Stochastic energy-exchange models of non-equilibrium.

Cristian Giardina’

Joint work with J. Kurchan (Paris), F. Redig (Delft) K.Vafayi (Eindhoven), G. Carinci, C. Giberti (Modena).
Fourier law \[ J = \kappa \nabla T \]

- 1D Hamiltonian models:
  - Oscillators chains (Lebowitz, Lieb, Rieder, 1967): \( \kappa \sim N \).
  - Non-linear oscillators chains (Lepri, Livi, Politi, Phys. Rep. 2003): \( \kappa \sim N^\alpha \), \( 0 < \alpha < 1 \)
  - Non-linear fluctuating hydrodynamics (van Beijeren 2012, Spohn 2013)
Stochastic energy exchange models

Kipnis, Marchioro, Presutti (1982):

Observables: Energies at every site \( z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N \)

Dynamics: Select a bond at random and **uniformly** redistribute the energy under the constraint of conserving the total energy.

\[
L^{KMP} f(z) = \sum_{i=1}^{N} \int_{0}^{1} dp \left[ f(z_1, \ldots, p(z_i + z_{i+1}), (1 - p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]
\]
Outline

1. From Hamiltonian to stochastics: a simple model.
2. Duality Theory:
   - Brownian Momentum Process (BMP).
   - Symmetric Inclusion Process (SIP).
4. Boundary driven systems.
5. A larger picture & “redistribution” models.
From Hamiltonian to stochastics
A simple Hamiltonian model (G., Kurchan)

\[
H(q, p) = \sum_{i=1}^{N} \frac{1}{2} (p_i - A_i)^2
\]

\[A = (A_1(q), \ldots, A_N(q)) \] “vector potential” in \(\mathbb{R}^N\).

\[
\frac{dq_i}{dt} = v_i
\]

\[
\frac{dv_i}{dt} = \sum_{j=1}^{N} B_{ij} v_j
\]

where

\[
B_{ij}(q) = \frac{\partial A_i(q)}{\partial q_j} - \frac{\partial A_j(q)}{\partial q_i}
\]

antisymmetric matrix containing the “magnetic fields”
Conservation laws

- **Conservation of Energy:**
  Even if the forces depend on velocities and positions, the model conserves the total (kinetic) energy

\[
\frac{d}{dt} \left( \sum_i \frac{1}{2} v_i^2 \right) = \sum_{i,j} B_{ij} v_i v_j = 0
\]

- **Conservation of Momentum:**
  If we choose the \( A_i(x) \) such that they are left invariant by the simultaneous translations \( x_i \rightarrow x_i + \delta \), then the quantity \( \sum_i p_i \) is conserved.
Example: discrete time dynamics with “magnetic kicks”

\[ q(t + 1) = q(t) + v(t) \]
\[ v(t + 1) = R(t + 1) \cdot v(t) \]

with \( R(t) \) a rotation matrix

\[
R(t + 1) = \begin{pmatrix}
\cos(B(q(t + 1))) & \sin(B(q(t + 1))) \\
-\sin(B(q(t + 1))) & \cos(B(q(t + 1)))
\end{pmatrix}
\]
Chaoticity properties of the map on $\mathbb{T}_2$

Figure: Poincare section with plane $q^{(2)} = 0$ of the map

\[
\begin{align*}
q_{t+1}^{(1)} &= q_t^{(1)} + v \cos(\beta_t) \\
q_{t+1}^{(2)} &= q_t^{(2)} + v \sin(\beta_t) \\
\beta_{t+1} &= \beta_t + B(q_t^{(1)}, q_t^{(2)})
\end{align*}
\]

with $v = \sqrt{v_1^2 + v_2^2}$, $\beta = \arctan(v_2/v_1)$, $B(q^{(1)}, q^{(2)}) = q^{(1)} + q^{(2)} - 2\pi$. 

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Numerical result

Thermal conductivity

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Duality theory
Duality

Definition

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega$ with generator $L$,

$(\xi_t)_{t \geq 0}$ Markov process on $\Omega_{dual}$ with generator $L_{dual}$
Duality

Definition

\((\eta_t)_{t \geq 0}\) Markov process on \(\Omega\) with generator \(L\),

\((\xi_t)_{t \geq 0}\) Markov process on \(\Omega_{\text{dual}}\) with generator \(L_{\text{dual}}\)

\(\xi_t\) is dual to \(\eta_t\) with duality function \(D : \Omega \times \Omega_{\text{dual}} \to \mathbb{R}\) if \(\forall t \geq 0\)

\[\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{\text{dual}}\]
Duality

**Definition**

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega$ with generator $L$,

$(\xi_t)_{t \geq 0}$ Markov process on $\Omega_{dual}$ with generator $L_{dual}$

$\xi_t$ is **dual** to $\eta_t$ with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$
\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}
$$

Equivalently

$$
LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)
$$

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How to find a dual process?

