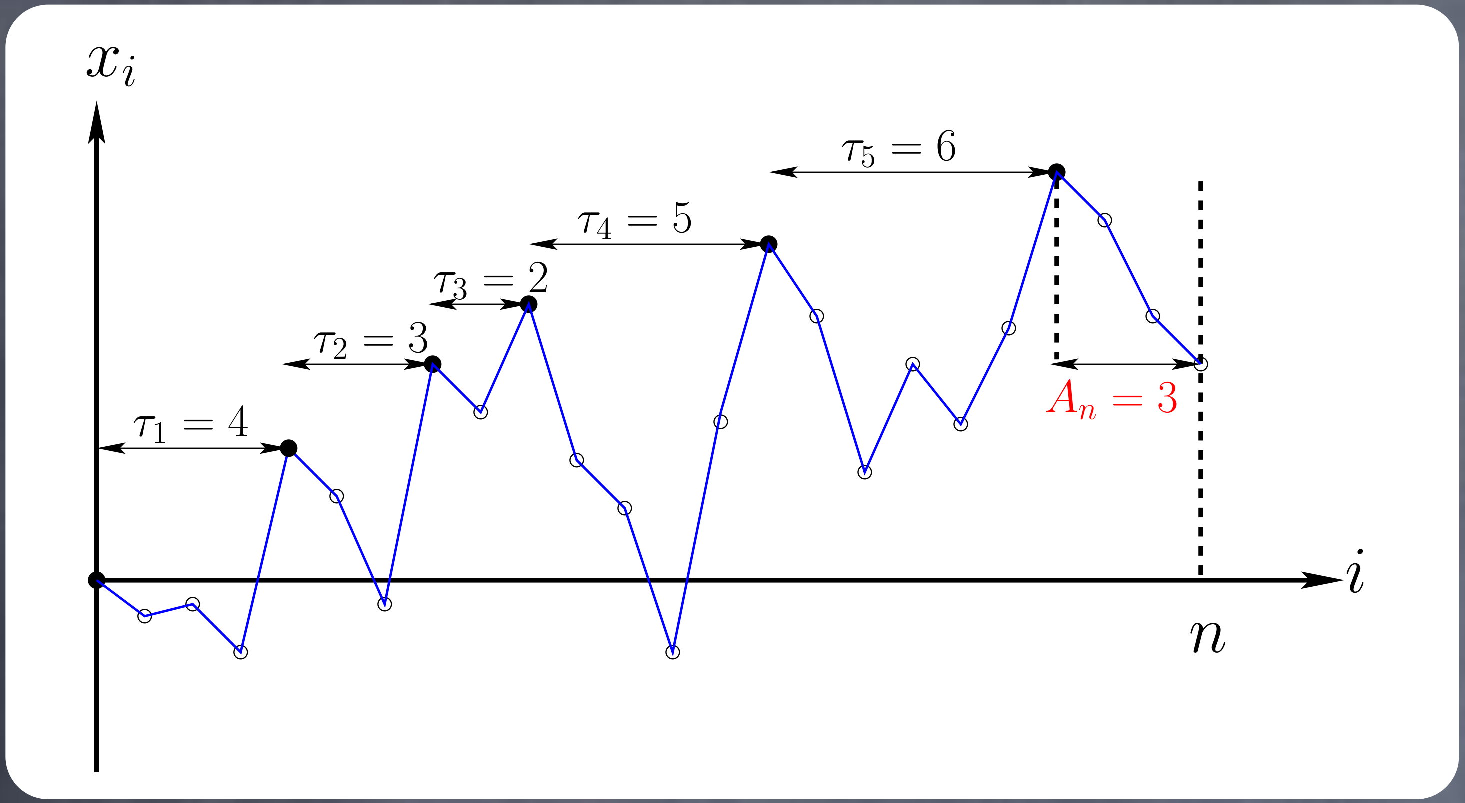
Renewal approach to records of RW



Joint distribution of

$$R_n, \tau_1, \tau_2, \cdots, \tau_{R_n - 1}, A_n \ ?$$

Renewal approach to records of RW



Joint distribution of

$R_n, \tau_1, \tau_2, \cdots, \tau_{R_n-1}, A_n$?

- RW is a Markov process $\Longrightarrow$ $\tau_1, \tau_2, \cdots, \tau_{R_n-1}, A_n$ are independent except for the global constraint

$$\sum_{i=1}^{R_n-1} \tau_i + A_n = n$$

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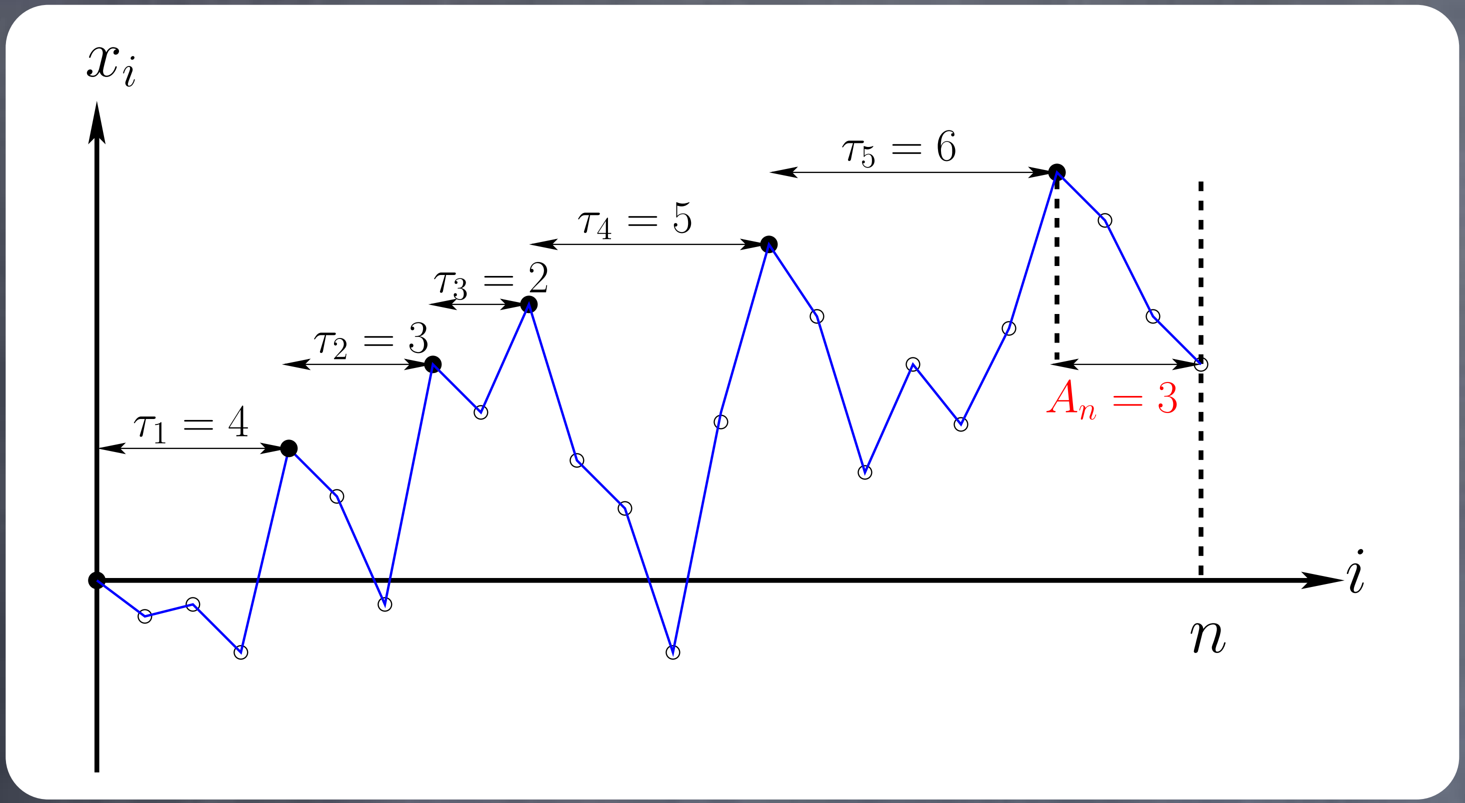

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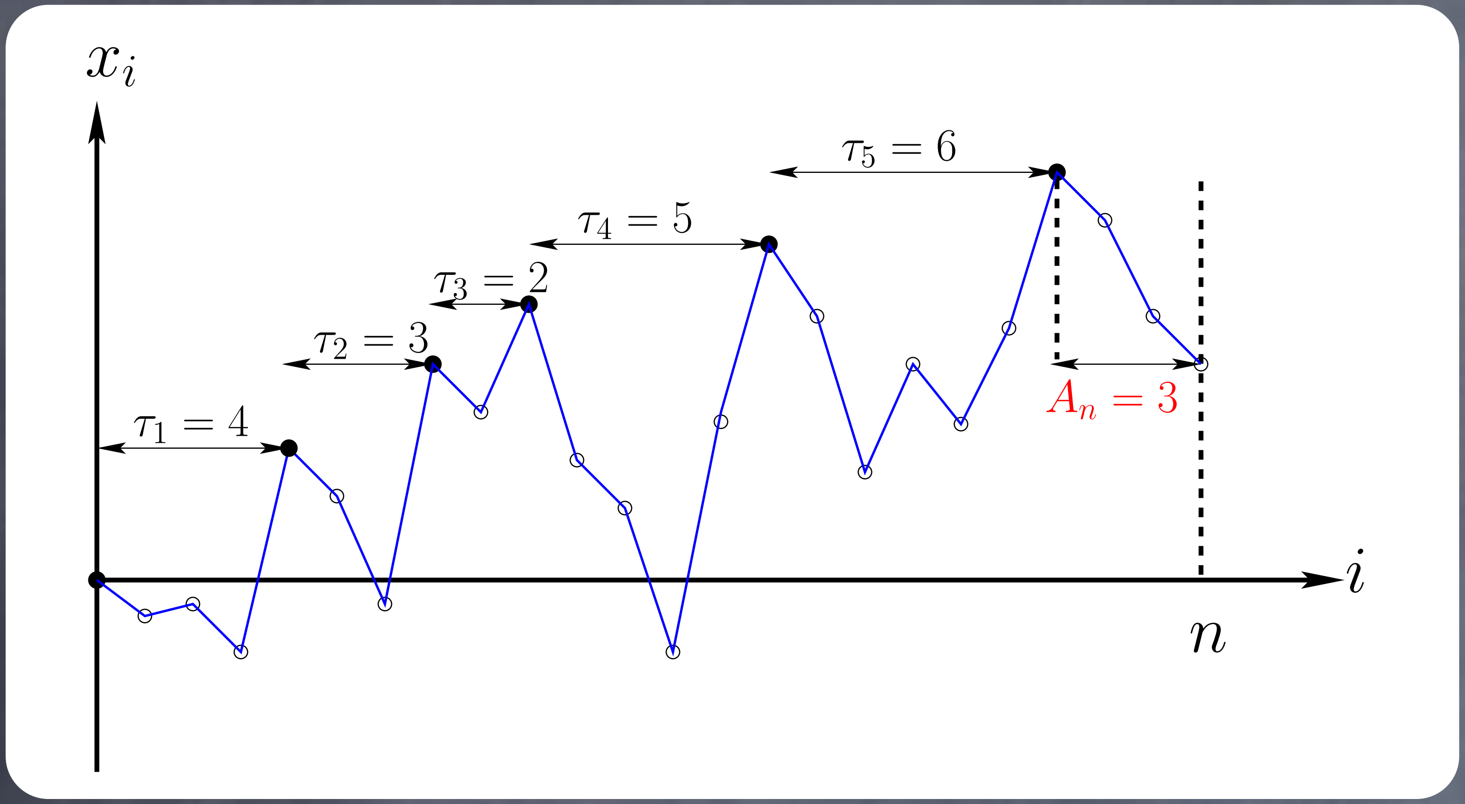
- RW is translationally invariant $\Longrightarrow$ τ_is are identical

