Analytical and numerical study of a realistic model for fish schools

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Collective motion in fish schools

Swarming, schooling, milling

Introduction

- Several models reproduce qualitatively the collective behaviors in fish schools, wild insect swarms, flocks of birds…

- The **Vicsek Model** (1995)

$$\phi_i(t+1) = \arg \sum_{\langle i,j \rangle_{R_0}}^{N} \mathrm{e}^{i\phi_j(t)} + \eta \xi_i(t)$$

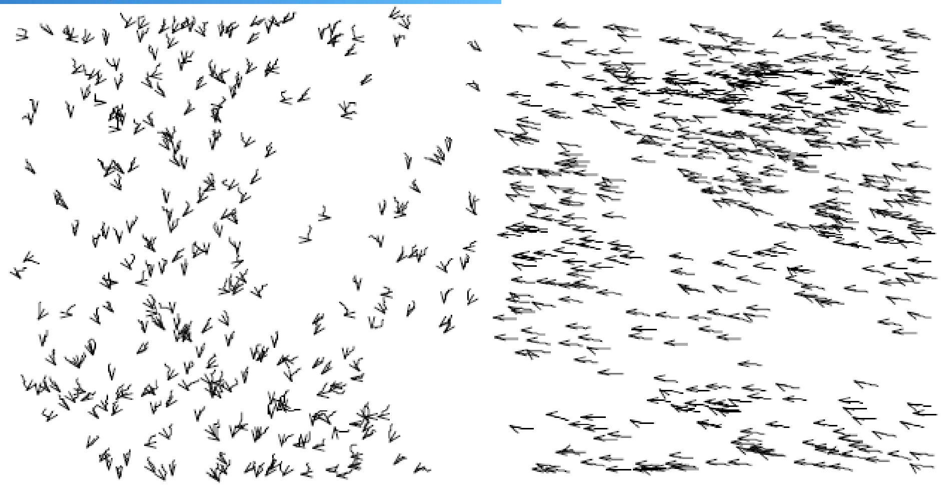
$$\mathbf{r_i}(t+1) = \mathbf{r_i}(t+1) + v_0 \mathbf{e}_{\phi_i(t+1)}$$

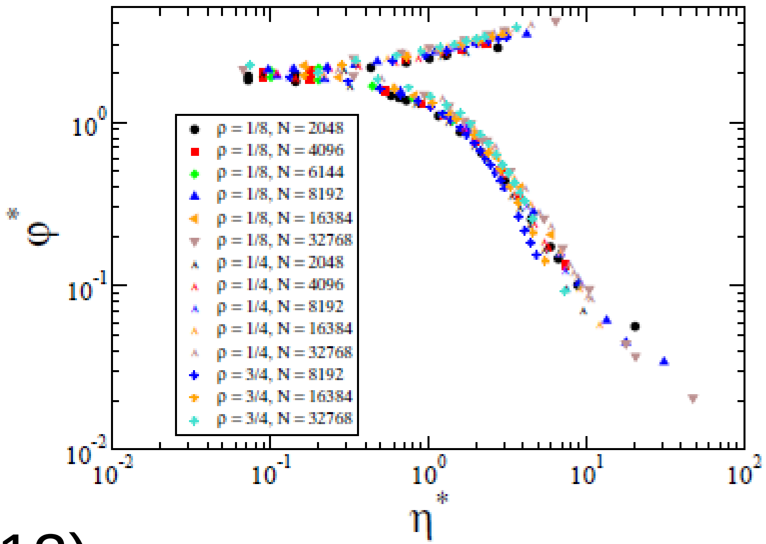
η, R_0, v_0 are the noise intensity, interaction radius, and velocity

Fixing R_0 and v_0, the control parameters are η or the density ρ

Order parameter $\phi = \frac{1}{N} \left| \sum_{i=1}^{N} \mathbf{e}_{\phi_i} \right|$

Vicsek Model



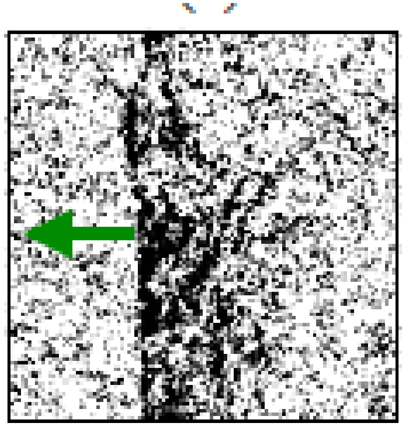


$\phi(\eta) \sim |\eta - \eta_c|^{\beta}$ (Baglietto *et al.* 2012)

$\beta \approx 0.45(3), \quad \nu \approx 1.6(3), \quad \gamma \approx 2.3(4)$

$$\phi^* = \phi(L)L^{\beta/\nu}, \quad \eta^* = \frac{|\eta - \eta_c|}{\sqrt{\rho}}$$

Second order nature of the transition **challenged by kinetic theory:** mode instability – stripes – destabilizes the long-range order just below the onset of flocking (Grégoire *et al.* 2004, Bertin *et al.* 2006, Chaté *et al.* 2007, Ihle 2010)

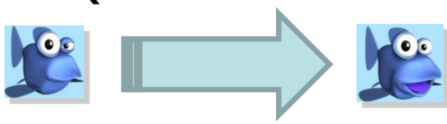


Experiments by the CRCA team

- Need for **realistic models** based on constraining and validating **experiments**
- These experiments (1, 2, 5… up to 30 fish) permit to identify **"individual laws"**, **"elementary interactions"**, and **"microscopic parameters"**