1. Write the generator in **abstract form**, i.e. as an element of a Lie algebra, using creation and annihilation operators.

2. Duality is related to a **change of representation**, i.e. new operators that satisfy the same algebra.

3. Self-duality is associated to **symmetries**, i.e. conserved quantities.
The method at work

Brownian momentum process

$\downarrow$

$SU(1,1)$ algebra

$\downarrow$

Inclusion process
Brownian momentum process (BMP) on two sites

Given \((x_i, x_j)\) \equiv \text{velocities of the couple} \((i, j)\)

\[
L_{i,j}^{BMP} f(x_i, x_j) = \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)
\]

- polar coordinates
  \[
  L_{i,j}^{BMP} = \frac{\partial^2}{\partial \theta_{ij}^2}
  \]

- Brownian motion for angle \(\theta_{i,j} = \arctan(x_j/x_i)\)

- total kinetic energy conserved:
  \[
  r_{i,j}^2 = x_i^2 + x_j^2
  \]
Brownian momentum process (BMP)

For a graph \( G = (V, E) \) let \( \Omega = \bigotimes_{i \in V} \Omega_i = \mathbb{R}^{|V|} \).
Configuration \( x = (x_1, \ldots, x_{|V|}) \in \Omega \)

Generator BMP

\[
L^{BMP} = \sum_{(i,j) \in E} L_{i,j}^{BMP} = \sum_{(i,j) \in E} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2
\]
Brownian momentum process (BMP)

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Brownian momentum process (BMP)

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**Generator BMP**

$$L_{\text{BMP}} = \sum_{(i,j) \in E} L_{i,j} = \sum_{(i,j) \in E} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

Stationary measures: Gaussian product measures

$$d\mu(x) = \prod_{i=1}^{|V|} \frac{e^{-\frac{x_i^2}{2T}}}{\sqrt{2\pi T}} \, dx_i$$
Symmetric Inclusion Process (SIP)

\[ \Omega_{dual} = \bigotimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^{|V|} \]

Configuration \( \xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega_{dual} \)
Symmetric Inclusion Process (SIP)

\[ \Omega_{\text{dual}} = \bigotimes_{i \in V} \Omega_{i}^{\text{dual}} = \{0, 1, 2, \ldots\}^{|V|} \]

Configuration \( \xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega_{\text{dual}} \)

Generator SIP

\[
L^{\text{SIP}} f(\xi) = \sum_{(i,j) \in E} L_{i,j}^{\text{SIP}} f(\xi)
\]

\[
= \sum_{(i,j) \in E} \xi_i \left( \xi_j + \frac{1}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \left( \xi_i + \frac{1}{2} \right) \xi_j [f(\xi^{j,i}) - f(\xi)]
\]

Stationary measures: product of Negative Binomial with \( r = 2 \)

\[
P_r(\xi_1 = n_1, \ldots, \xi_{|V|} = n_{|V|}) = |V| \prod_{i=1}^{\frac{n_i}{r}} p^{n_i} (1 - p)^{r n_i} \Gamma(r + n_i) \Gamma(r)
\]
Symmetric Inclusion Process (SIP)

\[ \Omega_{dual} = \bigotimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^{|V|} \]

Configuration \( \xi = (\xi_1, \ldots, \xi_{|V|}) \in \Omega_{dual} \)

Generator SIP

\[ L^\text{SIP} f(\xi) = \sum_{(i,j) \in E} L^\text{SIP}_{i,j} f(\xi) \]

\[ = \sum_{(i,j) \in E} \xi_i \left( \xi_j + \frac{1}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \left( \xi_i + \frac{1}{2} \right) \xi_j [f(\xi^{j,i}) - f(\xi)] \]

Stationary measures: product of Negative Binomial\((r, p)\) with \(r = 2\)

\[ P_r(\xi_1 = n_1, \ldots, \xi_{|V|} = n_{|V|}) = \prod_{i=1}^{|V|} \frac{p^{n_i}(1 - p)^r}{n_i!} \frac{\Gamma(r + n_i)}{\Gamma(r)} \]