while A_n has different statistics

Renewal approach to records of RW



Joint distribution of

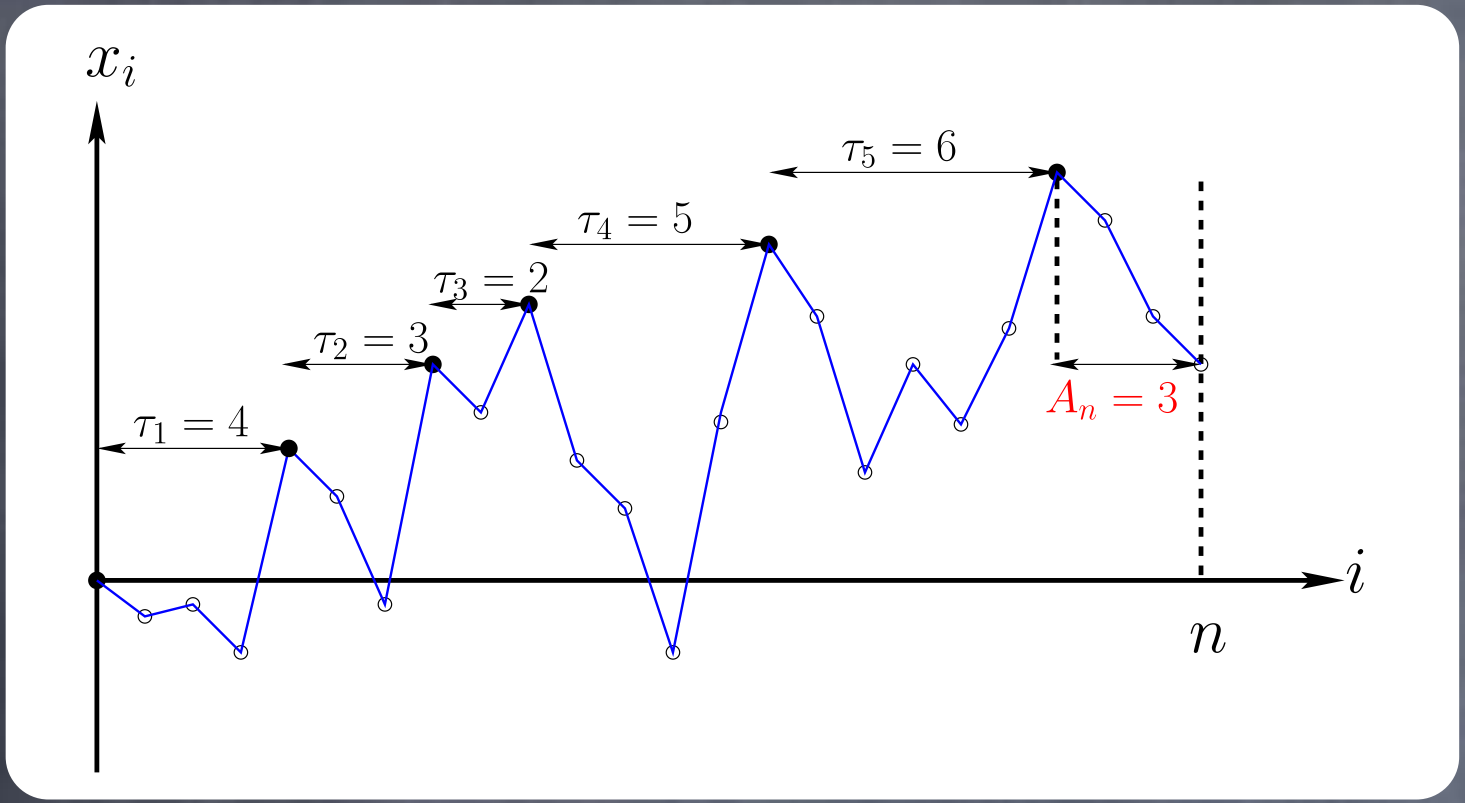
$$R_n, \tau_1, \tau_2, \cdots, \tau_{R_n - 1}, A_n \ ?$$

Two main objects:

- Persistence (or survival) probability
  $q_-(k) = \Pr(y_1 < y_0, y_2 < y_0, \cdots, y_k < y_0)$ indep. of y_0

- Distribution of first-passage time (from below)
  $f_-(k) = \Pr(y_1 < y_0, y_2 < y_0, \cdots, y_{k-1} < y_0, y_k > y_0)$
  $= q_-(k) - q_-(k-1)$ indep. of y_0

Renewal approach to records of RW



Joint distribution of

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$$\Pr(R_n = m, \tau_1 = \ell_1, \cdots, \tau_{m-1} = \ell_{m-1}, A_n = a) = P(\vec{\ell}, m, n)$$

$$\boxed{P(\vec{l}, m, n) = f_-(\ell_1) f_-(\ell_2) \cdots f_-(\ell_{m-1}) q_-(a) \delta\left(\sum_{k=1}^{m-1} \ell_k + a, n\right)}$$

first passage proba.

survival proba.

Proba. distribution of the number of records

$$P(m,n) = \Pr.(R_n = m) = \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \cdots \sum_{\ell_{m-1}=1}^{\infty} \sum_{a=0}^{\infty} P(\vec{l}, m, n)$$

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Generating function w.r.t. the number of steps

$$\sum_{n=0}^{\infty} P(m,n)z^n = \left(\sum_{\ell\geq 1} z^\ell f_-(\ell)\right)^{m-1} \sum_{a\geq 0} z^a q_-(a)$$

$$= \left[\tilde{f}_-(z)\right]^{m-1} \tilde{q}_-(z)$$

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(for symmetric jumps) $= \left[1-\sqrt{1-z}\right]^{m-1} \dfrac{1}{\sqrt{1-z}}$

Proba. distribution of the number of records

- By ``inverting'' the GF (for symmetric jumps):

Majumdar, Ziff `08

$$P(m,n) = \binom{2n-m+1}{n} 2^{-2n+m-1} \ , \ m \leq n+1$$

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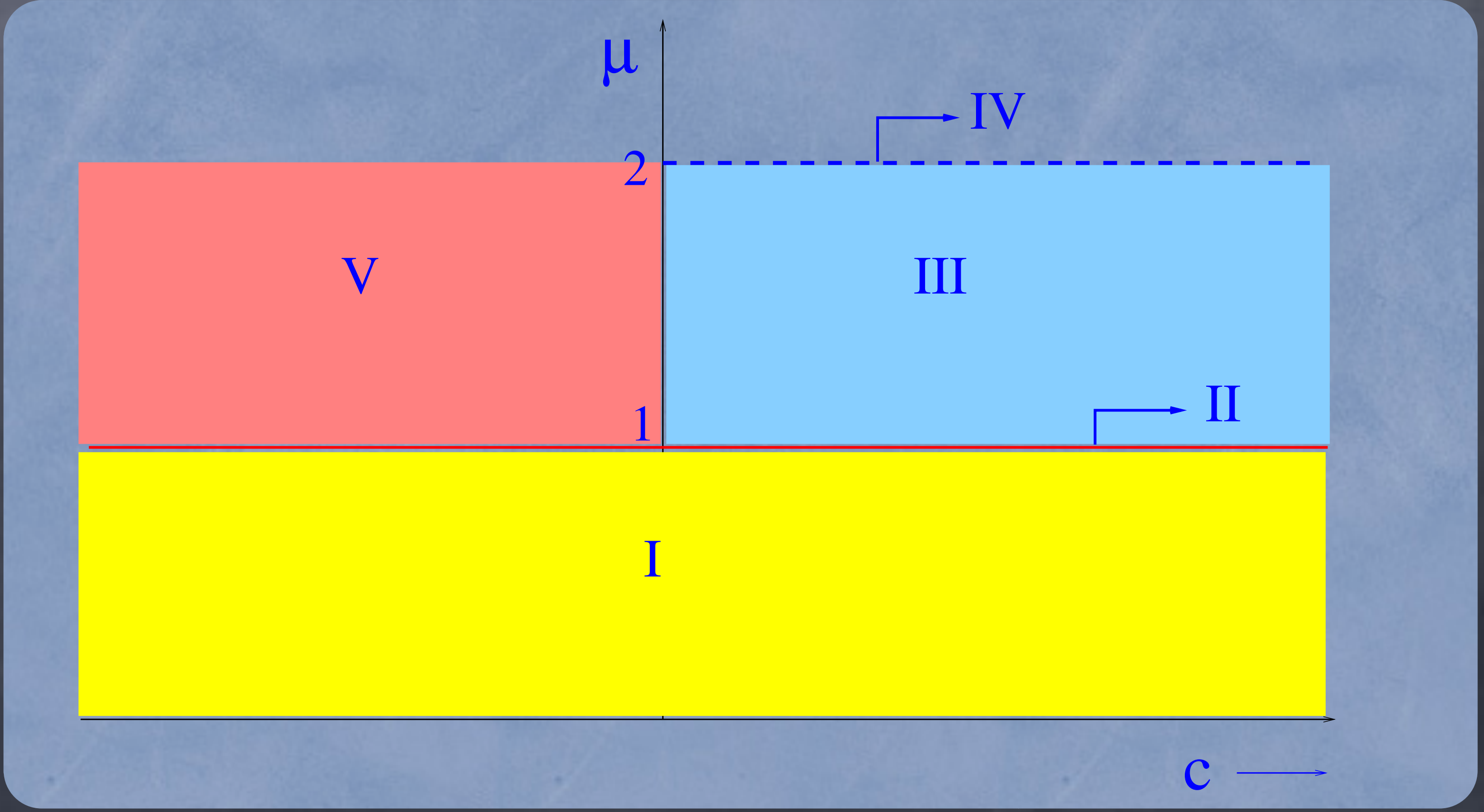
- RW with a drift

Majumdar, G. S., Wergen `12



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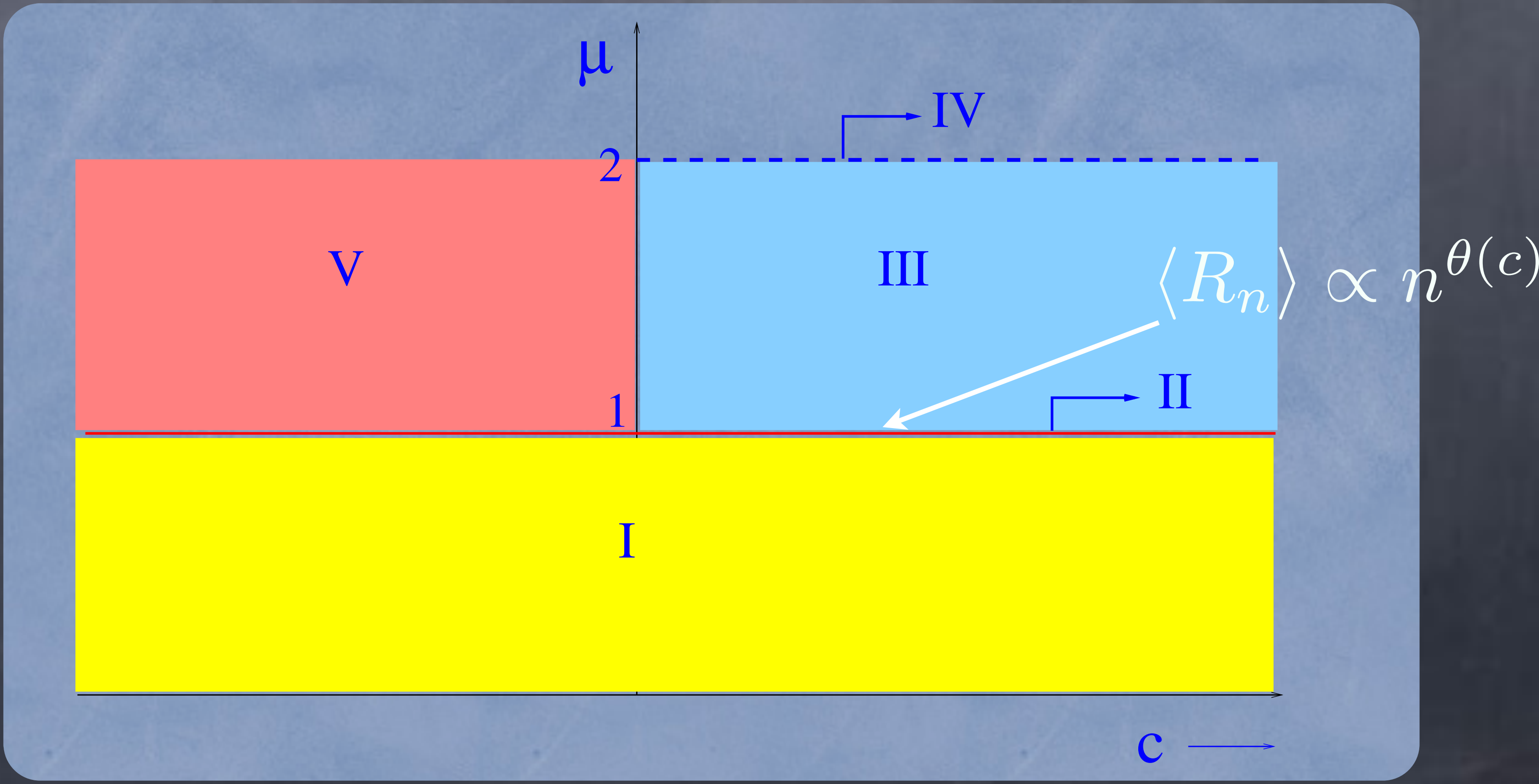

