Stochastic energy-exchange models of non-equilibrium.

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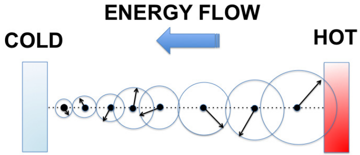
Joint work with J. Kurchan (Paris), F. Redig (Delft) K.Vafayi (Eindhoven), G. Carinci, C. Giberti (Modena).

Fourier law $J = \kappa \nabla T$



- 1D Hamiltonian models:
  - Oscillators chains (Lebowitz, Lieb, Rieder, 1967): $\kappa \sim N$.
  - Non-linear oscillators chains (Lepri, Livi, Politi, Phys. Rep. 2003): $\kappa \sim N^{\alpha}, \quad 0 < \alpha < 1$
  - Non-linear fluctuating hydrodynamics (van Beijeren 2012, Spohn 2013)

Stochastic energy exchange models

Kipnis, Marchioro, Presutti (1982):

Observables: Energies at every site $z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

Dynamics: Select a bond at random and **uniformly** redistribute the energy under the constraint of conserving the total energy.

$$L^{KMP} f(z) =$$
$$\sum_{i=1}^{N} \int_0^1 dp \left[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]$$

Outline

1. From Hamiltonian to stochastics: a simple model.
2. Duality Theory:
   - Brownian Momentum Process (BMP).
   - Symmetric Inclusion Process (SIP).
3. Self-duality (SIP).
4. Boundary driven systems.
5. A larger picture & “redistribution” models.

From Hamiltonian to stochastics

A simple Hamiltonian model (G., Kurchan)

$$H(q,p) = \sum_{i=1}^{N} \frac{1}{2}\Big(p_i - A_i\Big)^2$$

$A = (A_1(q), \ldots, A_N(q))$ "vector potential" in $\mathbb{R}^N$.

$$\begin{aligned} \frac{dq_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \sum_{j=1}^{N} B_{ij} v_j \end{aligned}$$

where

$$B_{ij}(q) = \frac{\partial A_i(q)}{\partial q_j} - \frac{\partial A_j(q)}{\partial q_i}$$

antisymmetric matrix containing the "magnetic fields"

Conservation laws

- *Conservation of Energy:*
Even if the forces depend on velocities and positions, the model conserves the total (kinetic) energy

$$\frac{d}{dt}\left(\sum_i \frac{1}{2} v_i^2\right) = \sum_{i,j} B_{ij} v_i v_j = 0$$

- *Conservation of Momentum:*
If we choose the $A_i(x)$ such that they are left invariant by the simultaneous translations $x_i \to x_i + \delta$, then the quantity $\sum_i p_i$ is conserved.

Example: discrete time dynamics with "magnetic kicks"

$$
\begin{aligned}
q(t+1) &= q(t) + v(t) \\
v(t+1) &= R(t+1) \cdot v(t)
\end{aligned}
$$

with $R(t)$ a rotation matrix

$$
R(t+1) = \begin{pmatrix} \cos(B(q(t+1))) & \sin(B(q(t+1))) \\ -\sin(B(q(t+1))) & \cos(B(q(t+1))) \end{pmatrix}
$$

Chaoticity properties of the map on $\mathbb{T}_2$

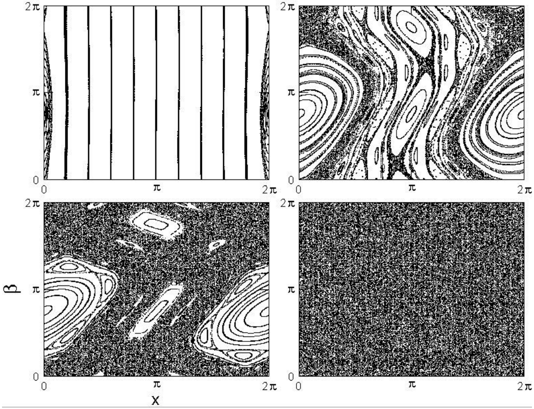
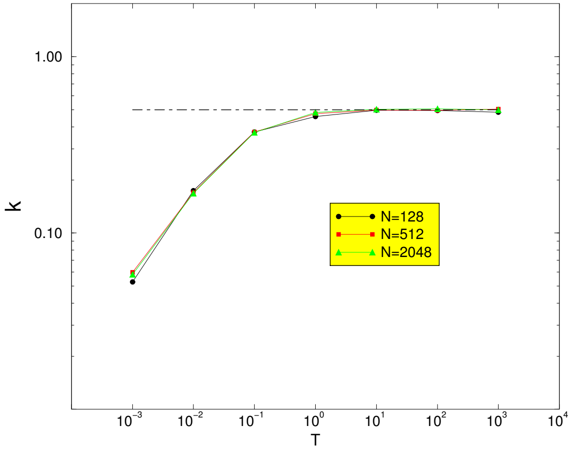