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Duality between BMP and SIP

**Theorem 1**

The process \( \{x(t)\}_{t \geq 0} \) with generator \( L = L^{BMP} \) and the process \( \{\xi(t)\}_{t \geq 0} \) with generator \( L_{dual} = L^{SIP} \) are dual on

\[
D(x, \xi) = \prod_{i \in V} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}
\]

Proof: An explicit computation gives

\[
L^{BMP} D(x, \xi)(x) = L^{SIP} D(x, \xi)(\xi)
\]
Duality between BMP and SIP

Theorem 1

The process \( \{x(t)\}_{t \geq 0} \) with generator \( L = L^{BMP} \) and the process \( \{\xi(t)\}_{t \geq 0} \) with generator \( L_{dual} = L^{SIP} \) are dual on

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\[
L^{BMP} D(\cdot, \xi)(x) = L^{SIP} D(x, \cdot)(\xi)
\]
Duality explained

**SU(1, 1) ferromagnetic quantum spin chain**

Abstract operator

\[ \mathcal{L} = \sum_{(i,j) \in E} \left( K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{1}{8} \right) \]

with \( \{K_i^+, K_i^-, K_i^o\}_{i \in V} \) satisfying SU(1, 1) commutation relations:

\[
[K_i^o, K_j^{\pm}] = \pm \delta_{i,j} K_i^{\pm} \hspace{1cm} [K_i^-, K_j^+] = 2\delta_{i,j} K_i^o
\]
Duality explained

\textit{SU}(1, 1) \textit{ferromagnetic quantum spin chain}

Abstract operator

\[ \mathcal{L} = \sum_{(i,j) \in E} \left( K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{1}{8} \right) \]

with \( \{ K_i^+, K_i^-, K_i^o \}_{i \in V} \) satisfying \textit{SU}(1, 1) commutation relations:

\[ [K_i^o, K_j^\pm] = \pm \delta_{i,j} K_i^\pm \quad [K_i^-, K_j^+] = 2\delta_{i,j} K_i^o \]

Duality between \( L^{BMP} \) e \( L^{SIP} \) corresponds to two different representations of the operator \( \mathcal{L} \).
Duality fct is the intertwiner.
**SU(1, 1) structure**

### Continuous representation

\[
\begin{align*}
K_i^+ &= \frac{1}{2} x_i^2 \\
K_i^- &= \frac{1}{2} \frac{\partial^2}{\partial x_i^2} \\
K_i^0 &= \frac{1}{4} \left( x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)
\end{align*}
\]

satisfy commutation relations of the \(SU(1, 1)\) Lie algebra

\[
\begin{align*}
[K_i^0, K_i^\pm] &= \pm K_i^\pm \\
[K_i^-, K_i^+] &= 2 K_i^0
\end{align*}
\]
SU(1, 1) structure

Continuous representation

\[ K^+_i = \frac{1}{2} x_i^2 \quad K^-_i = \frac{1}{2} \frac{\partial^2}{\partial x_i^2} \]

\[ K^o_i = \frac{1}{4} \left( x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right) \]

satisfy commutation relations of the SU(1, 1) Lie algebra

\[ [K^o_i, K^\pm_i] = \pm K^\pm_i \quad [K^-_i, K^+_i] = 2K^o_i \]

In this representation

\[ \mathcal{L} = \mathcal{L}^{BMP} = \sum_{(i,j) \in E} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 \]
**SU(1, 1) structure**

**Discrete representation**

\[
\mathcal{K}_i^+ |\xi_i\rangle = \left( \xi_i + \frac{1}{2} \right) |\xi_i + 1\rangle \\
\mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\
\mathcal{K}_i^0 |\xi_i\rangle = \left( \xi_i + \frac{1}{4} \right) |\xi_i\rangle
\]
**SU(1, 1) structure**

**Discrete representation**

\[
\mathcal{K}_i^+ |\xi_i\rangle = \left( \xi_i + \frac{1}{2} \right) |\xi_i + 1\rangle \\
\mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\
\mathcal{K}_i^0 |\xi_i\rangle = \left( \xi_i + \frac{1}{4} \right) |\xi_i\rangle
\]