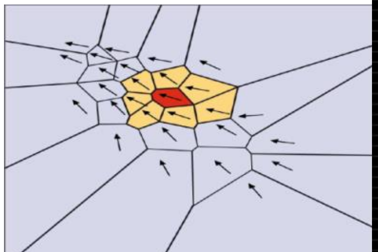
Some important aspects

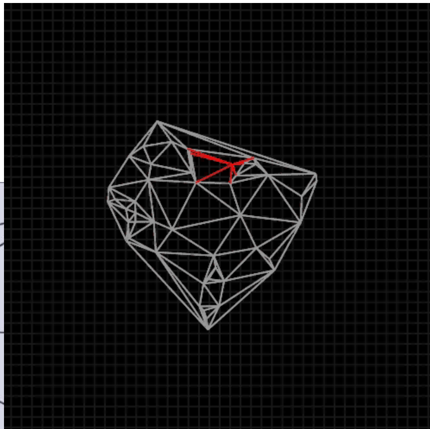
- Forces mediated by **vision** are in general **not conservative** (no law of action-reaction)



- Are forces really **additive** (a finite amount of information can be treated)? Instead, forces may be an **average** over the local environment

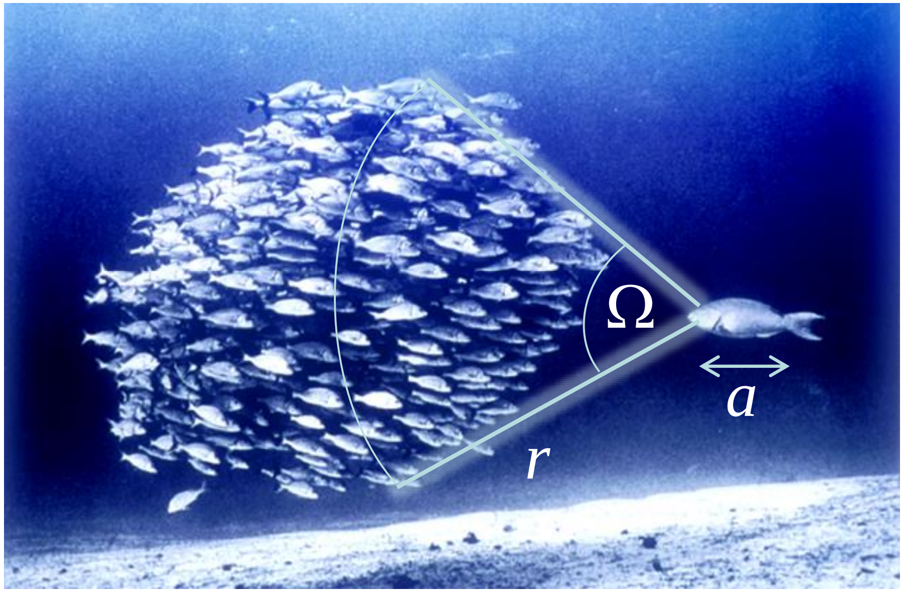
- Do fish (or birds) interact through **metric** or **topologic** (Voronoï diagram) forces? (not crucial in a tank)





Long-range attractive interactions?

- Additive (?) attractive force are mediated by **vision** and should be a linear function of the **(solid) angle spread** of the (group of) fish



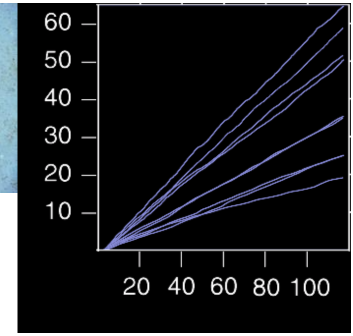
$$F(r) \sim \Omega \sim \left(\frac{a}{r}\right)^{d-1}$$

- **Long-range** attractive (~gravitation) force… but **screened** by obstacles

Basic model validated by CRCA experiments

on Barred Flagtail (Kuhlia Mugil), and more recently, Hemigrammus

J. Gautrais et al., J. Math. Biol. (2009); Plos Comput. Biology (2012)

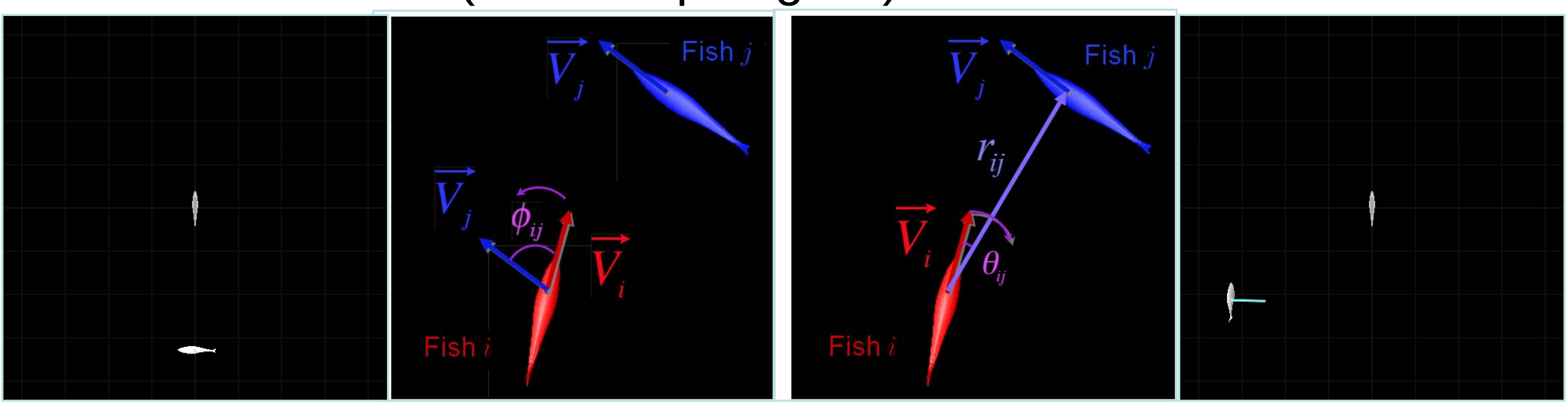


- **Constant velocity** $v \sim 0.1 - 0.6$ m/s
- Individual (2D) angular velocity evolving according to an **Ornstein-Uhlenbeck** process