Figure : Poincare section with plane $q^{(2)} = 0$ of the map $\begin{cases} q^{(1)}_{t+1} = & q^{(1)}_t + v\cos(\beta_t) \\ q^{(2)}_{t+1} = & q^{(2)}_t + v\sin(\beta_t) \\ \beta_{t+1} = & \beta_t + B(q^{(1)}_t, q^{(2)}_t) \end{cases}$

with $v = \sqrt{v_1^2 + v_2^2}$, $\quad \beta = \arctan(v_2/v_1)$, $\quad B(q^{(1)}, q^{(2)}) = q^{(1)} + q^{(2)} - 2\pi$.

Numerical result



Thermal conductivity

Duality theory

Duality

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

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ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

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Equivalently

$$LD(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$

How to find a dual process?

1. Write the generator in abstract form , i.e. as an element of a Lie algebra, using creation and annihilation operators.
2. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra.
3. Self-duality is associated to symmetries, i.e. conserved quantities.

The method at work

Brownian momentum process

$\downarrow$

SU(1,1) algebra

$\downarrow$

Inclusion process

Brownian momentum process (BMP) on two sites

Given $(x_i, x_j) \equiv$ velocities of the couple (i, j)

$$L_{i,j}^{BMP} f(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)$$

- polar coordinates $$L_{i,j}^{BMP} = \frac{\partial^2}{\partial \theta_{ij}^2}$$
- Brownian motion for angle $\theta_{i,j} = \arctan(x_j / x_i)$
- total kinetic energy conserved: $r_{i,j}^2 = x_i^2 + x_j^2$

Brownian momentum process (BMP)

For a graph $G = (V, E)$ let $\Omega = \otimes_{i \in V} \Omega_i = \mathbb{R}^{|V|}$.
Configuration $x = (x_1, \ldots, x_{|V|}) \in \Omega$

Generator BMP

$$L^{BMP} = \sum_{(i,j) \in E} L_{i,j}^{BMP} = \sum_{(i,j) \in E} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2$$

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Stationary measures: Gaussian product measures

$$d\mu(x) = \prod_{i=1}^{|V|} \frac{e^{-\frac{x_i^2}{2T}}}{\sqrt{2\pi T}} \, dx_i$$

Symmetric Inclusion Process (SIP)

$\Omega_{dual} = \otimes_{i \in V} \Omega_i^{dual} = \{0, 1, 2, \dots\}^{|V|}$
Configuration $\xi = (\xi_1, \dots, \xi_{|V|}) \in \Omega_{dual}$

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Generator SIP

$$
\begin{aligned}
L^{SIP} f(\xi) &= \sum_{(i,j) \in E} L_{i,j}^{SIP} f(\xi) \\
&= \sum_{(i,j) \in E} \xi_i \left(\xi_j + \frac{1}{2}\right) [f(\xi^{i,j}) - f(\xi)] + \left(\xi_i + \frac{1}{2}\right) \xi_j \, [f(\xi^{j,i}) - f(\xi)]
\end{aligned}
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\end{aligned}
$$

Stationary measures: product of Negative Binomial(r, p) with $r = 2$

$$
\mathbb{P}_r(\xi_1 = n_1, \ldots, \xi_{|V|} = n_{|V|}) = \prod_{i=1}^{|V|} \frac{p^{n_i} (1-p)^r}{n_i!} \frac{\Gamma(r + n_i)}{\Gamma(r)}
$$