In a canonical base

\[
\mathcal{K}_i^+ = \begin{pmatrix}
0 \\
\frac{1}{2} \\
\cdot \\
\cdot \\
3 \\
\cdot \\
\cdot
\end{pmatrix}
\quad
\mathcal{K}_i^- = \begin{pmatrix}
0 & 1 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
2 & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot
\end{pmatrix}
\quad
\mathcal{K}_i^0 = \begin{pmatrix}
\frac{1}{4} & 0 & \cdot \\
\cdot & \frac{5}{4} & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\]
SU(1, 1) structure

Discete representation

\[ \mathcal{K}_i^+ |\xi_i\rangle = \left( \xi_i + \frac{1}{2} \right) |\xi_i + 1\rangle \]
\[ \mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \]
\[ \mathcal{K}_i^0 |\xi_i\rangle = \left( \xi_i + \frac{1}{4} \right) |\xi_i\rangle \]

In this representation

\[ \mathcal{L} f(\xi) = L^{SIP} f(\xi) \]
\[ = \sum_{(i,j) \in E} \xi_i \left( \xi_j + \frac{1}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \left( \xi_i + \frac{1}{2} \right) \xi_j [f(\xi^{j,i}) - f(\xi)] \]
\( SU(1, 1) \) structure

**Intertwiner**

\[
\begin{align*}
K_i^+ D_i(\cdot, \xi_i)(x_i) &= \mathcal{K}_i^+ D_i(x_i, \cdot)(\xi_i) \\
K_i^- D_i(\cdot, \xi_i)(x_i) &= \mathcal{K}_i^- D_i(x_i, \cdot)(\xi_i) \\
K_i^0 D_i(\cdot, \xi_i)(x_i) &= \mathcal{K}_i^0 D_i(x_i, \cdot)(\xi_i)
\end{align*}
\]

From the creation operators

\[
\frac{x_i^2}{2} D_i(x_i, \xi_i) = \left( \xi_i + \frac{1}{2} \right) D(x, \xi_i + 1)
\]

Therefore

\[
D_i(x_i, \xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} D_i(x_i, 0)
\]
Self-duality
Markov chain with finite state space
Markov chain with finite state space

1. Matrix formulation of self-duality ($L_{dual} = L$)

\[ LD = DL^T \]
Markov chain with finite state space

1. Matrix formulation of self-duality \( L_{\text{dual}} = L \)

\[
LD = DL^T
\]

Indeed
Markov chain with finite state space

1. Matrix formulation of self-duality ($L_{dual} = L$)

$$LD = DL^T$$

Indeed

$$LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi)$$
Markov chain with finite state space

1. Matrix formulation of self-duality ($L_{dual} = L$)

$$LD = DL^T$$

Indeed

$$\sum_{\eta'} L(\eta, \eta') D(\eta', \xi) = LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi)$$
Markov chain with finite state space

1. Matrix formulation of self-duality ($L_{dual} = L$)

$$LD = DL^T$$

Indeed

$$\sum_{\eta'} L(\eta, \eta')D(\eta', \xi) = LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi) = \sum_{\xi'} L(\xi, \xi')D(\eta, \xi')$$
Self-Duality

2. trivial self-duality $\iff$ reversible measure $\mu$

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$
Self-Duality

2. trivial self-duality $\Longleftrightarrow$ reversible measure $\mu$

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$

Indeed
Self-Duality

2. trivial self-duality $\iff$ reversible measure $\mu$

\[
d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}
\]

Indeed

\[
Ld(\eta, \xi) = dL^T(\eta, \xi)
\]
Self-Duality

2. trivial self-duality $\iff$ reversible measure $\mu$

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$

Indeed

$$\frac{L(\eta, \xi)}{\mu(\xi)} = Ld(\eta, \xi) = dL^T(\eta, \xi)$$
Self-Duality

2. trivial self-duality $\iff$ reversible measure $\mu$

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta,\xi}$$

Indeed

$$\frac{L(\eta, \xi)}{\mu(\xi)} = Ld(\eta, \xi) = dL^T(\eta, \xi) = \frac{L(\xi, \eta)}{\mu(\eta)}$$
Self-Duality

3. $S$: symmetry of the generator, i.e. $[L, S] = 0$, 
   $\mathbf{d}$: trivial self-duality function, 
   $\longrightarrow \quad D = S \mathbf{d}$ self-duality function.
Self-Duality

3. $S$: symmetry of the generator, i.e. $[L, S] = 0$,
   $d$: trivial self-duality function,
   $\rightarrow$ $D = Sd$ self-duality function.