$$\frac{d^2\phi_i}{dt^2} = \frac{d\omega_i}{dt} = -\frac{1}{\tau}(\omega_i - \omega_i^*) + \sigma\,\eta(t), \quad \frac{d\mathbf{r}_i}{dt} = v\mathbf{e}_{\phi_i}$$

$$(\tau \sim \xi / v;\ \sigma \sim \hat{\sigma} v)$$

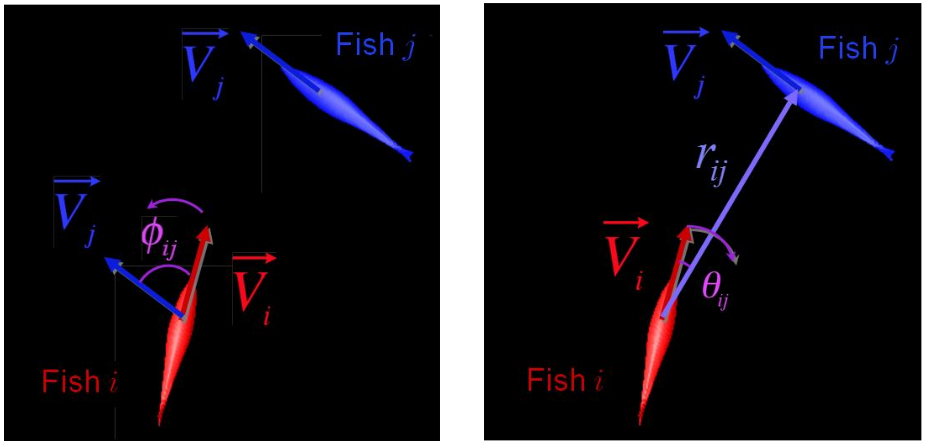
- The target angular velocity includes the effect of **alignment** and **attraction** (metric/topological) forces

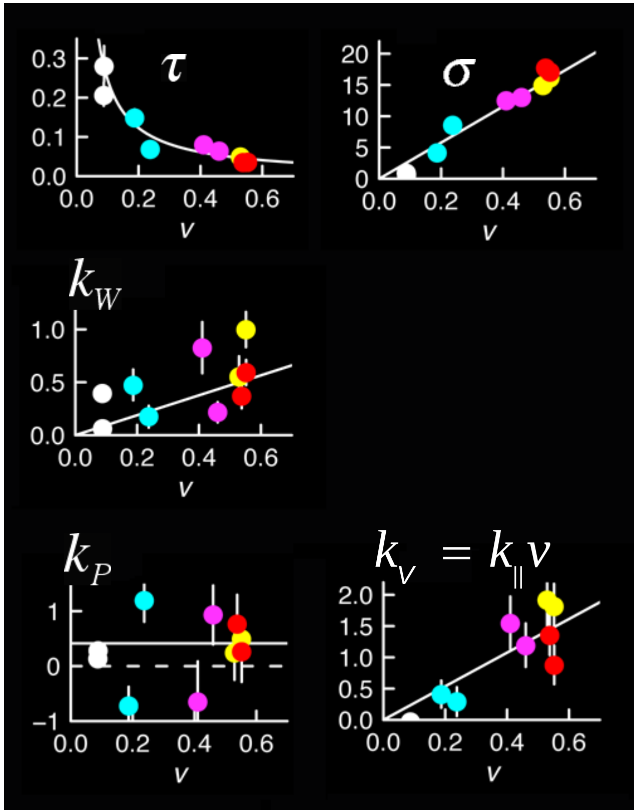


Basic model validated by CRCA experiments

- Topological **alignment** force $\sim v$ and **attraction** force $\sim r_{ij}$
- Phenomenological effect of **vision angle**

$$\omega^*_{j\to i} = \left[k_{\parallel} v \sin(\phi_j - \phi_i) + k_P r_{ij} \sin(\theta_{ij}) \right] \times \left[1 + \varepsilon \cos(\theta_{ij}) \right]$$