Duality between BMP and SIP

Theorem 1

The process $\{x(t)\}_{t\geq 0}$ with generator $L = L^{BMP}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $L_{dual} = L^{SIP}$ are dual on

$$D(x,\xi) = \prod_{i\in V} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

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Proof: An explicit computation gives

$$L^{BMP} D(\cdot,\xi)(x) = L^{SIP} D(x,\cdot)(\xi)$$

Duality explained

$SU(1,1)$ ferromagnetic quantum spin chain

Abstract operator

$$\mathscr{L} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{1}{8} \right)$$

with $\{K_i^+, K_i^-, K_i^o\}_{i\in V}$ satisfying $SU(1,1)$ commutation relations:

$$[K_i^o, K_j^{\pm}] = \pm\delta_{i,j} K_i^{\pm} \qquad [K_i^-, K_j^+] = 2\delta_{i,j} K_i^o$$

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Duality between L^{BMP} e L^{SIP} corresponds to two different representations of the operator $\mathscr{L}$.
Duality fct is the intertwiner.

$SU(1,1)$ structure

Continuous representation

$$K_i^+ = \frac{1}{2} x_i^2 \qquad K_i^- = \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$K_i^o = \frac{1}{4}\left(x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)$$

satisfy commutation relations of the $SU(1,1)$ Lie algebra

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In this representation

$$\mathscr{L} = L^{BMP} = \sum_{(i,j)\in E}\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right)^2$$

$SU(1,1)$ structure

Discrete representation

$$\mathcal{K}_i^+|\xi_i\rangle = \left(\xi_i + \frac{1}{2}\right)|\xi_i + 1\rangle$$

$$\mathcal{K}_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$

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In a canonical base

$$\mathcal{K}_i^+ = \begin{pmatrix} 0 & & & \\ \frac{1}{2} & \ddots & & \\ & \frac{3}{2} & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad \mathcal{K}_i^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad \mathcal{K}_i^0 = \begin{pmatrix} \frac{1}{4} & 0 & & \\ & \frac{5}{4} & \ddots & \\ & & \frac{9}{4} & \ddots \\ & & & \ddots \end{pmatrix}$$

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In this representation

$$\begin{aligned}\mathscr{L}f(\xi) &= L^{SIP}f(\xi) \\ &= \sum_{(i,j)\in E} \xi_i\left(\xi_j + \frac{1}{2}\right)[f(\xi^{i,j}) - f(\xi)] + \left(\xi_i + \frac{1}{2}\right)\xi_j[f(\xi^{j,i}) - f(\xi)]\end{aligned}$$

$SU(1,1)$ structure

Intertwiner

$$K_i^+ D_i(\cdot,\xi_i)(x_i) = \mathcal{K}_i^+ D_i(x_i,\cdot)(\xi_i)$$
$$K_i^- D_i(\cdot,\xi_i)(x_i) = \mathcal{K}_i^- D_i(x_i,\cdot)(\xi_i)$$
$$K_i^o D_i(\cdot,\xi_i)(x_i) = \mathcal{K}_i^o D_i(x_i,\cdot)(\xi_i)$$

From the creation operators

$$\frac{x_i^2}{2} D_i(x_i,\xi_i) = \left(\xi_i + \frac{1}{2}\right) D(x,\xi_i+1)$$

Therefore

$$D_i(x_i,\xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i-1)!!} D_i(x_i,0)$$

Self-duality

Markov chain with finite state space

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1. Matrix formulation of self-duality ($L_{dual} = L$)

$$\mathbf{LD} = \mathbf{DL}^T$$

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$$LD(\cdot,\xi)(\eta) = LD(\eta,\cdot)(\xi)$$

Markov chain with finite state space

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Markov chain with finite state space

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2. trivial self-duality $\Longleftrightarrow$ reversible measure μ

$$\mathbf{d}(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta,\xi}$$

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3. S: symmetry of the generator, i.e. $[\mathbf{L}, \mathbf{S}] = 0$,
d: trivial self-duality function,

$$\longrightarrow \quad \mathbf{D} = \mathbf{S}\mathbf{d} \text{ self-duality function.}$$

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$\mathbf{LD}$

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Self-duality is related to the action of a symmetry.