Indeed
Self-Duality

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   $d$: trivial self-duality function,
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Indeed

$LD$
3. $S$: symmetry of the generator, i.e. $[L, S] = 0$, 
   $d$: trivial self-duality function, 
   $\rightarrow \quad D = Sd$ self-duality function.

Indeed

$$LD = L_{Sd}$$
Self-Duality

3. S: symmetry of the generator, i.e. $[L, S] = 0$,
   d: trivial self-duality function,
   $\rightarrow D = Sd$ self-duality function.

Indeed

$LD = LSh = SLd$
Self-Duality

3. \( S \): symmetry of the generator, i.e. \([L, S] = 0\),
\( d \): trivial self-duality function,
\[\longrightarrow D = Sd \text{ self-duality function.}\]

Indeed

\[LD = L(Sd) = SLd = SdL^T\]
Self-Duality

3. $S$: symmetry of the generator, i.e. $[L, S] = 0$,
   $d$: trivial self-duality function,
   $\rightarrow D = Sd$ self-duality function.

Indeed

$$LD = LSc = SLd = SdT = DL^T$$
Self-Duality

3. \( S \): symmetry of the generator, i.e. \([L, S] = 0\),
\( d \): trivial self-duality function,
\[\longrightarrow\quad D = Sd \text{ self-duality function.}\]

Indeed

\[
LD = L\text{Sd} = S\text{Ld} = SdL^T = DL^T
\]

Self-duality is related to the action of a symmetry.
Theorem 2
The process with generator $L^{SIP}$ is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \xi_i \right)}$$
Self-duality of the SIP process

**Theorem 2**

The process with generator $L^{SIP}$ is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \xi_i \right)}$$

**Proof:**

$$[L^{SIP}, \sum_i K_i^0] = [L^{SIP}, \sum_i K_i^+] = [L^{SIP}, \sum_i K_i^-] = 0$$

Self-duality fct related to the symmetry $S = e^{\sum_i K_i^+}$
Boundary driven systems.
Brownian Momentum Process with reservoirs

\[ L^{res}_L = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \]

\[ L^{res}_R = T_R \frac{\partial^2}{\partial x_N^2} - x_N \frac{\partial}{\partial x_N} \]
Inclusion Process with absorbing reservoirs

\[ L^\text{abs}_1 f(\xi) = 2\xi_1 \left( f(\xi^{1,0}) - f(\xi) \right) \]
Duality between BMP with reservoirs and SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{\text{dual}} = \mathbb{N}^{N+2}$
Duality between BMP with reservoirs and SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

**Theorem 3**

The process $\{x(t)\}_{t \geq 0}$ with generator $L^{BMP, res}$ is dual to the process $\{\bar{\xi}(t)\}_{t \geq 0}$ with generator $L^{SIP, abs}$ on

$$D(x, \bar{\xi}) = T_L^{\xi_0} \left( \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$
CONSEQUENCES OF DUALITY

- From continuous to discrete:
  Interacting diffusions (BMP) studied via particle systems (SIP).

- From many to few:
  \( n \)-points correlation functions of \( N \) particles using \( n \) dual walkers
  **Remark:** \( n \ll N \)

- From reservoirs to absorbing boundaries:
  Stationary state of dual process described by absorption probabilities at the boundaries
Proposition

Let $\mathbb{P}_{\bar{\xi}}(a, b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi})$. Then

$$
\mathbb{E}(D(x, \bar{\xi})) = \sum_{a,b} T^a_L T^b_R \mathbb{P}_{\bar{\xi}}(a, b)
$$
Proposition

Let $\mathbb{P}_{\tilde{\xi}}(a, b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \tilde{\xi})$. Then

$$\mathbb{E}(D(x, \tilde{\xi})) = \sum_{a,b} T^a_L T^b_R \mathbb{P}_{\tilde{\xi}}(a, b)$$

Proof:

$$\mathbb{E}(D(x, \tilde{\xi})) = \lim_{t \to \infty} \mathbb{E}_{x_0}(D(x_t, \tilde{\xi}))$$
Proposition

Let $\mathbb{P}_{\bar{\xi}}(a, b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi})$. Then

$$\mathbb{E}(D(x, \bar{\xi})) = \sum_{a,b} T^a_L T^b_R \mathbb{P}_{\bar{\xi}}(a, b)$$