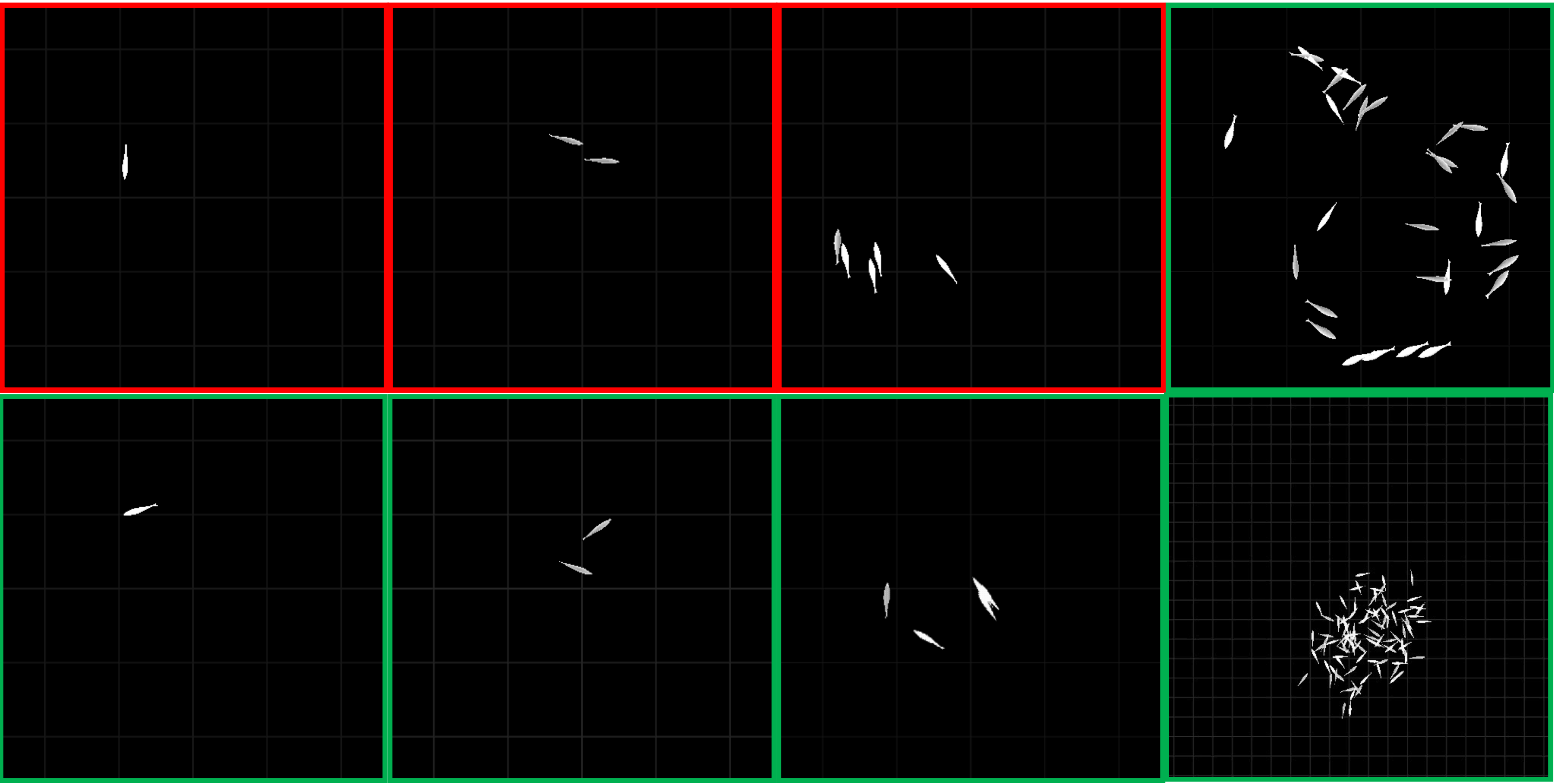


- + repulsive interaction with the **wall** (k_W)
- **Averaging**

$$\omega^*_i = \frac{1}{N_i} \sum_{\langle j,i \rangle} \omega^*_{j\to i} \quad (N_i \sim 6)$$

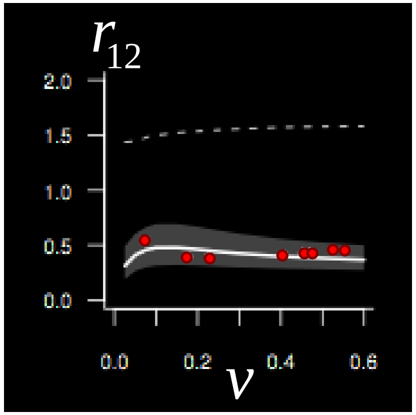
Basic model validated by CRCA experiments

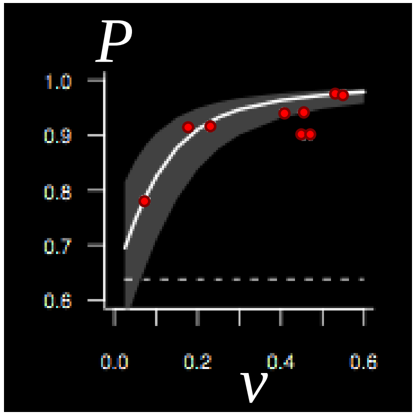
- **Experiments** *vs* **model simulations**
  Swarming to schooling transition as the velocity (and hence, **alignment**) is increased



Basic model validated by CRCA experiments

- Mean fish distance r_{12} and magnetization P *vs* velocity v

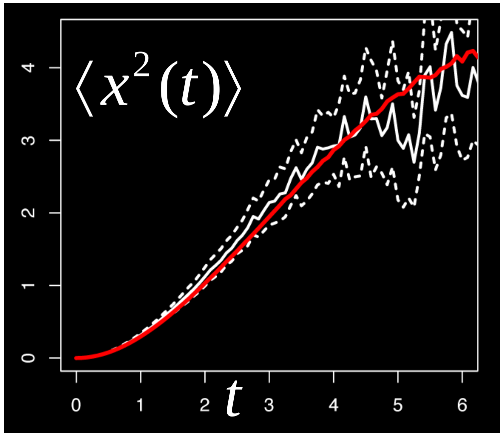




Order parameter $P = \dfrac{1}{N}\left|\sum_{i=1}^{N} \mathbf{e}_{\phi_i}\right|$