Self-duality of the SIP process

Theorem 2

The process with generator L^{SIP} is self-dual on functions

$$D(\eta, \xi) = \prod_{i \in V} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \xi_i\right)}$$

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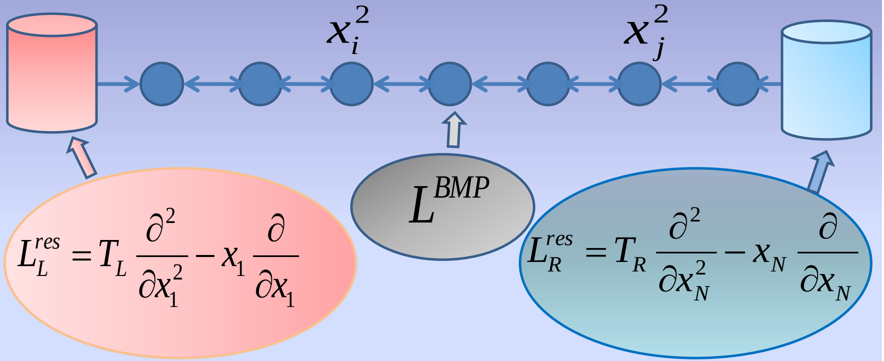
Proof:

$$[L^{SIP}, \sum_i K_i^o] = [L^{SIP}, \sum_i K_i^+] = [L^{SIP}, \sum_i K_i^-] = 0$$

Self-duality fct related to the simmetry $S = e^{\sum_i K_i^+}$

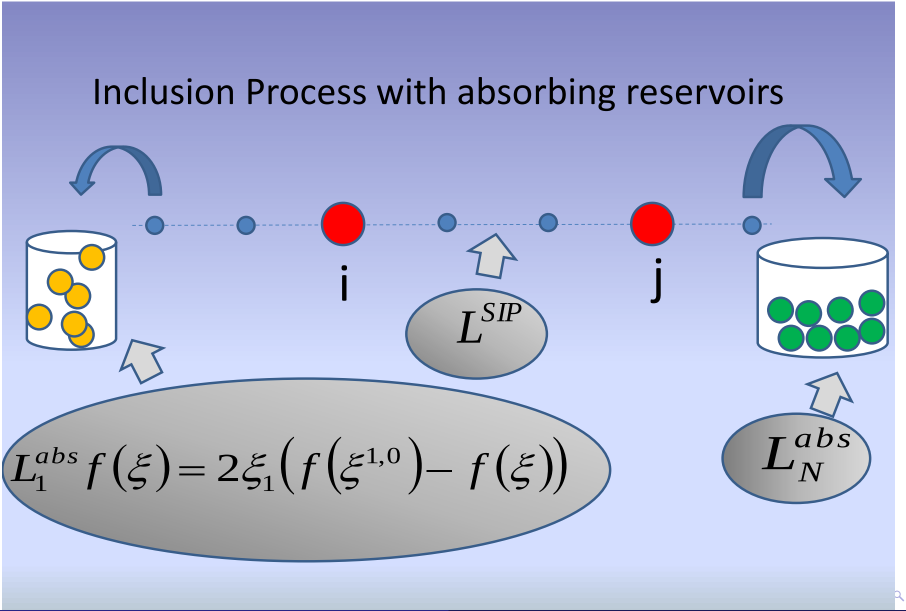
Boundary driven systems.

Brownian Momentum Process with reservoirs



$$L_L^{res} = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1}$$

$$L_R^{res} = T_R \frac{\partial^2}{\partial x_N^2} - x_N \frac{\partial}{\partial x_N}$$


Inclusion Process with absorbing reservoirs
i
j
L^{SIP}
$L_1^{abs} f(\xi) = 2\xi_1 \left(f(\xi^{1,0}) - f(\xi) \right)$
L_N^{abs}

Duality between BMP with reservoirs and SIP with absorbing boundaries

Configurations $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

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Theorem 3

The process $\{x(t)\}_{t\geq 0}$ with generator $L^{BMP,res}$ is dual to the process $\{\bar{\xi}(t)\}_{t\geq 0}$ with generator $L^{SIP,abs}$ on

$$D(x, \bar{\xi}) = T_L^{\xi_0} \left(\prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!} \right) T_R^{\xi_{N+1}}$$

CONSEQUENCES OF DUALITY

- From continuous to discrete:
  Interacting diffusions (BMP) studied via particle systems (SIP).