Proof:

$$\mathbb{E}(D(x, \bar{\xi})) = \lim_{t \to \infty} \mathbb{E}_{x_0}(D(x_t, \bar{\xi}))$$

$$= \lim_{t \to \infty} \mathbb{E}_{\bar{\xi}}(D(x_0, \xi_t))$$
Proposition

Let \( P_{\xi}(a, b) = P(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi}) \). Then

\[
E(D(x, \bar{\xi})) = \sum_{a,b} T_L^a T_R^b \ P_{\xi}(a, b)
\]

Proof:

\[
E(D(x, \bar{\xi})) = \lim_{t \to \infty} E_x(0)(D(x_t, \bar{\xi}))
\]

\[
= \lim_{t \to \infty} E_{\bar{\xi}}(D(x_0, \bar{\xi}_t))
\]

using \( D(x, \bar{\xi}) = T_L^{\xi_0} \left( \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}} \)
Proposition

Let $\mathbb{P}_{\bar{\xi}}(a, b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi})$. Then

$$\mathbb{E}(D(x, \bar{\xi})) = \sum_{a,b} T^a_L T^b_R \mathbb{P}_{\bar{\xi}}(a, b)$$

Proof:

$$\mathbb{E}(D(x, \bar{\xi})) = \lim_{t \to \infty} \mathbb{E}_{x_0}(D(x_t, \bar{\xi}))$$

$$= \lim_{t \to \infty} \mathbb{E}_{\bar{\xi}}(D(x_0, \bar{\xi}_t))$$

using $D(x, \bar{\xi}) = T^\xi_0 \left( \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T^\xi_{N+1}$

$$= \mathbb{E}_{\bar{\xi}}(T^\xi_0(\infty) T^\xi_{N+1}(\infty))$$
Temperature profile

\[ \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 \]

site \( i \) \[ \Rightarrow 1 \text{ SIP walker} \ (X_t)_{t \geq 0} \text{ with } X_0 = i \]
Temperature profile

\[ \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 \]

site \( i \) \n
\[ \Rightarrow 1 \text{ SIP walker } (X_t)_{t \geq 0} \text{ with } X_0 = i \]

\[ \mathbb{E} \left( x_i^2 \right) = T_L \mathbb{P}_i(X_\infty = 0) + T_R \mathbb{P}_i(X_\infty = N + 1) \]
Temperature profile

\[ \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 \]

site \(i\) \(\Rightarrow\) 1 SIP walker \((X_t)_{t \geq 0}\) with \(X_0 = i\)

\[ \mathbb{E}(x_i^2) = T_L \mathbb{P}_i(X_\infty = 0) + T_R \mathbb{P}_i(X_\infty = N + 1) \]

\[ \mathbb{E}(x_i^2) = T_L + \left( \frac{T_R - T_L}{N + 1} \right) i \]

\[ \langle J \rangle = \mathbb{E}(x_{i+1}^2) - \mathbb{E}(x_i^2) = \frac{T_R - T_L}{N + 1} \quad \text{Fourier’s law} \]
Energy covariance

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ ⇒ $D(x, \vec{\xi}) = x_i^2 x_j^2$

site $i$ ➔ site $j$ ➔
Energy covariance

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ \quad \Rightarrow \quad D(x, \vec{\xi}) = x_i^2 x_j^2$

site $i \rightarrow$ site $j \rightarrow$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$
Inclusion Process with absorbing reservoirs

\[ L_{1}^{abs} f(\xi) = 2\xi_{1} \left( f(\xi^{1,0}) - f(\xi) \right) \]
\[ E(x_i^2 x_j^2) = T_L^2 \, P(\text{red}) + T_R^2 \, P(\text{green}) + T_L T_R \, (P(\text{yellow}) + P(\text{blue})) \]
Energy covariance

\[ \mathbb{E} \left( x_i^2 x_j^2 \right) - \mathbb{E} \left( x_i^2 \right) \mathbb{E} \left( x_j^2 \right) = \frac{2i(N + 1 - j)}{(N + 3)(N + 1)^2} (T_R - T_L)^2 \geq 0 \]

**Remark**: up to a sign, covariance is the same in the boundary driven Exclusion Process.
A larger picture & redistribution models

(i). Brownian Energy Process $BEP(m)$

(ii). Instantaneous thermalization

(iii). Symmetric exclusion (SEP(n))
(i) Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

\[ z_i(t) = x_i^2(t) \]
(i) Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

\[ z_i(t) = x_i^2(t) \]

evolve with

**Generator**

\[
L^{BEP} = \sum_{(i,j) \in E} z_i z_j \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{1}{2} (z_i - z_j) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)
\]
Generalized Brownian Energy Process: BEP(m)

\[ L^{BMP}(m) = \sum_{(i,j) \in E} \sum_{\alpha,\beta = 1}^{m} \left( x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 \]
Generalized Brownian Energy Process: BEP$(m)$

\[
L^{BMP}(m) = \sum_{(i,j) \in E} \sum_{\alpha, \beta = 1}^{m} \left( x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2
\]