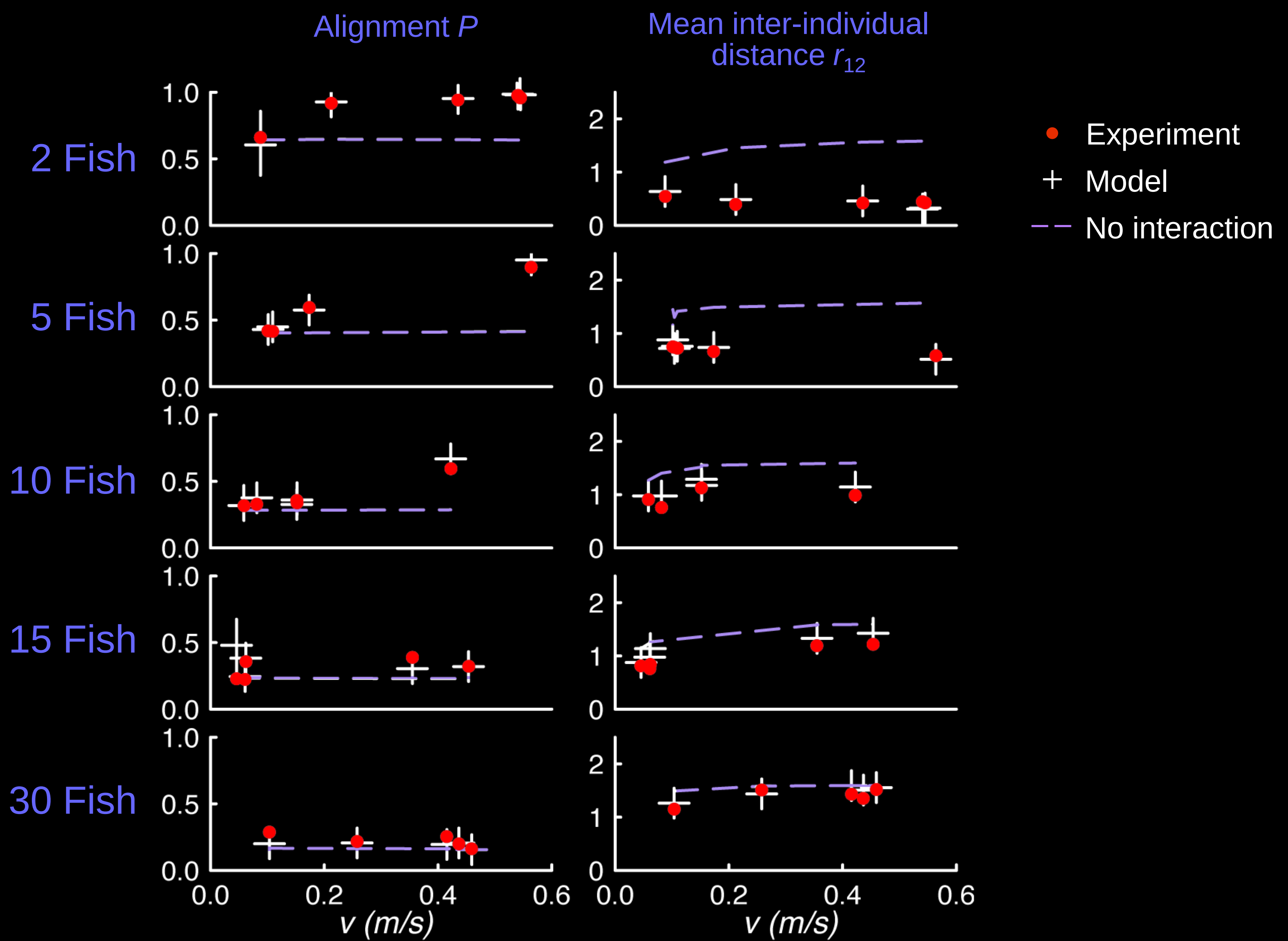
(Polarization)

- Mean square displacement in a tank



Empirical investigation of fish schooling

Comparison between model predictions and experimental data



Dimensionless equations of motion

- $$\alpha \frac{d^2\phi_i}{dt^2} + \frac{d\phi_i}{dt} + \sqrt{2}\eta_i = \frac{1}{N_i}\sum_{\langle i,j\rangle} \omega^*_{j\to i}, \quad \frac{d\mathbf{r}_i}{dt} = \mathbf{e}_{\phi_i} = (\cos\phi_i, \sin\phi_i)$$

$$\omega^*_{j\to i} = \left[\beta\sin(\phi_j - \phi_i) + \gamma r_{ij}\sin(\theta_{ij})\right]\times\left[1 + \varepsilon\cos(\theta_{ij})\right]$$

$\mathbf{r}_i$ = position of fish i; ϕ_i = angle of fish i velocity with respect to the horizontal

θ_{ij} = angle view of fish i looking at fish j; r_{ij} = distance between fish i and j

For $v = 0.24$ m/s, $\tau = \xi / v = 0.1$ s, $2/\tau_0 = (\xi\hat{\sigma})^2 \approx 0.48\ \mathrm{s}^{-1}$

$$\alpha = \frac{\tau}{\tau_0} \approx 0.024, \quad \beta = \frac{k_\parallel \xi}{\alpha} \approx 2.7, \quad \gamma = \frac{k_P \xi \tau}{\alpha^2} \approx 1.7$$

- Alignment $\omega^*_{j\to i} = -\dfrac{\partial V}{\partial\phi_i}(\phi_i - \phi_j)$, with $V(\phi) = -\beta\cos\phi$

  (XY model, **in-between $d = 2$ and mean-field**)

- **Inertial effects** on the angle dynamics are negligible
- $\varepsilon = 1$ in numerical simulations (**no milling phase** for $\varepsilon = 0$)

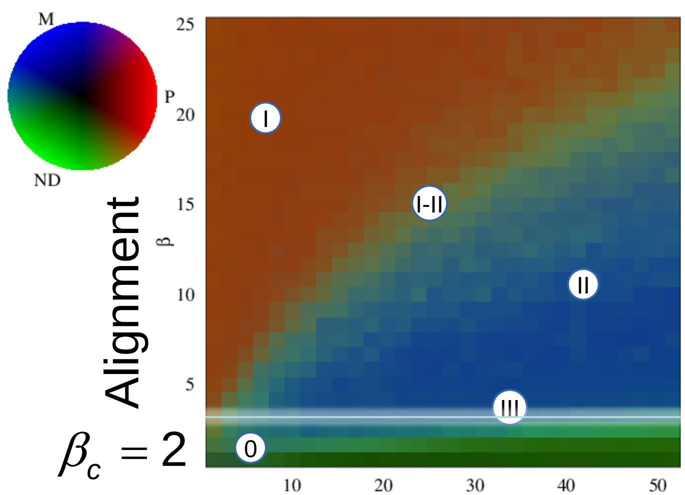
Phase diagram without a tank

$(\alpha = 0.024;\ \varepsilon = 1;\ \beta - \gamma$ *plane*$)$

DC, UL, SN, CS, HC & GT, New J. Phys. (2014)

- **Order parameters :** Polarization $P = \frac{1}{N}\left|\sum_{i=1}^{N} \mathbf{e}_{\phi_i}\right|$

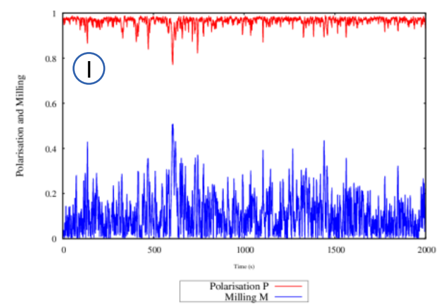
  Milling $M = \frac{1}{N}\left|\sum_{i=1}^{N} \mathbf{e}_{\phi_i} \times \mathbf{e}_{\mathbf{r}_i}\right|$