- From many to few:
  n-points correlation functions of N particles using n dual walkers
  Remark: $n \ll N$

- From reservoirs to absorbing boundaries:
  Stationary state of dual process described by absorption probabilities at the boundaries

Proposition

Let $\mathbb{P}_{\bar{\xi}}(a,b) = \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b \mid \xi(0) = \bar{\xi})$. Then

$$\mathbb{E}(D(x,\bar{\xi}\,)) = \sum_{a,b} T_L^a T_R^b \;\; \mathbb{P}_{\bar{\xi}}(a,b)$$

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Proof:

$$\mathbb{E}(D(x,\bar{\xi}\,)) = \lim_{t\to\infty} \mathbb{E}_{x_0}(D(x_t,\bar{\xi}\,))$$

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$$\mathbb{E}(D(x,\bar{\xi}\,)) = \sum_{a,b} T_L^a T_R^b \;\mathbb{P}_{\bar{\xi}}(a,b)$$

Proof:

$$\begin{aligned}\mathbb{E}(D(x,\bar{\xi}\,)) &= \lim_{t\to\infty} \mathbb{E}_{x_0}(D(x_t,\bar{\xi}\,)) \\ &= \lim_{t\to\infty} \;\mathbb{E}_{\bar{\xi}}(D(x_0,\bar{\xi}_t\,))\end{aligned}$$

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$$= \mathbb{E}_{\bar{\xi}}(\,T_L^{\xi_0(\infty)} T_R^{\xi_{N+1}(\infty)})$$

Temperature profile

$$\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2$$

site i $\nearrow$

$\Rightarrow$ 1 SIP walker $(X_t)_{t \geq 0}$ with $X_0 = i$

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$$\mathbb{E}(x_i^2) = T_L + \left(\frac{T_R - T_L}{N+1}\right) i$$

$$\langle J \rangle = \mathbb{E}(x_{i+1}^2) - \mathbb{E}(x_i^2) = \frac{T_R - T_L}{N+1} \quad \textit{Fourier's law}$$

Energy covariance

$$\text{If } \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad D(x, \vec{\xi}) = x_i^2 x_j^2$$