The energies \( z_i(t) = \sum_{\alpha = 1}^{m} x_{i,\alpha}^2(t) \) evolve with

**Generator**

\[
L^{BEP}(m) = \sum_{(i,j) \in E} z_i z_j \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - m \left( z_i - z_j \right) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)
\]

Stationary measures: product $\text{Gamma}(\frac{m}{2}, \theta)$
Adding-up \( SU(1, 1) \) spins

\[
\mathcal{L}^{(m)} = \sum_{(i,j) \in E} \left( K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^0 K_j^0 + \frac{m^2}{8} \right)
\]

\( \{ K_i^+, K_i^-, K_i^0 \}_{i \in \mathcal{V}} \) satisfy \( SU(1, 1) \)
Adding-up \( SU(1, 1) \) spins

\[
\mathcal{L}^{(m)} = \sum_{(i,j) \in E} \left( \mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^0 \mathcal{K}_j^0 + \frac{m^2}{8} \right)
\]

\( \{ \mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^0 \}_{i \in V} \) satisfy \( SU(1, 1) \)

\[
\begin{align*}
\mathcal{K}_i^+ &= z_i \\
\mathcal{K}_i^- &= z_i \partial_{z_i}^2 + \frac{m}{2} \partial_{z_i} \\
\mathcal{K}_i^0 &= z_i \partial_{z_i} + \frac{m}{4}
\end{align*}
\]

\[
\begin{align*}
\mathcal{K}_i^+ |\xi_i\rangle &= (\xi_i + \frac{m}{2}) |\xi_i + 1\rangle \\
\mathcal{K}_i^- |\xi_i\rangle &= \xi_i |\xi_i - 1\rangle \\
\mathcal{K}_i^0 |\xi_i\rangle &= (\xi_i + m) |\xi_i\rangle
\end{align*}
\]
Generalized Symmetric Inclusion Process: SIP(m)

**Generator**

\[
L_{SIP(m)} f(\xi) = \sum_{(i,j) \in E} \xi_i \left( \xi_j + \frac{m}{2} \right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left( \xi_i + \frac{m}{2} \right) [f(\xi^{j,i}) - f(\xi)]
\]
Duality between BEP(m) and SIP(m)

**Theorem 4**

The process \( \{z(t)\}_{t \geq 0} \) with generator \( L^{\text{BEP}(m)} \) and the process \( \{\xi(t)\}_{t \geq 0} \) with generator \( L^{\text{SIP}(m)} \) are dual on

\[
D(z, \xi) = \prod_{i \in V} z_i^{\xi_i} z_i \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + \xi_i\right)}
\]
(ii) Redistribution models

Generator

\[ L^{\text{KMP}} f(z) = \]
\[ \sum_i \int_0^1 dp[f(z_1, \ldots, p(z_i + z_{i+1}), (1 - p)(z_i + z_{i+1}), \ldots, z_N) - f(z)] \]

KMP model is an instantaneous thermalization limit of BEP(2).
Instantaneous thermalization limit

\[ L_{i,j}^{IT} f(z_i, z_j) \]
Instantaneous thermalization limit

\[ L_{i,j}^{T} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{t L_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j) \]
Instantaneous thermalization limit

\[ L_{i,j}^{IT} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{t L_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j) \]

\[ = \int dz'_i dz'_j \rho^{(m)}(z'_i, z'_j \mid z'_i + z'_j = z_i + z_j) [f(z'_i, z'_j) - f(z_i, z_j)] \]
Instantaneous thermalization limit

\[
L_{i,j}^{IT} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{tL_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j)
\]

\[
= \int dz'_i dz'_j \rho^{(m)}(z'_i, z'_j | z'_i + z'_j = z_i + z_j)[f(z'_i, z'_j) - f(z_i, z_j)]
\]

\[
= \int_0^1 dp \nu^{(m)}(p) \left[ f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j) \right]
\]
Instantaneous thermalization limit

\[ L_{i,j}^{IT} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{tL_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j) \]

\[ = \int dz_i' dz_j' \rho^{(m)}(z_i', z_j' \mid z_i' + z_j' = z_i + z_j) [f(z_i', z_j') - f(z_i, z_j)] \]

\[ = \int_0^1 dp \nu^{(m)}(p) [f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j)] \]

\[ X, Y \sim \text{Gamma} \left( \frac{m}{2}, \theta \right) \text{ i.i.d.} \quad \implies \quad P = \frac{X}{X + Y} \sim \text{Beta} \left( \frac{m}{2}, \frac{m}{2} \right) \]