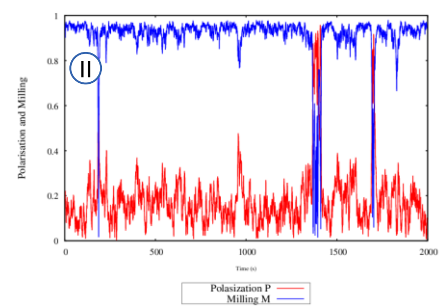


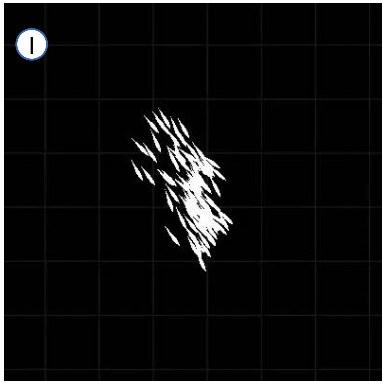
Alignment

$\beta_c = 2$

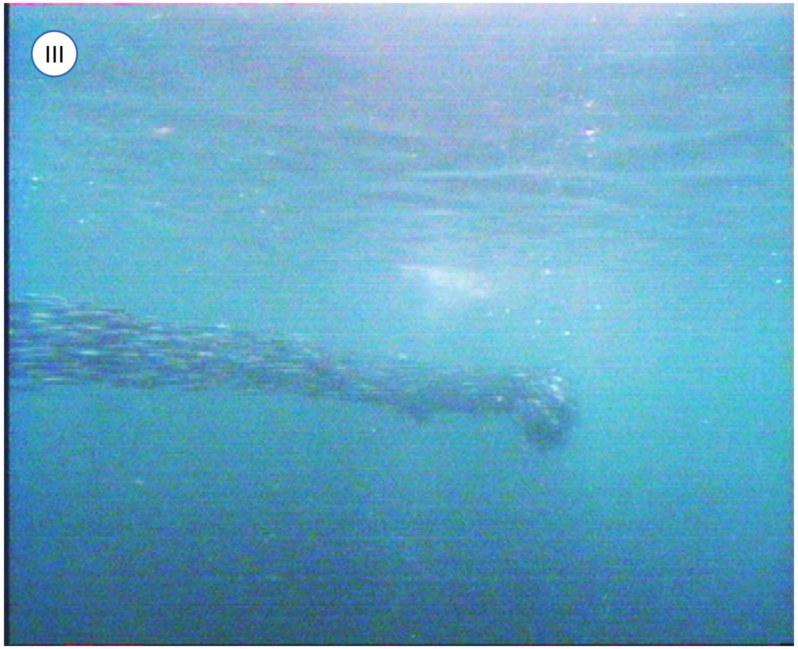
Attraction



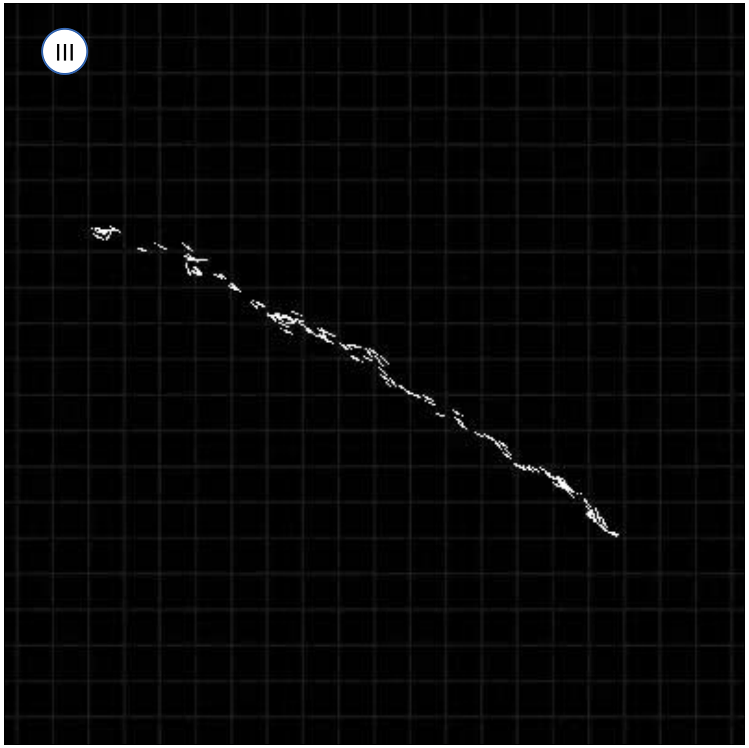




Existence of a third narrow elongated phase for $\gamma \gg \beta$; observed in some fish schools



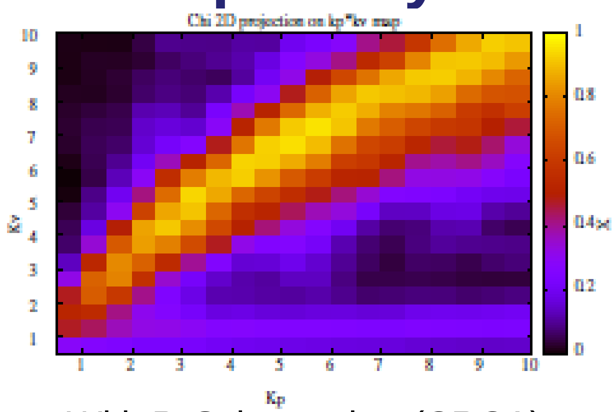
School of Atlantic herring (*Clupea harengus*)
Photo courtesy of P. Brehmer - IRD