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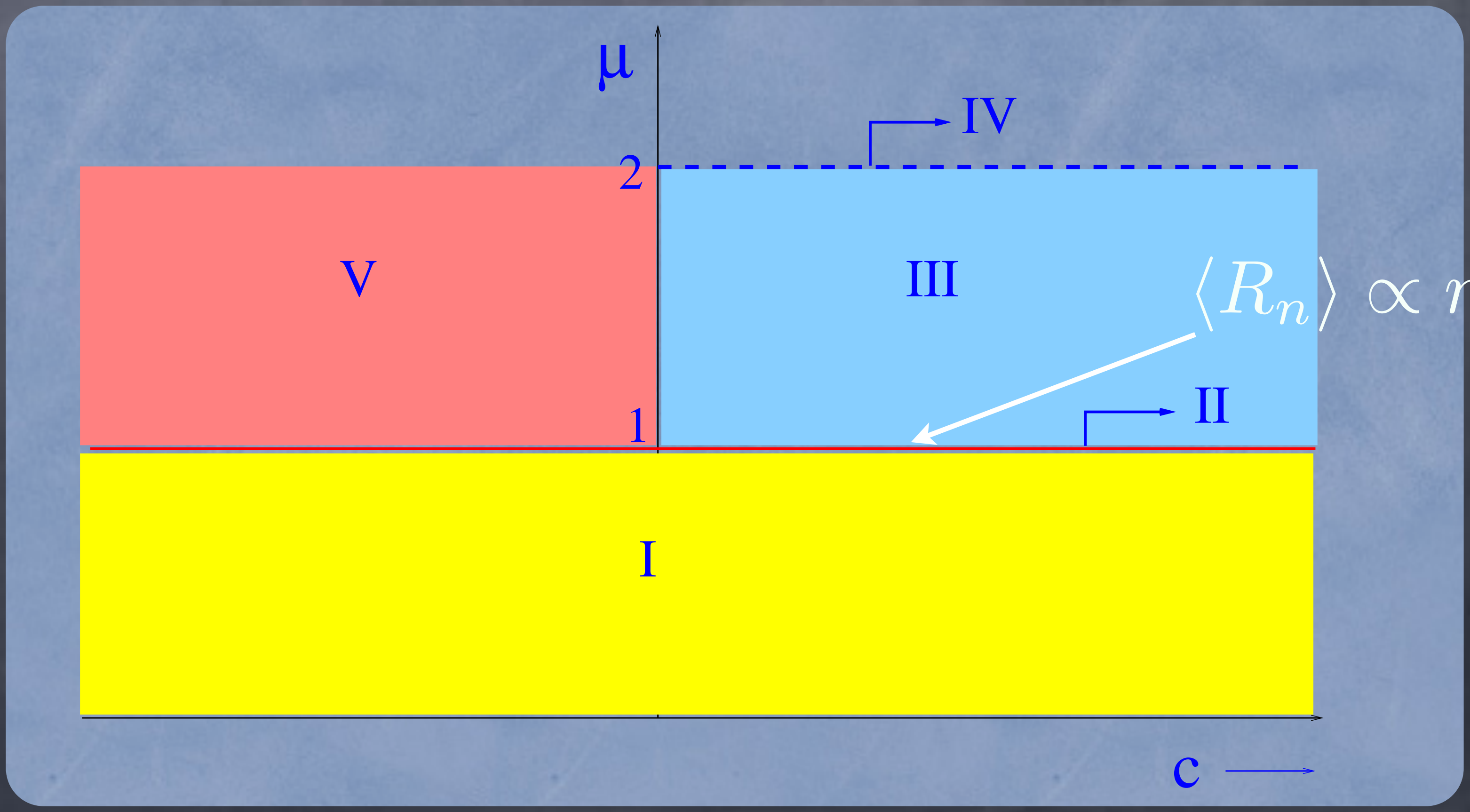
- RW with a drift   Majumdar, G. S., Wergen '12

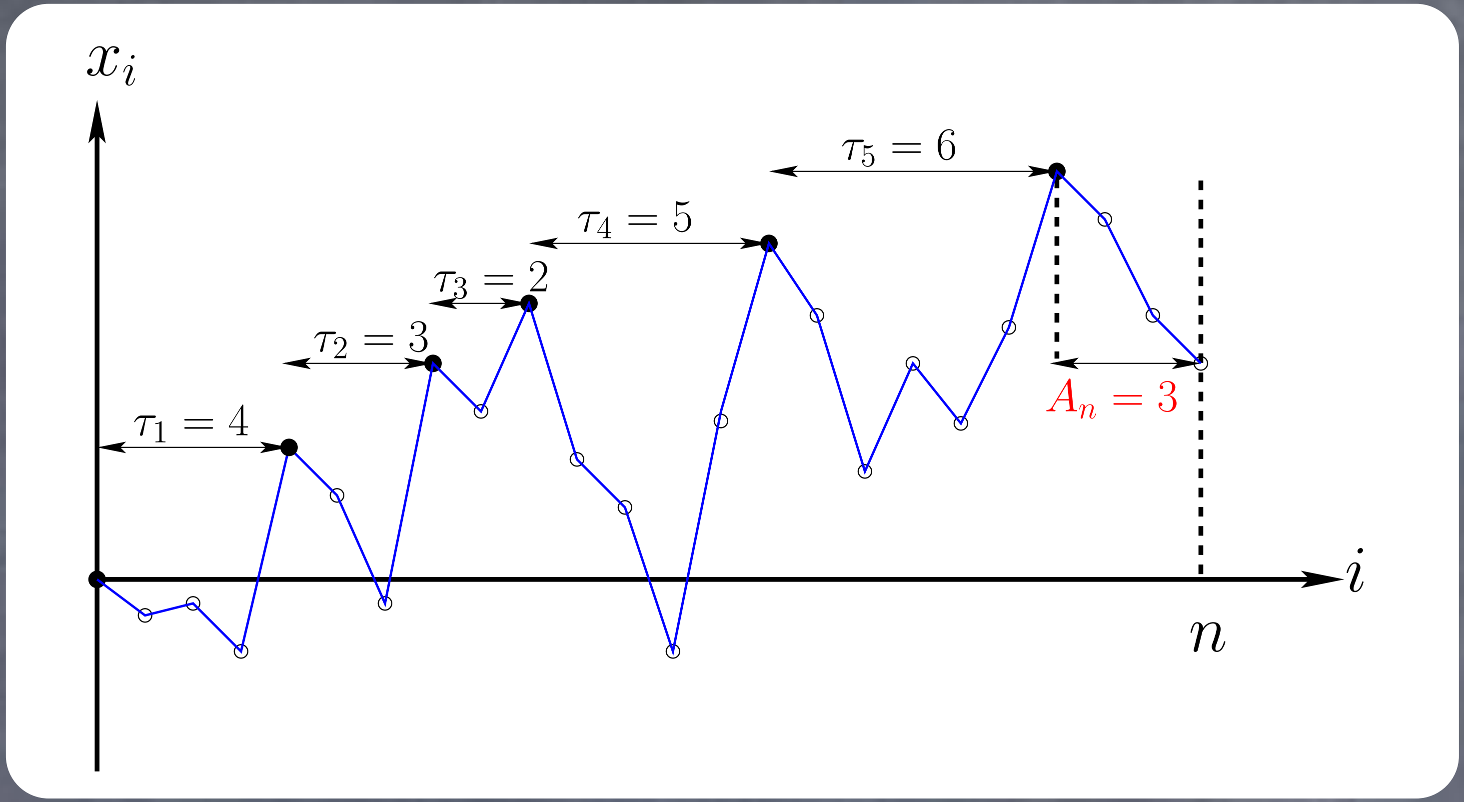
$$P(m,n) = \frac{1}{n^{\theta(c)}} g_c\left(\frac{m}{n^{\theta(c)}}\right)$$

e.g. for $\mu = 1 \; , \; \theta(c) = 1/3$

$$g_c(x) = 3^{2/3} \mathrm{Ai}\left(\frac{x}{3^{1/3}}\right)$$



Statistics of the ages of records



sym. RW

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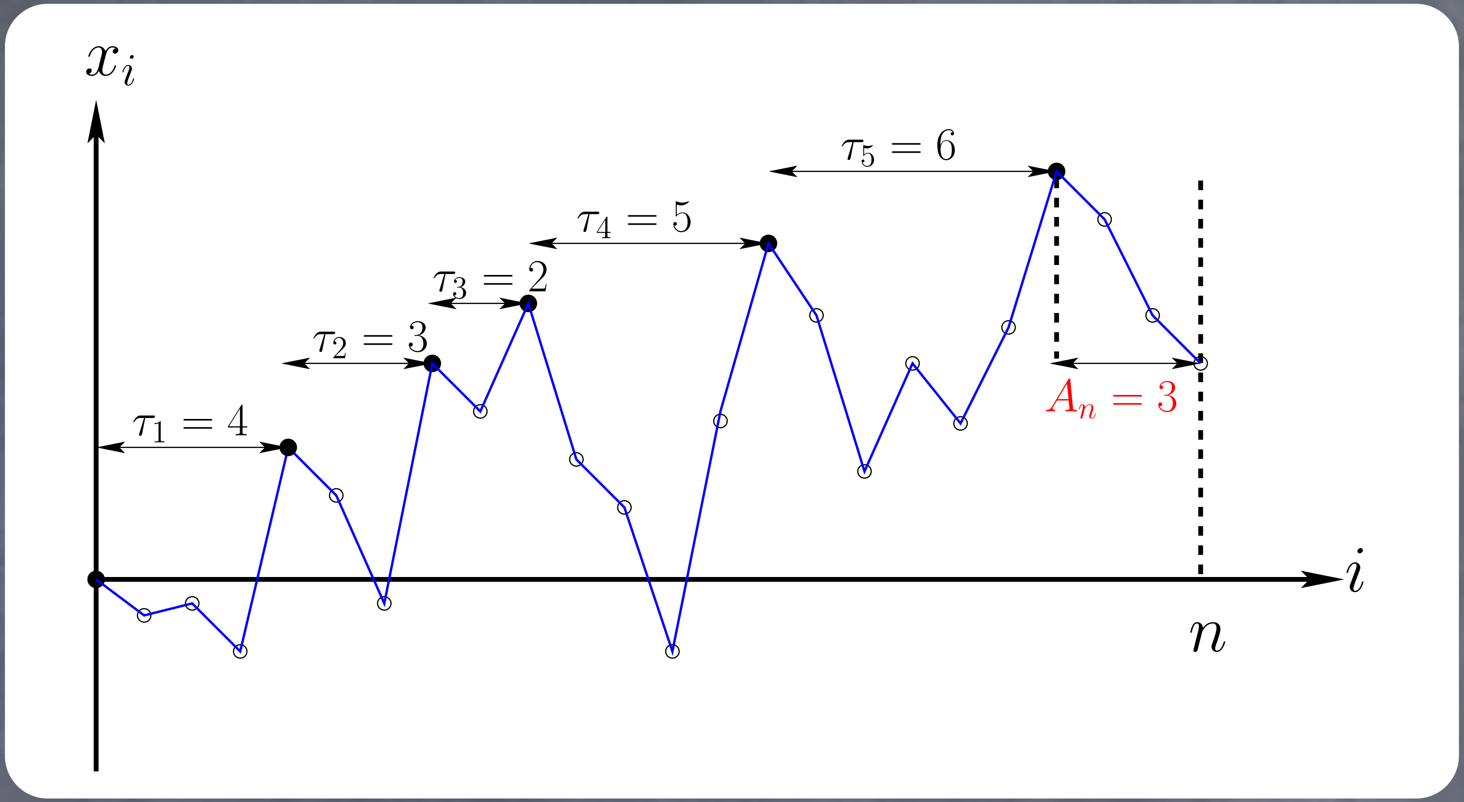

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- Typical age of a record: $$\ell_{\text{typ}} \sim \frac{n}{\langle R_n \rangle} \sim \sqrt{n}$$