For \( m = 2 \): uniform redistribution
(iii) Generalized Symmetric Exclusion Process, SEP(n) [Schütz]

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$
(iii) Generalized Symmetric Exclusion Process, SEP(n) [Schütz]

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$

$$L^{\text{SEP}(n)}f(\xi) = \sum_{(i,j) \in E} \xi_i (n - \xi_j)[f(\xi^{i,j}) - f(\xi)] + (n - \xi_i)\xi_j[f(\xi^{j,i}) - f(\xi)]$$
(iii) Generalized Symmetric Exclusion Process, SEP(n) [Schütz]

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$

$$L^{SEP(n)} f(\xi) = \sum_{(i,j) \in E} \xi_i (n - \xi_j)[f(\xi^{i,j}) - f(\xi)] + (n - \xi_i)\xi_j[f(\xi^{j,i}) - f(\xi)]$$

Stationary measures: product with marginals Binomial(n,p)
Generalized Symmetric Exclusion Process: SEP(n)

\[ \mathcal{L}^{(n)} = \sum_{(i,j) \in E} \left( J_i^+ J_i^- + J_i^- J_i^+ + 2J_i^o J_j^o - \frac{n^2}{2} \right) \]

\{ J_i^+, J_i^-, J_i^o \} satisfy SU(2) commutation relations

\[ [J_i^o, J_j^\pm] = \pm \delta_{i,j} J_i^\pm \quad [J_i^-, J_j^+] = -2 \delta_{i,j} J_i^o \]
Generalized Symmetric Exclusion Process: SEP(n)

\[ \mathcal{L}^{(n)} = \sum_{(i,j) \in E} \left( J_i^+ J_j^- + J_i^- J_j^+ + 2J_i^o J_j^o - \frac{n^2}{2} \right) \]

\{J_i^+, J_i^-, J_i^o\} satisfy SU(2) commutation relations

\[ [J_i^o, J_j^\pm] = \pm \delta_{i,j} J_i^\pm \quad [J_i^-, J_j^+] = -2\delta_{i,j} J_i^o \]

\[
\begin{cases}
J_i^+ |\xi_i\rangle = (n - \xi_i) |\xi_i + 1\rangle \\
J_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\
J_i^o |\xi_i\rangle = (\xi_i - \frac{n}{2}) |\xi_i\rangle
\end{cases}
\]
Self-duality of the SEP(n) process

Theorem 5

The process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(n + 1 - \xi_i)}{\Gamma(n + 1)}$$
Self-duality of the SEP(n) process

Theorem 5

The process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(n + 1 - \xi_i)}{\Gamma(n + 1)}$$

Proof:

$$[L^{SEP(n)}, \sum_i J_i^o] = [L^{SEP(n)}, \sum_i J_i^+] = [L^{SEP(n)}, \sum_i J_i^-] = 0$$

Self-duality corresponds to the action of the symmetry $S = e^{\sum_i J_i^+}$
The INCLUSION process is self-dual on

$$D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} + \xi_i \right)}$$
Summary of Self-duality

Theorem 6

The INCLUSION process is self-dual on

$$D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left(\frac{m}{2}\right)}{\Gamma \left(\frac{m}{2} + \xi_i\right)}$$

The INDEPENDENT WALKERS process is self-dual on

$$D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!}$$
Summary of Self-duality

Theorem 6

The INCLUSION process is self-dual on

\[ D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} + \xi_i \right)} \]

The INDEPENDENT WALKERS process is self-dual on

\[ D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!} \]

The EXCLUSION process is self-dual on

\[ D(\eta, \xi) = \prod_i \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( n + 1 - \xi_i \right)}{\Gamma (n + 1)} \]
Perspectives

Energy/particle redistribution models [JSP (2013)]
Instantaneous thermalization limit of inclusion process, independent walkers, exclusion process

Mathematical population genetics: [arXiv:1302.3206]
e.g. Multi-type Wright Fisher diffusion, Moran model

Bulk-driven models: [work in progress]
Asymmetric processes and q-deformed algebras