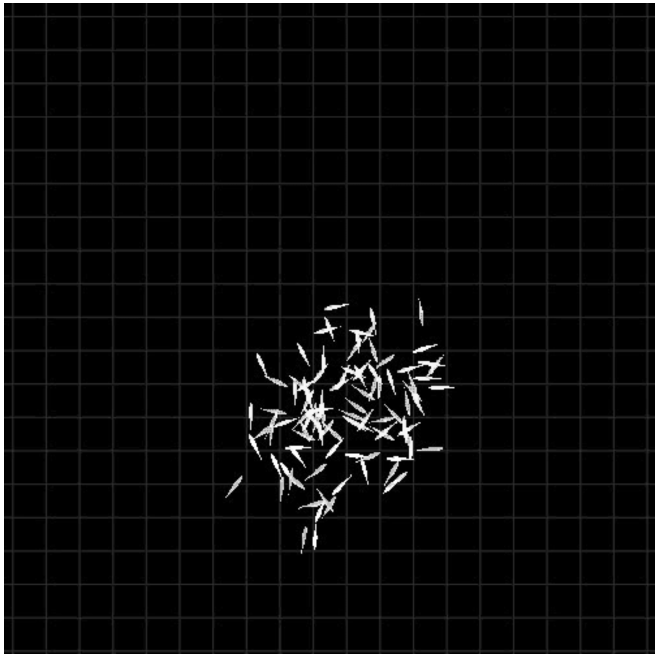
Phase diagram without a tank

$(\alpha = 0.024;\ \varepsilon = 1;\ \beta - \gamma$ *plane*$)$

- Experimental parameters $\beta \approx 2.5,\ \gamma \approx 1.7$ lie not far from the **transition line**: real fishes can slightly modify their velocity to go **from swarming to schooling** (notably in the presence of a **predator**)
- Divergence of the polarization **susceptibility** near the transition line



With P. Schumacher (CRCA)



- Swarming transition near the **mean-field transition line** (see hereafter) $\beta_c = 2\ \ (\varepsilon = 0)$

Mean-field theory

- **Variables:**

  Coordinates $\mathbf{r} = (r, \theta)$ *vs* the center of mass of the school

  Velocity angle ϕ

  Continuous density distribution of fish $\rho(r, \theta, \phi)$

- **Local order parameter**

  $$\mathbf{M}(r,\theta) = (M_x(r,\theta), M_y(r,\theta)) = M_0(\cos\phi_0, \sin\phi_0)$$

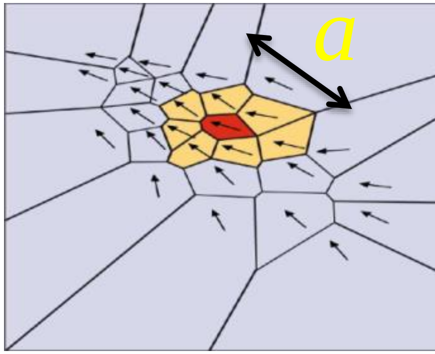
  $M_x(r,\theta) = \langle \cos\phi \rangle$, $M_y(r,\theta) = \langle \sin\phi \rangle$ (averages at fixed r and θ)

  Uniform schooling phase $(\phi_0 = 0)$: $\mathbf{M}(r,\theta) = (M_0, 0)$

  Isotropic milling phase $(\phi_0 = \theta + \pi/2)$: $\mathbf{M}(r,\theta) = M_0(-\sin\theta, \cos\theta)$

Mean-field theory (attraction force)

- If the density is smooth enough, the attractive force between fishes acts as an effective attraction force toward the center of mass ($\varepsilon = 0$)



$$\omega_{\mathrm{A}}^{*}(\mathbf{r}) = \gamma \frac{\int_{r'<a} r \sin(\theta' - \phi + \theta)\rho(\mathbf{r}+\mathbf{r}')r'\,dr'\,d\theta'}{\int_{r'<a} \rho(\mathbf{r}+\mathbf{r}')r'\,dr'\,d\theta'}$$

such that $\langle N_i \rangle = \int_{r'<a} \rho(\mathbf{r}+\mathbf{r}')r'\,dr'\,d\theta' \approx \pi\rho_0 a^2 = 6$

Expanding the top integral and assuming $|\nabla\rho|/\rho \sim \dfrac{r}{r_0^2}$,

$$\omega_{\mathrm{A}}^{*}(\mathbf{r}) = \frac{3}{2\pi\rho_0 r_0^2}\gamma r \sin(\phi - \theta)$$

which tends to **align the velocity to the direction** $\theta + \pi$

Mean-field theory equations of motion (ε=0)

$$\alpha \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \sqrt{2}\eta = \omega^* = \beta M_0 \sin(\phi_0 - \phi) + \gamma(r)\sin(\phi - \theta)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{e}_\phi - M_0 \mathbf{e}_{\phi_0}, \text{ with } M_0 \mathbf{e}_{\phi_0} = \langle \mathbf{e}_\phi \rangle_{r,\theta}$$

$$\frac{dr}{dt} = \cos(\phi - \theta) - M_0 \cos(\phi_0 - \theta), \; r\frac{d\theta}{dt} = \sin(\phi - \theta) - M_0 \sin(\phi_0 - \theta)$$

$$\alpha_{Exp.} \approx 0.024, \; \beta_{Exp.} \approx 2.7, \; \gamma(r)/r \sim \gamma_{Exp.} \approx 1.7 \quad \text{(+wall of the tank)}$$

Fokker - Planck equation ($\alpha = 0$)

$$\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{\partial\phi^2} - \frac{\partial}{\partial\phi}\left[\omega^*\rho\right] - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(\sin(\phi-\theta) - M_0 \sin(\phi_0 - \theta)\right)\rho\right]$$

$$-\frac{\partial}{\partial r}\left[\left(\cos(\phi-\theta) - M_0\cos(\phi_0-\theta)\right)\rho\right]$$