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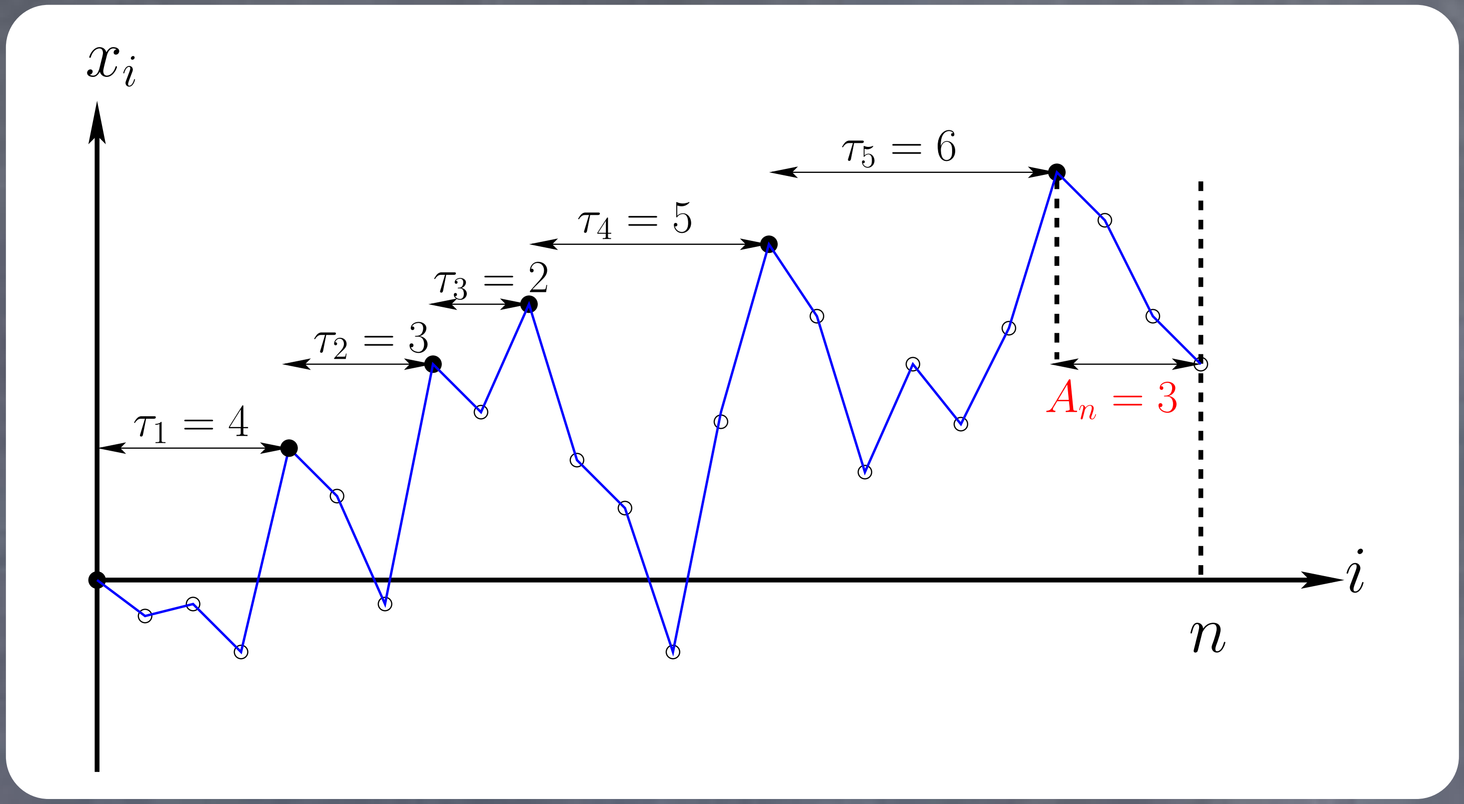

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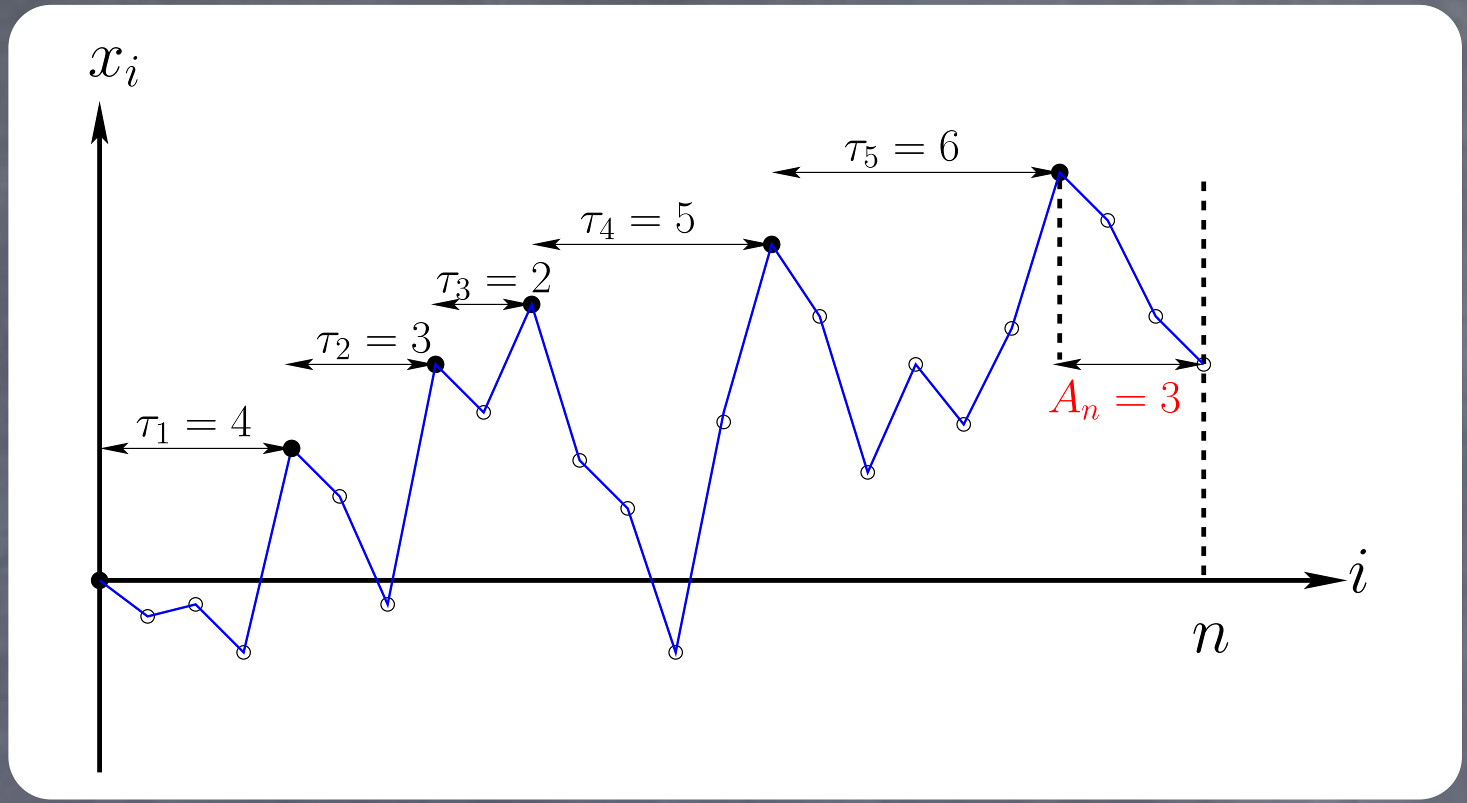

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Godrèche, Majumdar, G. S., `14

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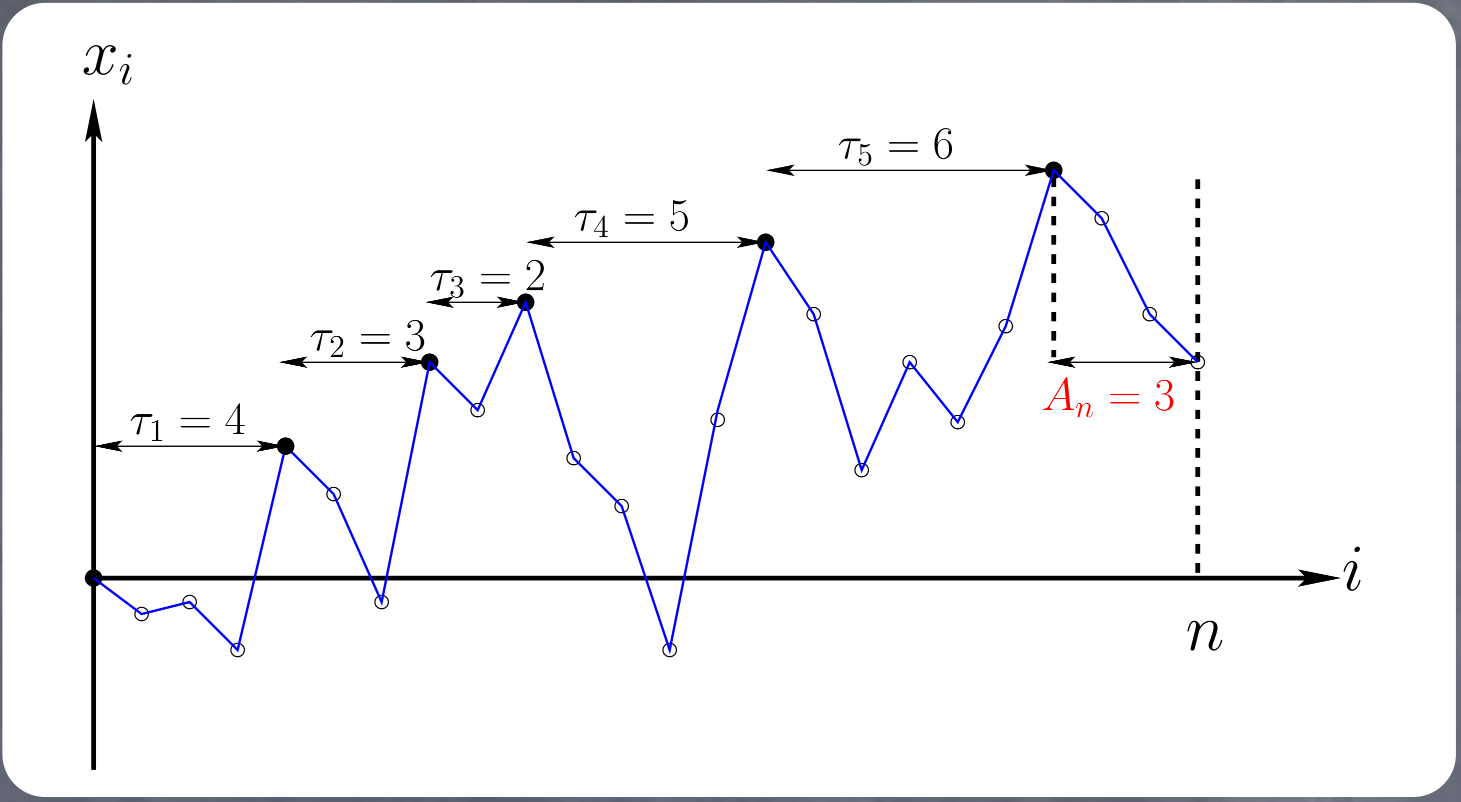

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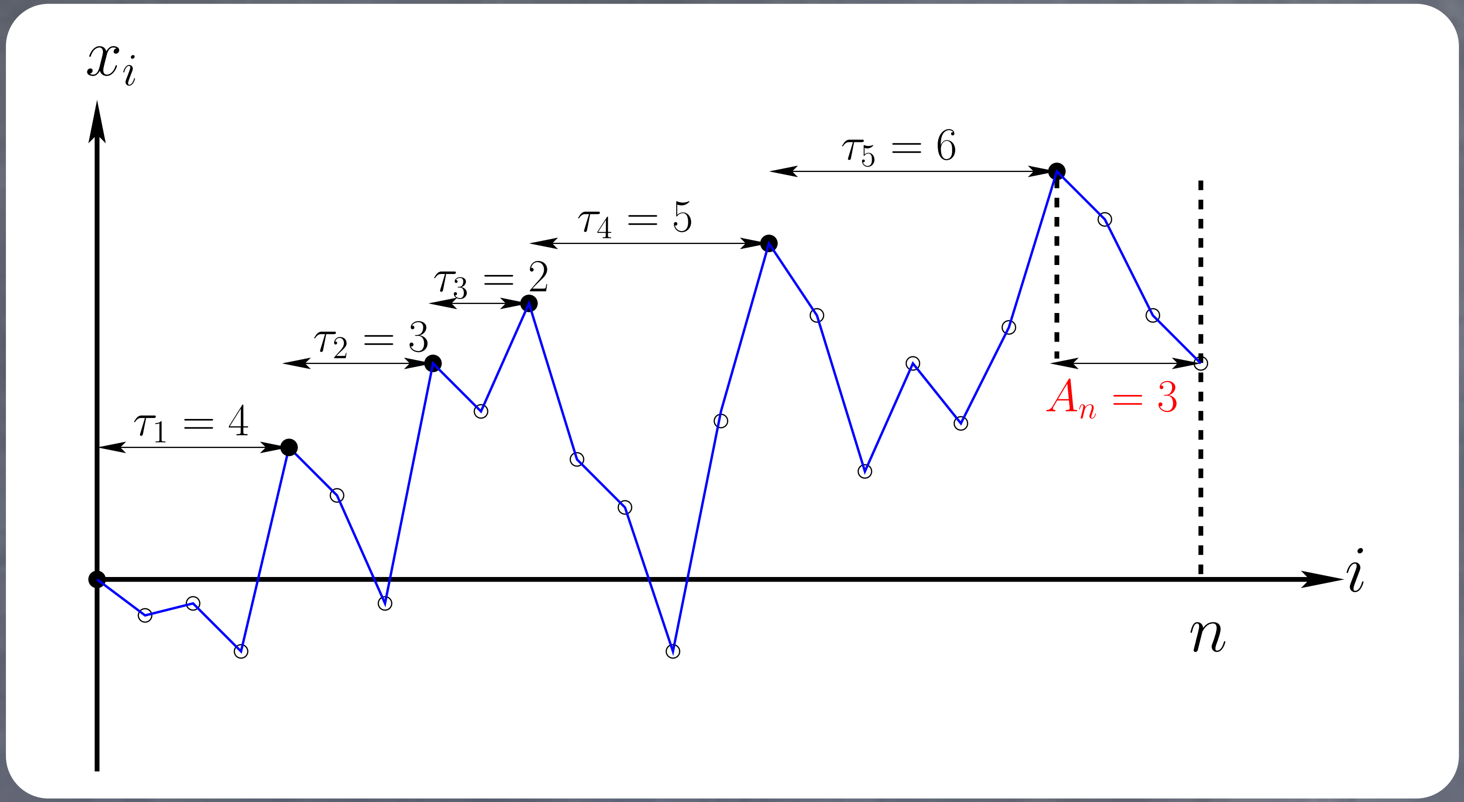

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$$Q(n) = \Pr\left[A_n \geq \max(\tau_1, \tau_2, \cdots, \tau_{m-1})\right]$$ = ?