Diffusion coefficient of a single fish

$$\alpha \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \sqrt{2}\eta = 0, \qquad \frac{d\mathbf{r}}{dt} = \mathbf{e}_\phi$$

$$\mathrm{C}(t) = \frac{1}{2}\langle[\phi(t)-\phi(0)]^2\rangle = t - \alpha\left[1-\exp(-t/\alpha)\right]$$

$$D(\alpha) = \lim_{t\to\infty}\langle[\mathbf{r}(t)-\mathbf{r}(0)]^2\rangle / t = 2\int_0^\infty \exp\left[-C(t)\right]dt$$

$$\sim_{\alpha\to 0} 2$$

$$\sim_{\alpha\to\infty} \sqrt{2\pi\alpha}$$

Expressing length and time in the original units with $\alpha = \dfrac{\tau}{\tau_0} = \dfrac{\tau(\xi\hat{\sigma})^2}{2}$

$$D_0 = v^2\tau_0 D(\alpha) = v\xi\frac{D(\alpha)}{\alpha} \qquad (\hat{\sigma} \sim \text{phase noise})$$

$$\sim_{\alpha\to 0} 2v\xi\times\alpha^{-1} \to_{\hat{\sigma}\to 0} +\infty$$

Weak phase noise ⇒ large D_0

$$\sim_{\alpha\to\infty} \sqrt{2\pi}v\xi\times\alpha^{-1/2} \to_{\hat{\sigma}\to\infty} 0$$

Strong phase noise ⇒ small D_0

Mean-field theory

- **Exact solution for $M_0 = 0$ (swarming)**

$$\rho(r,\theta,\phi) = \frac{1}{Z} r \exp\left[-\int_0^r \gamma(r')dr' \right]$$

- **Exact solution for $\gamma(r) = 0$**

  (space irrelevant; schooling / swarming)

$$\rho(\phi) = \frac{1}{Z} \exp\left[\beta M_0 \cos\phi \right], \quad M_0 = \langle \cos\phi \rangle$$

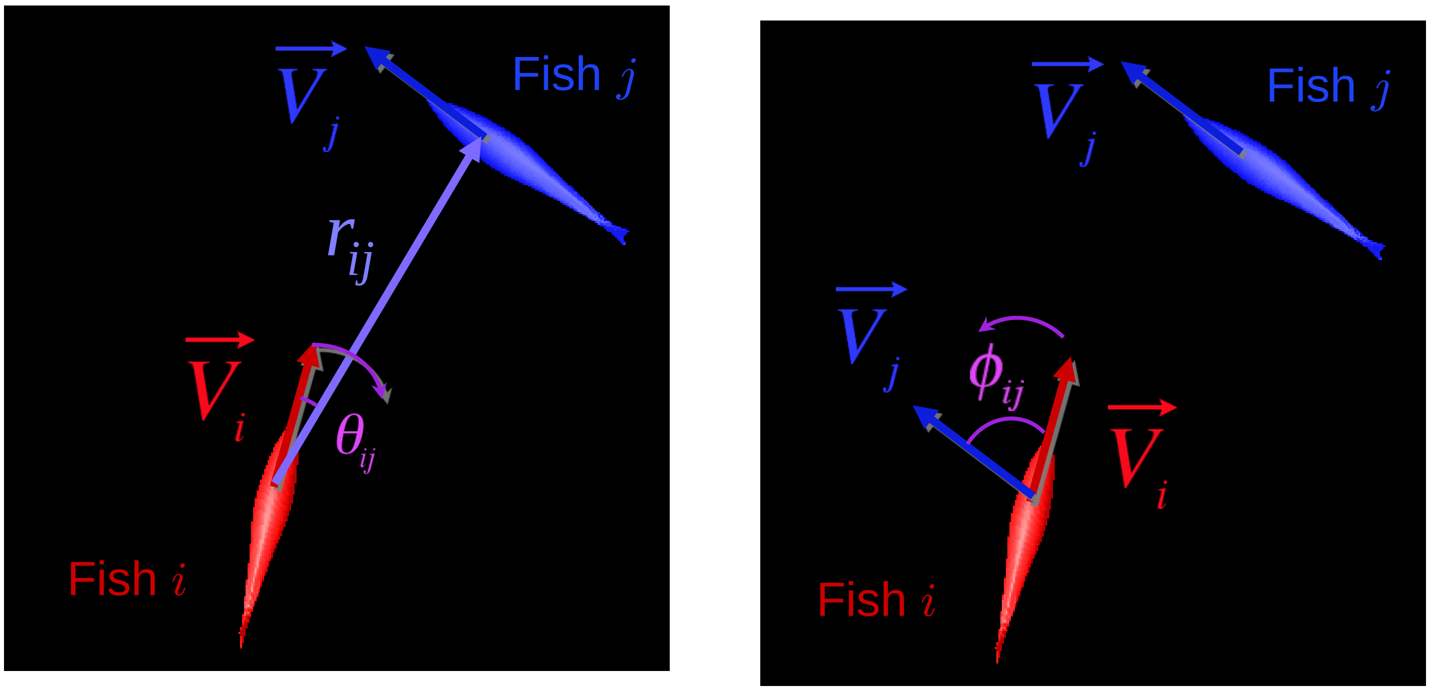
Complete analogy with the **HMF model** (Antoni & Ruffo 1995)

i.e. the XY model with all spins interacting with each other

$M_0 \sim \sqrt{(\beta - \beta_c)}$, with $\beta_c = 2$

Conclusion

- Realistic model for fish schools **validated by experiments**
- The general issue of **topologic/metric** force: relevance of **long-range interactions?** (~self-gravitating Brownian particles → school cohesion at low noise; Chavanis & CS)
- The **milling phase** is present when **vision effects** are taken into account, along with a narrow **elongated phase**
- **Biologically relevant** parameters are close to the swarming/schooling transition line, where fish can quickly adjust to their environment
- Introduction of a **mean-field theory**
  - Including the effect of vision (non conservative attractive force) to reproduce the milling phase
  - Allowing for non uniform/non isotropic order parameter
  - Time evolution (instability modes, dynamical transitions…)



$\vec{V}_j$
Fish j
r_{ij}
$\vec{V}_i$
θ_{ij}
Fish i
$\vec{V}_j$
Fish j
$\vec{V}_j$
ϕ_{ij}
$\vec{V}_i$
Fish i