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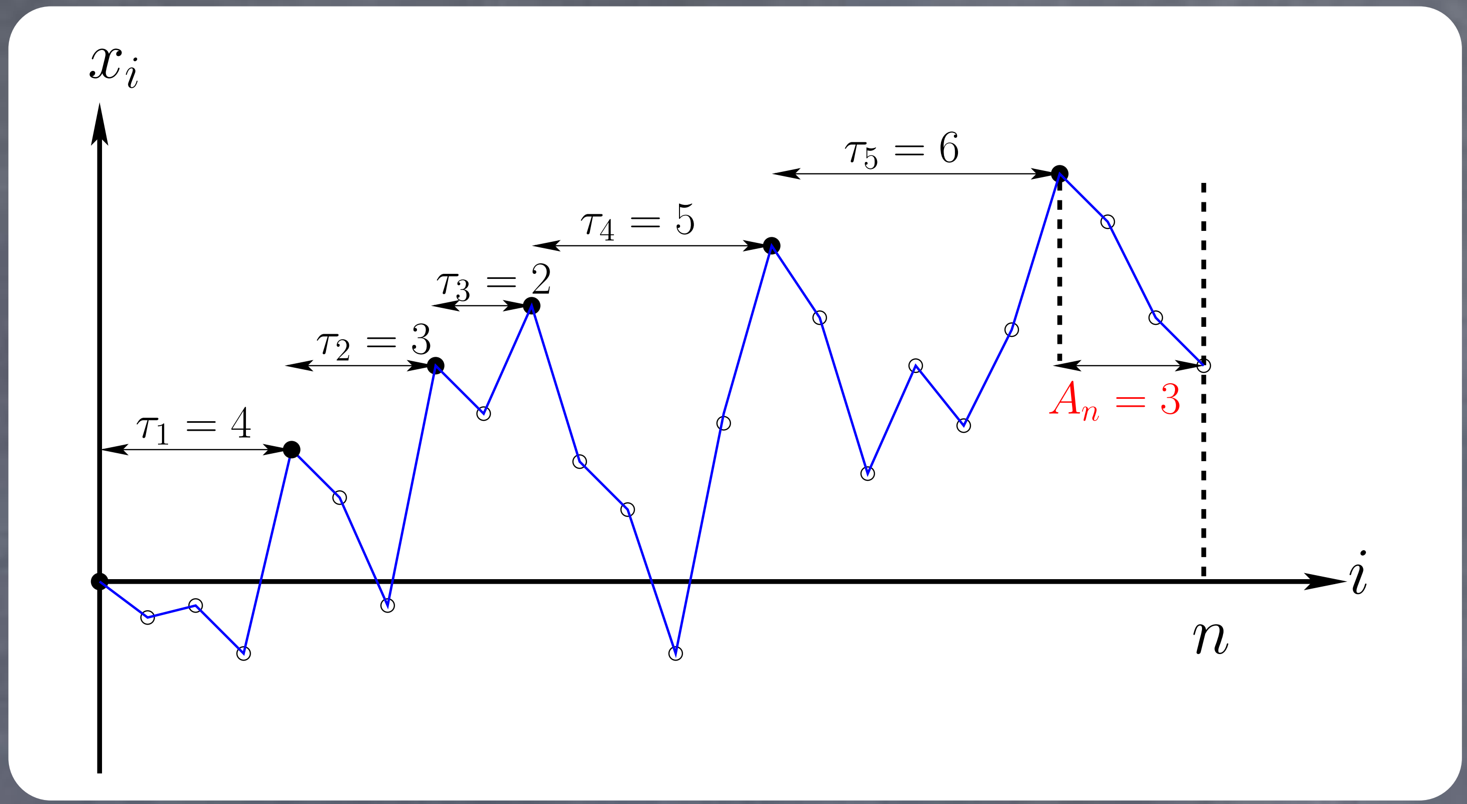

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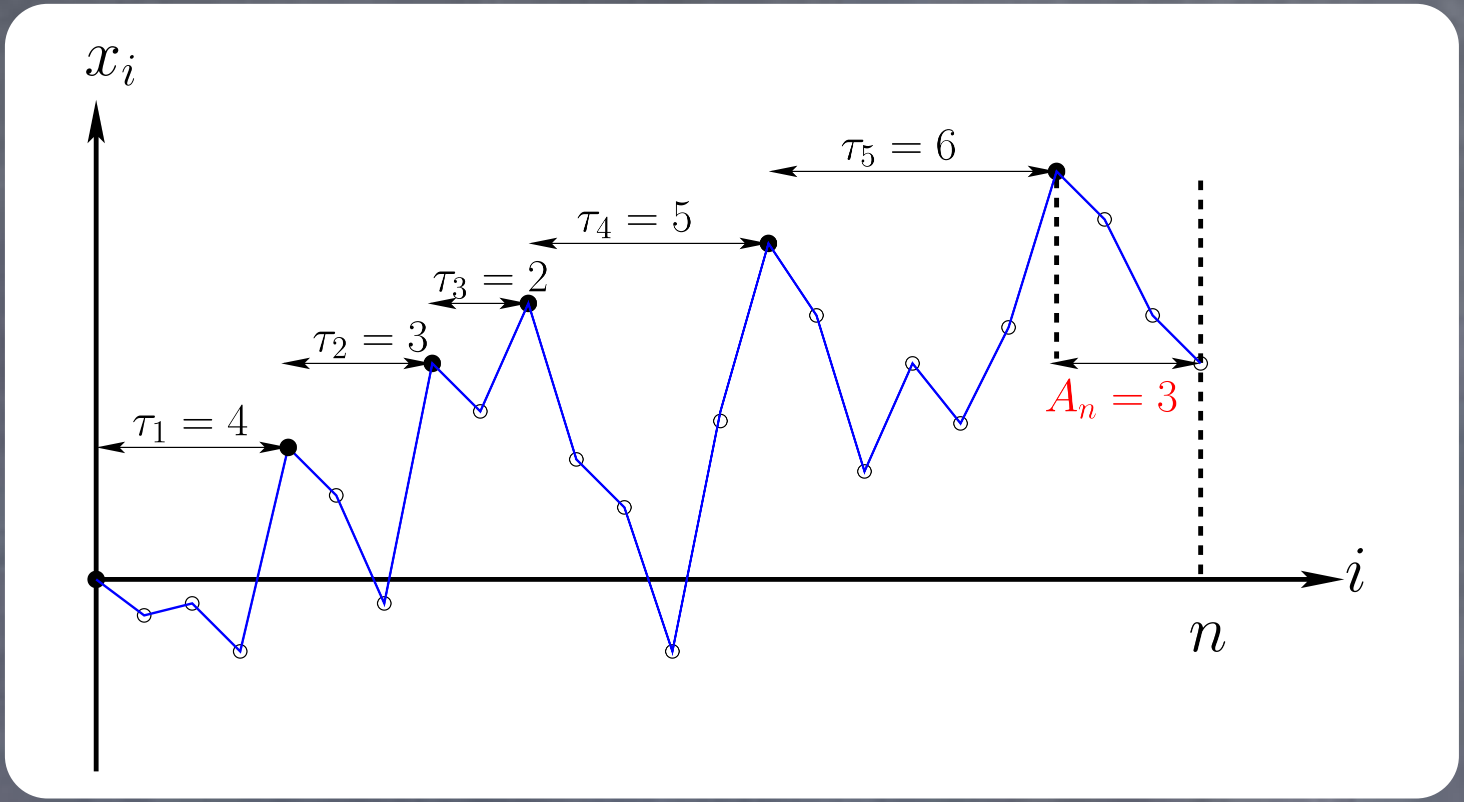

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$$q_-(k) = \frac{1}{2^{2k}}\binom{2k}{k}, \; f_-(k) = q_-(k) - q_-(k-1)$$

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One finds

$$\sum_{n\geq 0} z^n Q(n) = 1 + \frac{1}{2}z + \frac{5}{8}z^2 + \frac{5}{8}z^3 + \frac{81}{128}z^4 + \frac{5}{8}z^5 + \frac{161}{256}z^6 + \cdots$$

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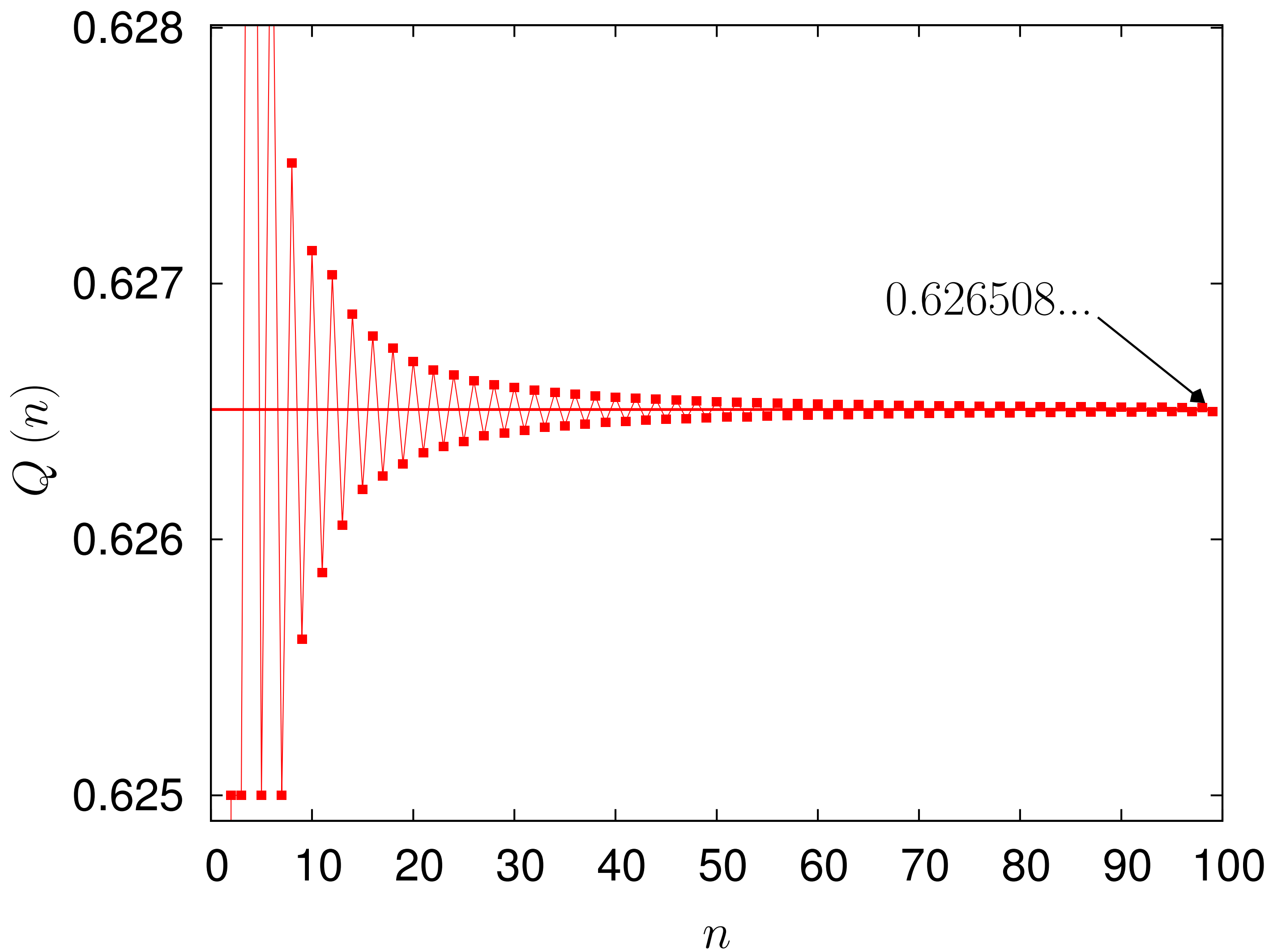

Godrèche, Majumdar, G. S.,`14

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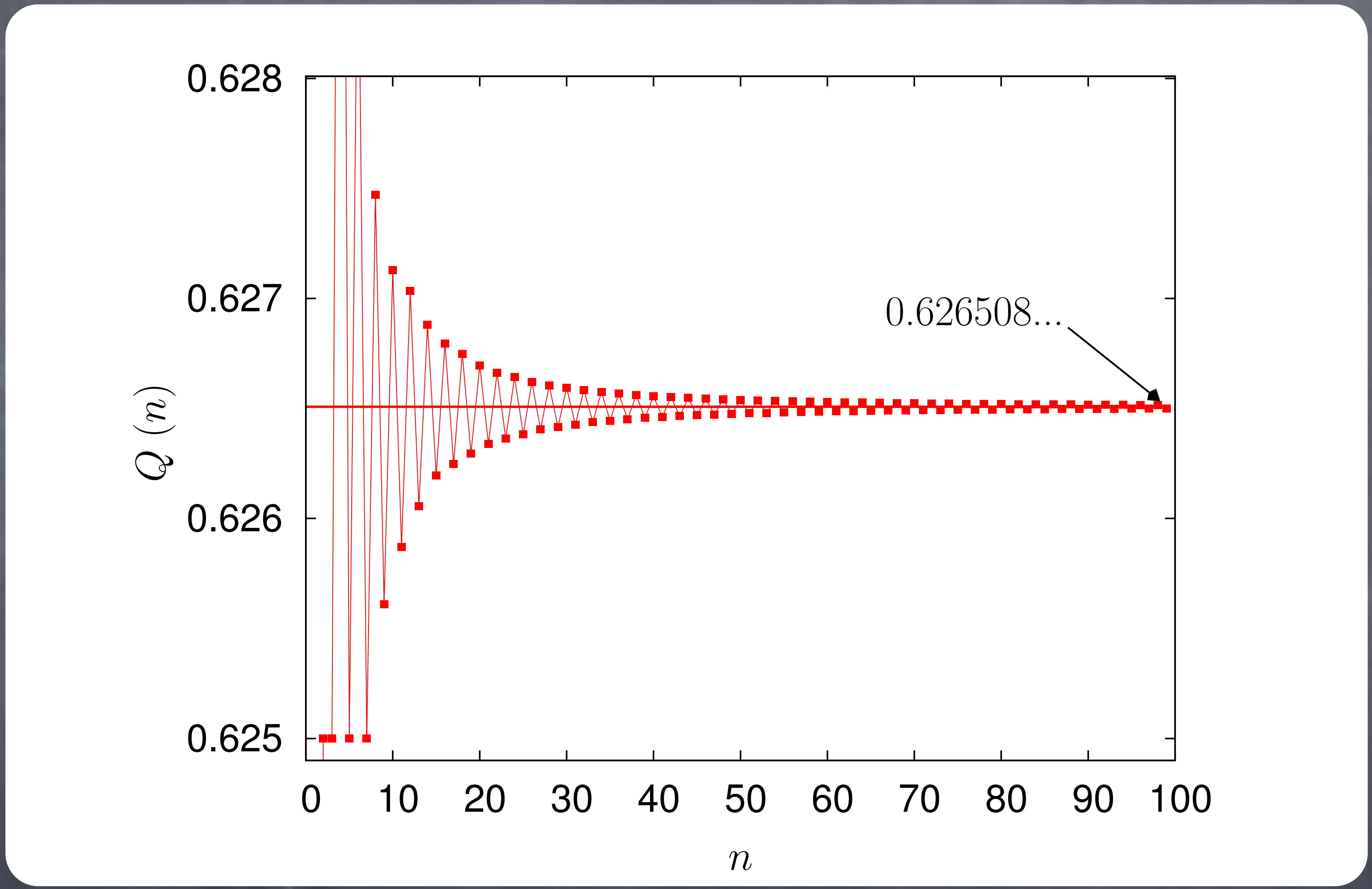

Godrèche, Majumdar, G. S.,`14

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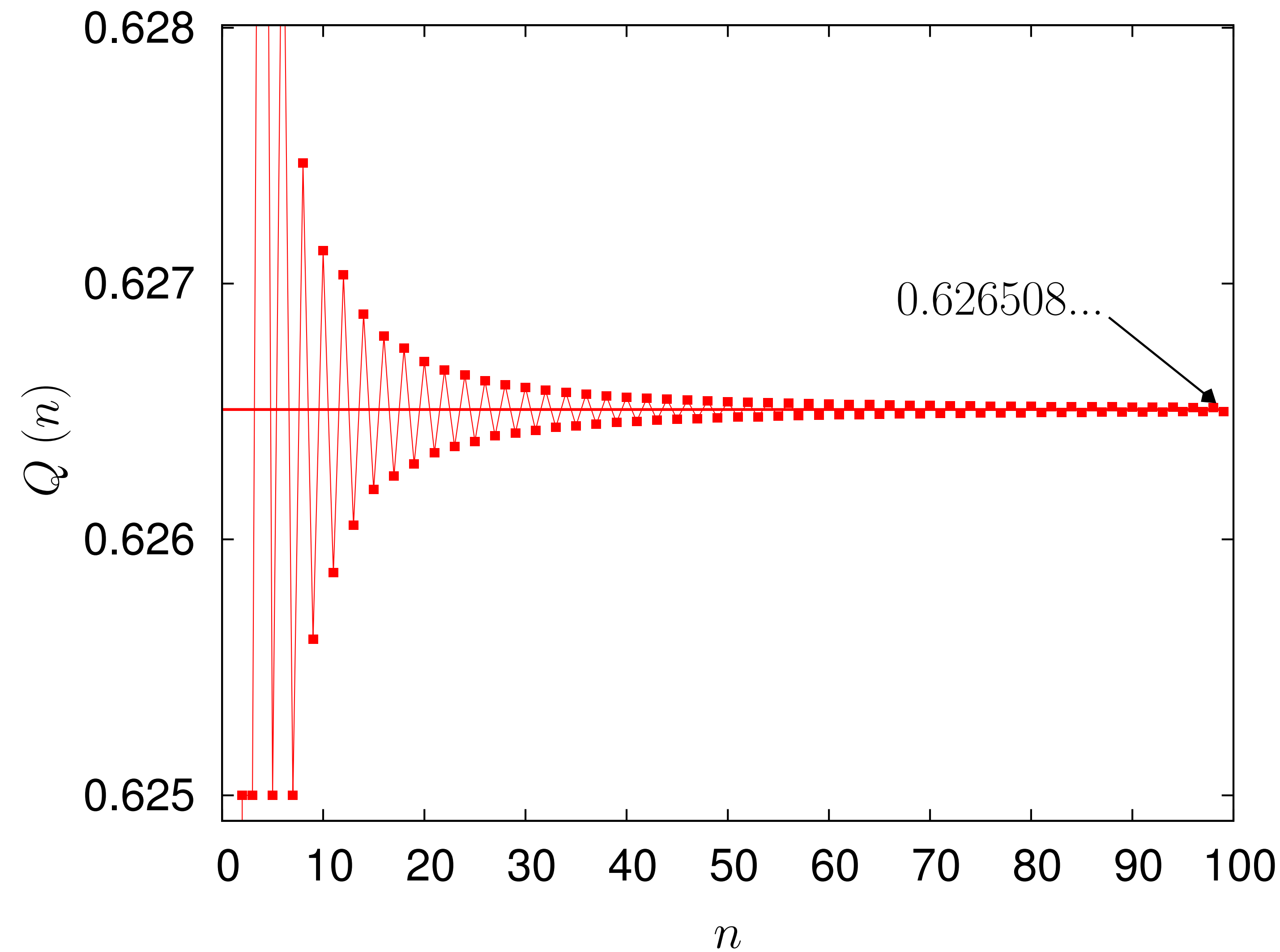

Godrèche, Majumdar, G. S.,`14

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Pitman, Yor, `97

Godrèche, Majumdar, G. S., `09

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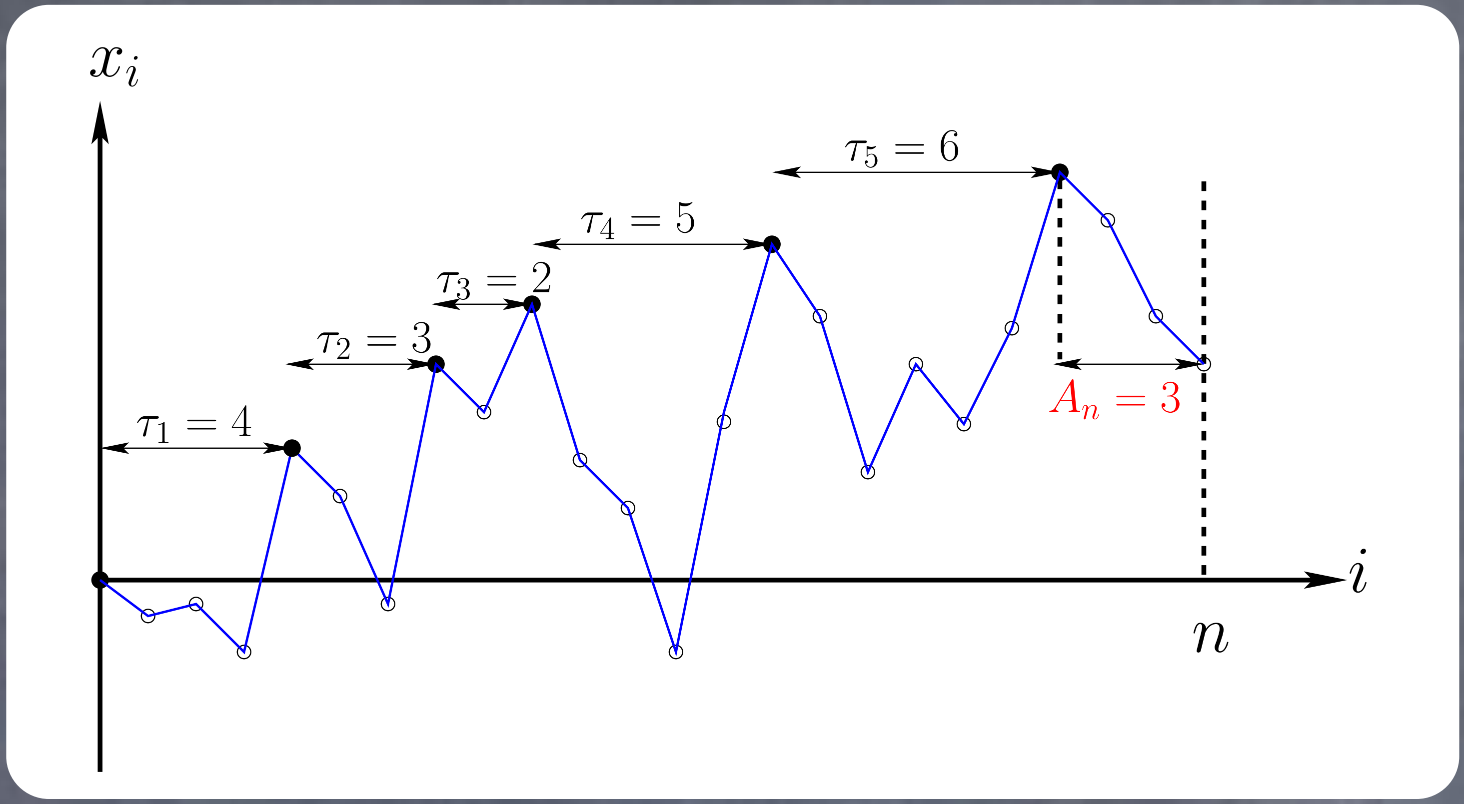
New observable...new universal constant



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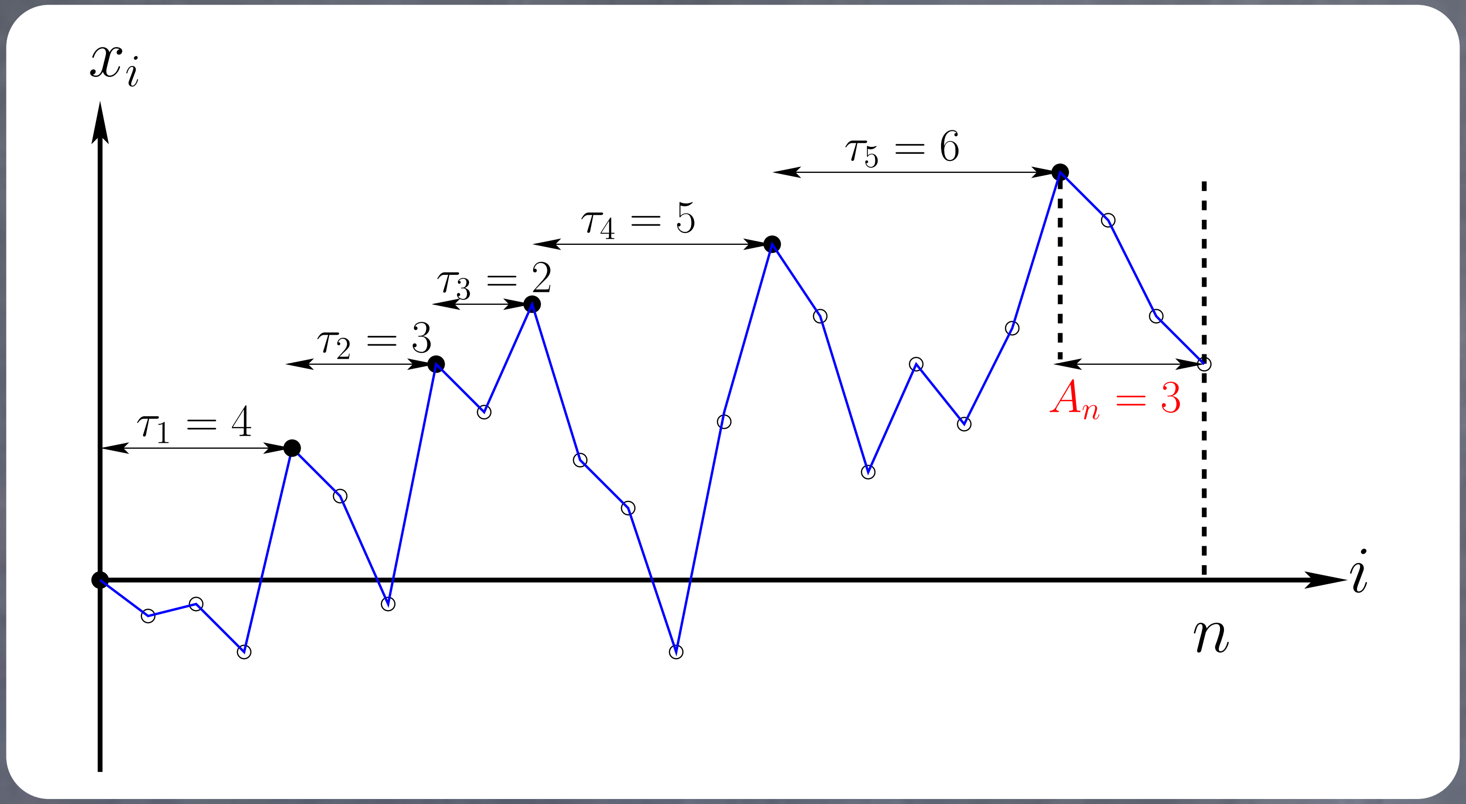

$$Q_1(n) = \Pr[\tau_1 \geq \max(\tau_2, \cdots, \tau_{m-1}, A_n)]$$

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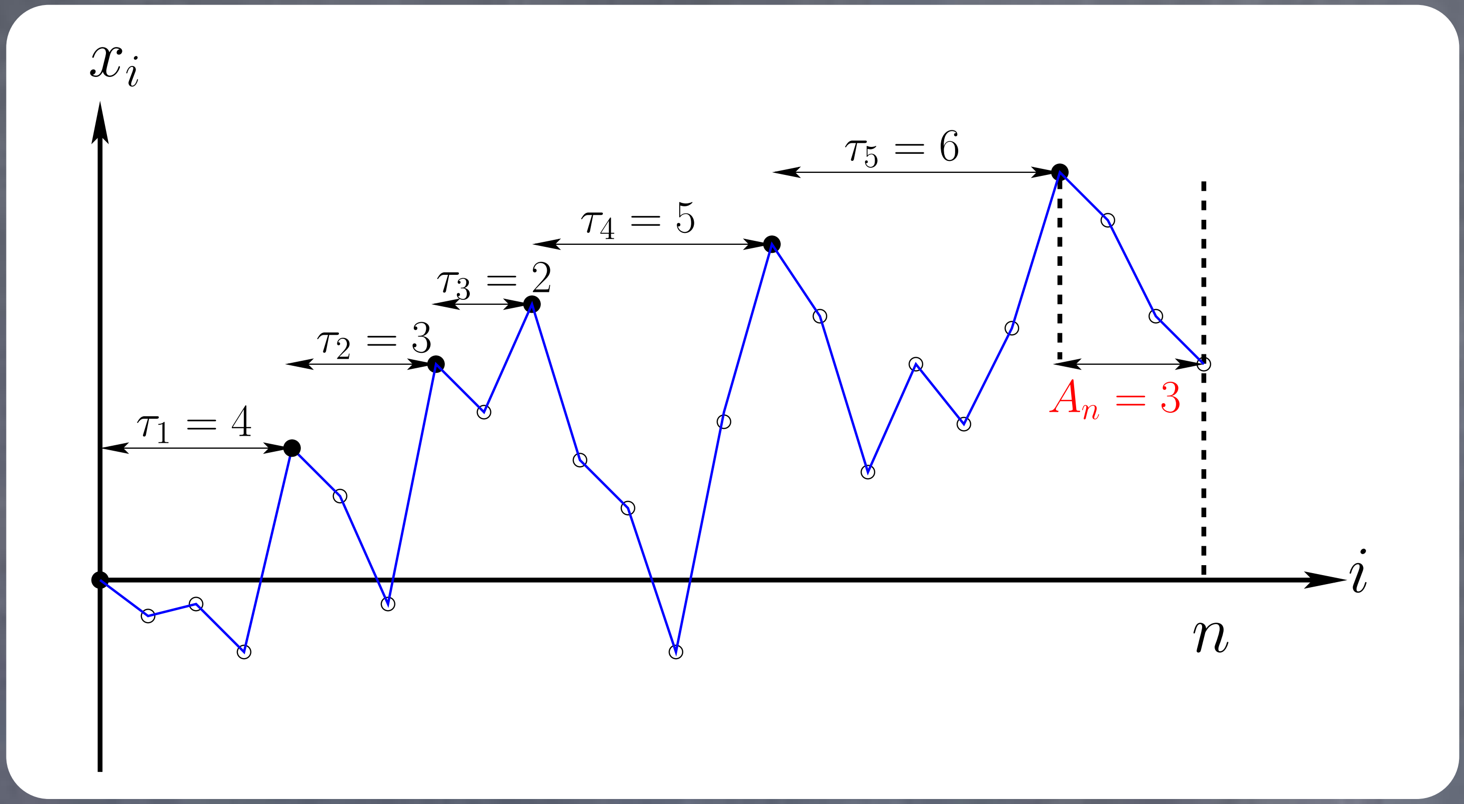

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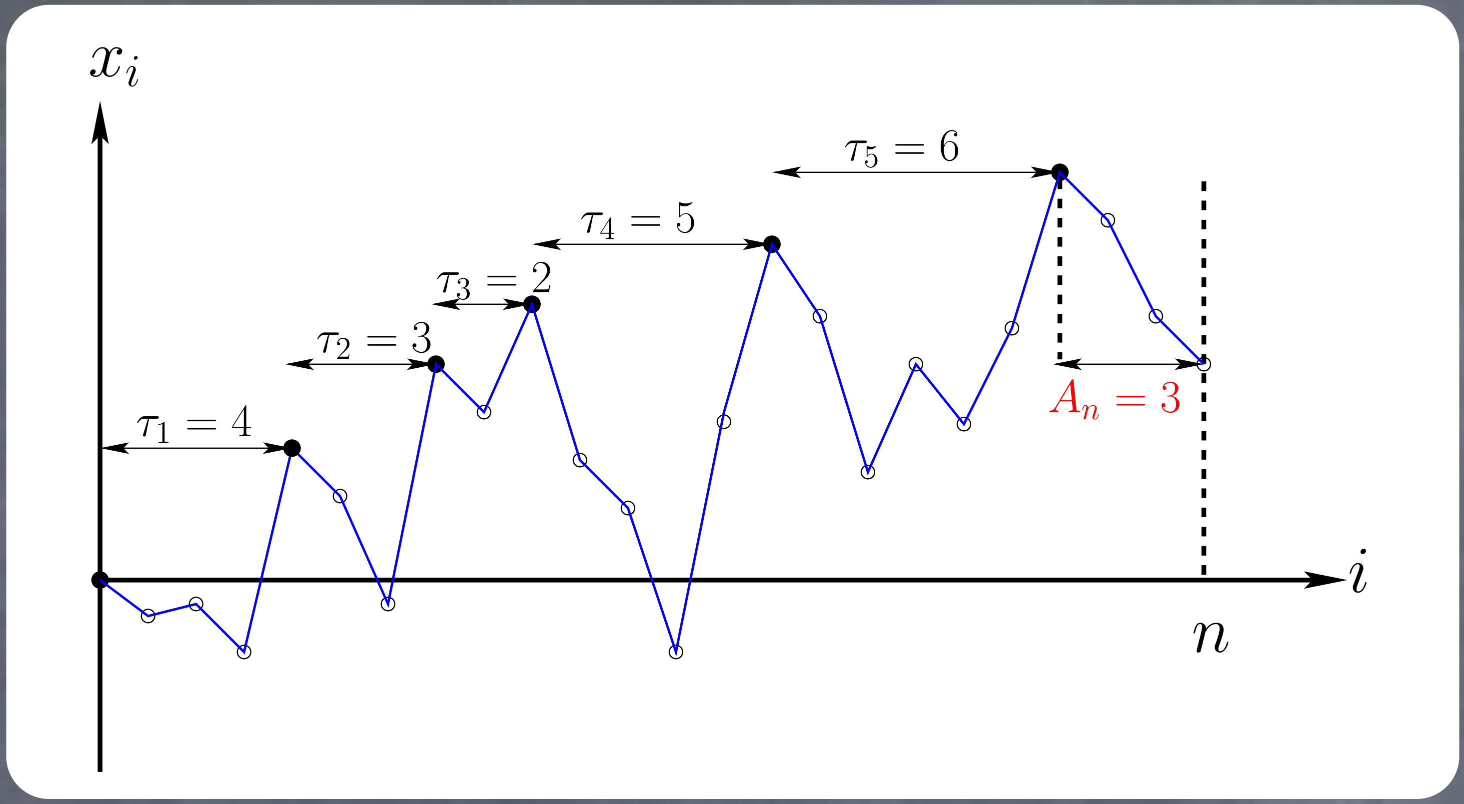

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$$\sim \frac{C_1}{\sqrt{n}}$$

$$C_1 = \frac{1}{\sqrt{\pi}}\left(1 + \frac{1}{2}\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\mathrm{erf}(\sqrt{x})}{1+\sqrt{\pi x}\,\mathrm{e}^x\,\mathrm{erf}(\sqrt{x})}\right) = 0.962641\ldots$$

Godrèche, Majumdar, G. S.,`14

Conclusions

- Exact results for records of strongly correlated time series

  see arXiv:1305.0639 for a short review

- Universal records statistics for (symmetric) RWs

- Extension to multiparticle systems Wergen, Majumdar, G. S. `12

- Extension to Continuous Time Random Walks (CTRWs)

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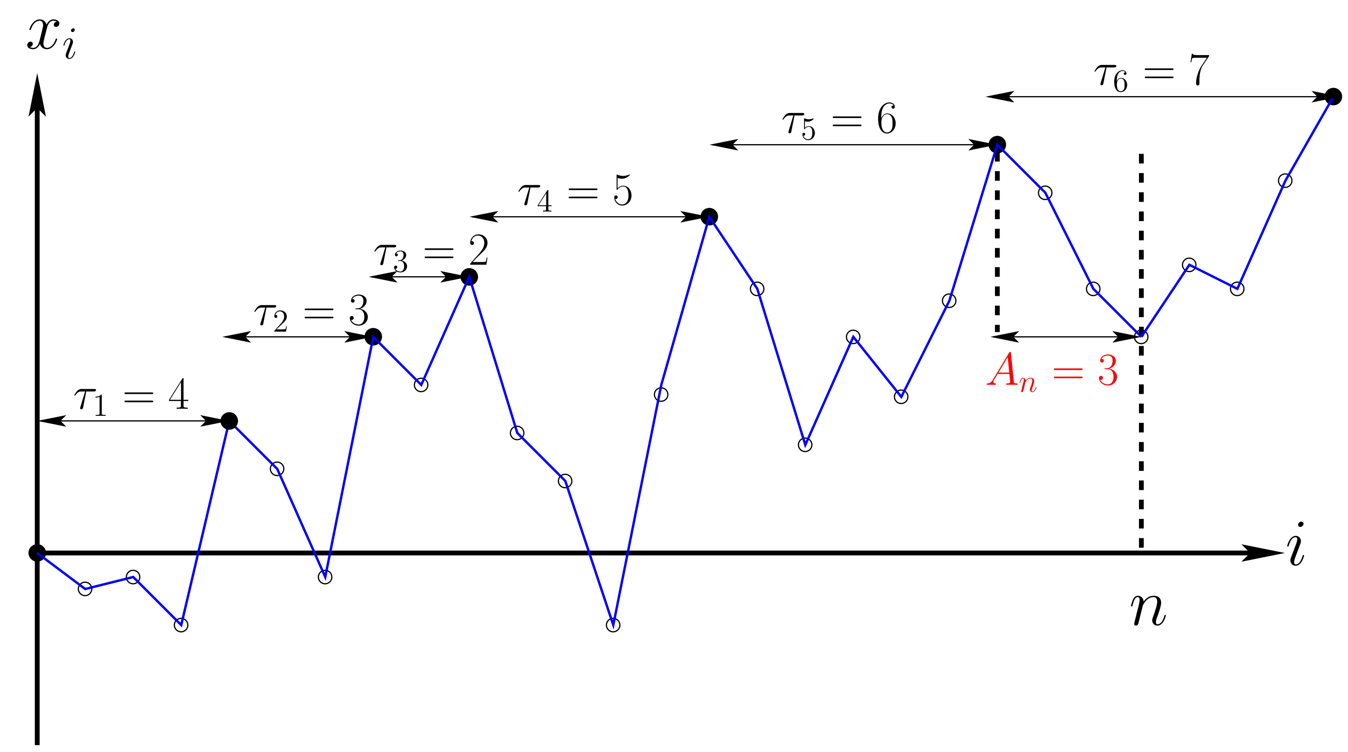
- High sensitivity to the definition of the age of the last record

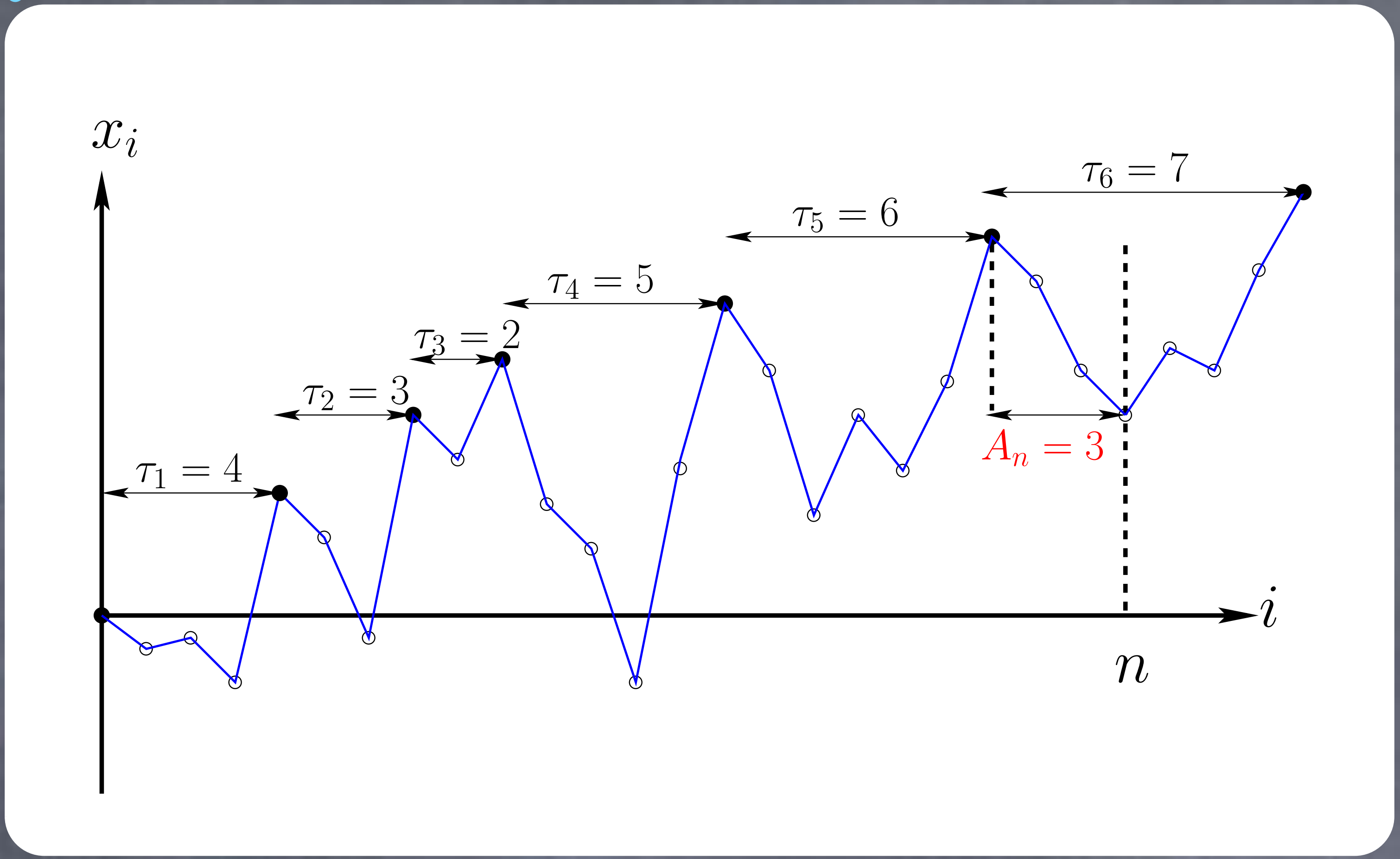
Sensitivity to the definition of the age of the last record

Godrèche, Majumdar, G. S., `14



Sensitivity to the definition of the age of the last record

Godrèche, Majumdar, G. S., `14



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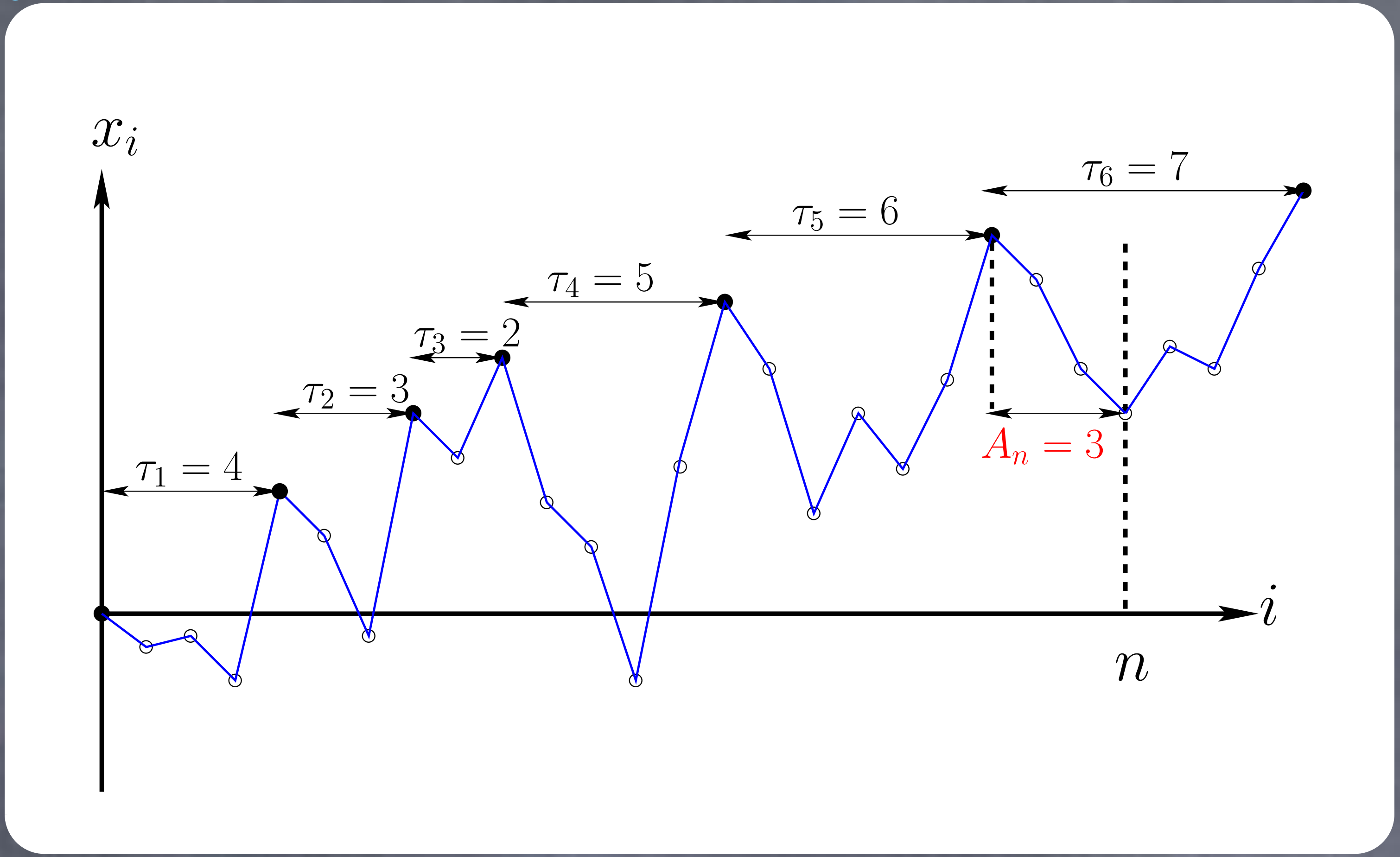

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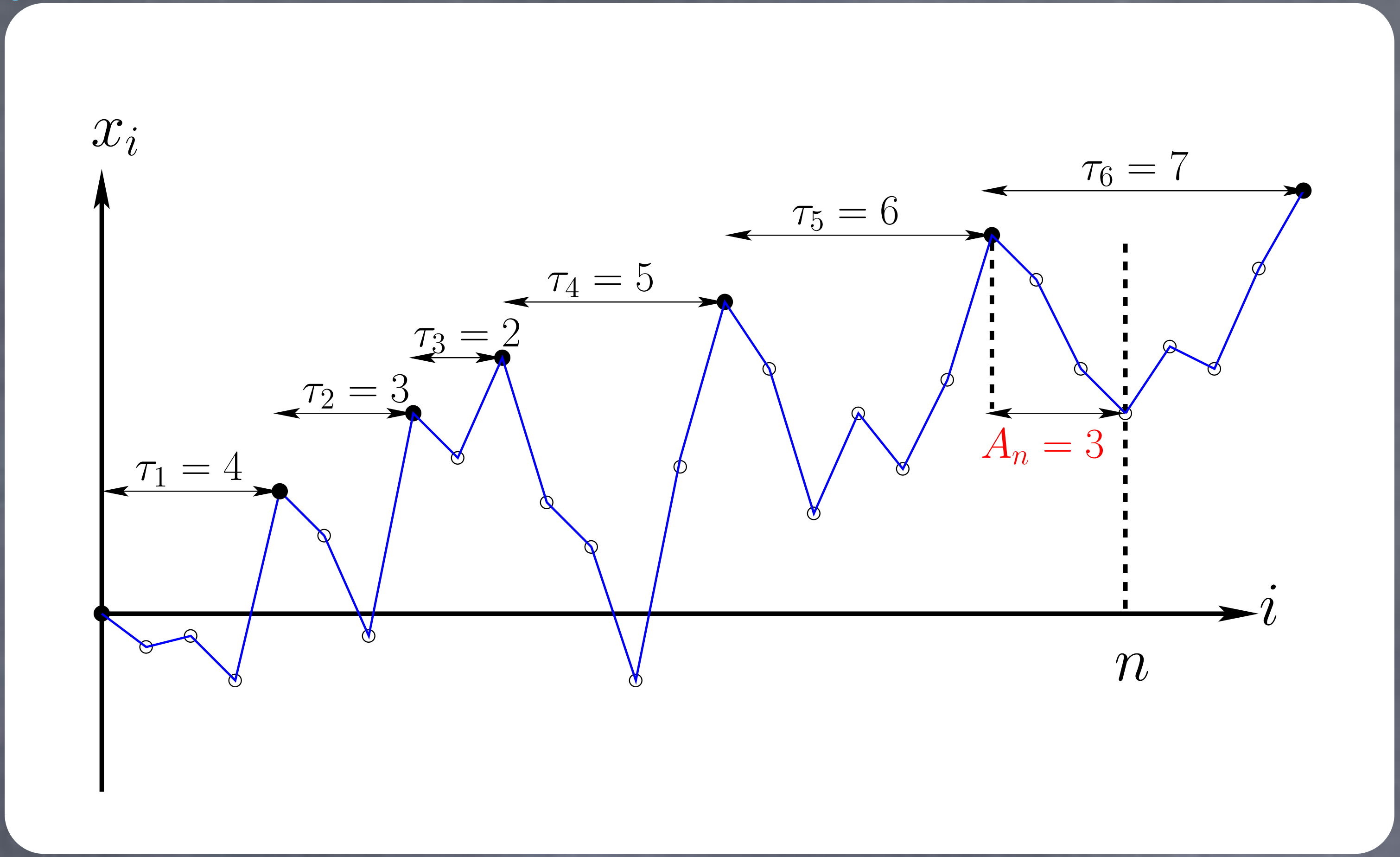

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$$Q^{\mathrm{II}}(\infty) = \frac{1}{2}\int_0^\infty \mathrm{d}x \, \frac{\mathrm{e}^x - 1}{x + \sqrt{\pi} x^{3/2} \, \mathrm{e}^x \, \mathrm{erf}(\sqrt{x})} = 0.800310\ldots \neq 0.626508\ldots$$