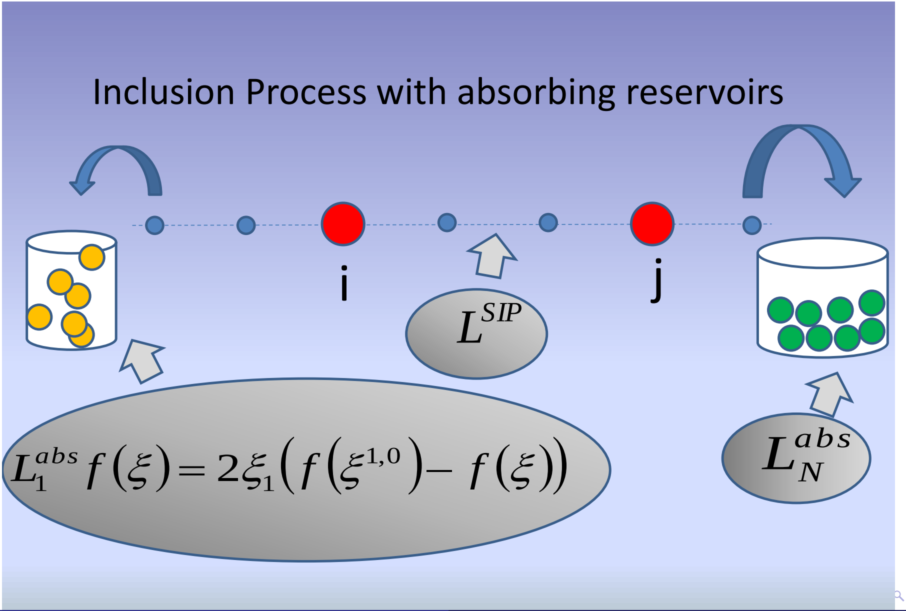
site i $\nearrow$ site j $\nearrow$

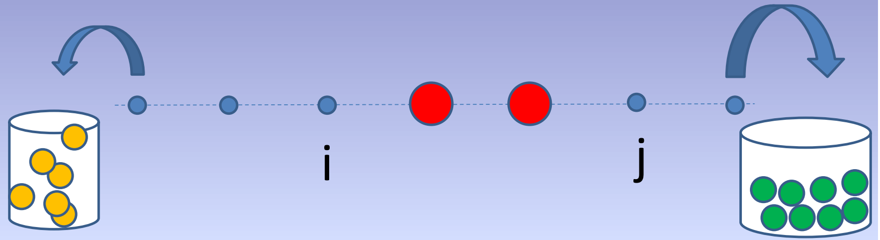
Energy covariance

$$\text{If } \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad D(x, \vec{\xi}) = x_i^2 x_j^2$$

site i $\nearrow$ site j $\nearrow$

In the dual process we initialize two
SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$


Inclusion Process with absorbing reservoirs
i
j
L^{SIP}
$L_1^{abs} f(\xi) = 2\xi_1 \left(f(\xi^{1,0}) - f(\xi) \right)$
L_N^{abs}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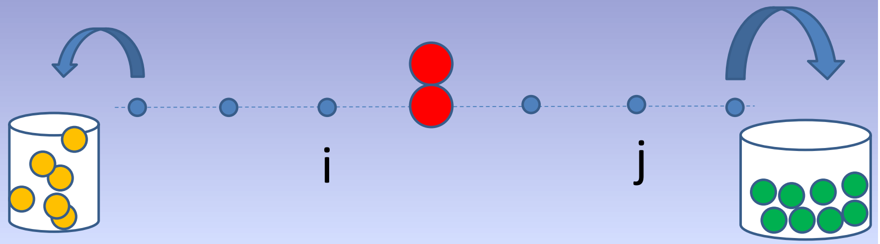
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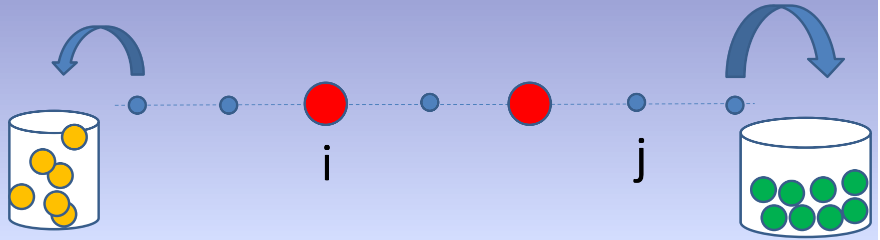
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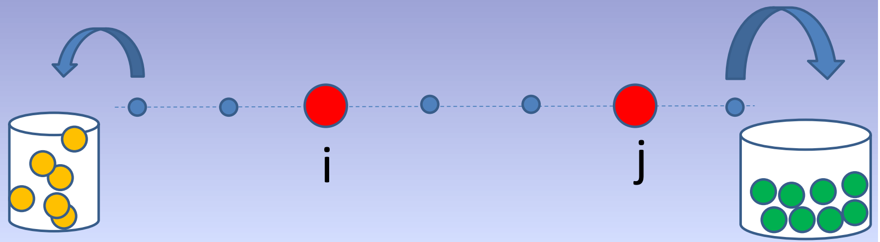
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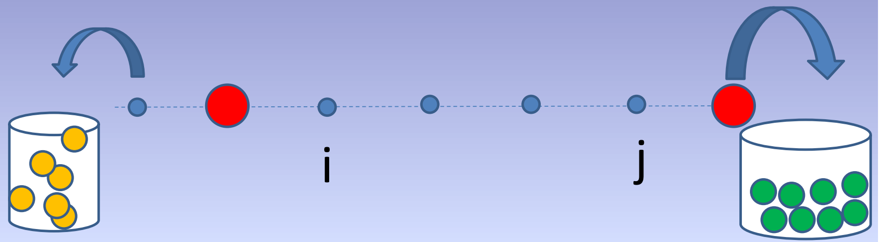
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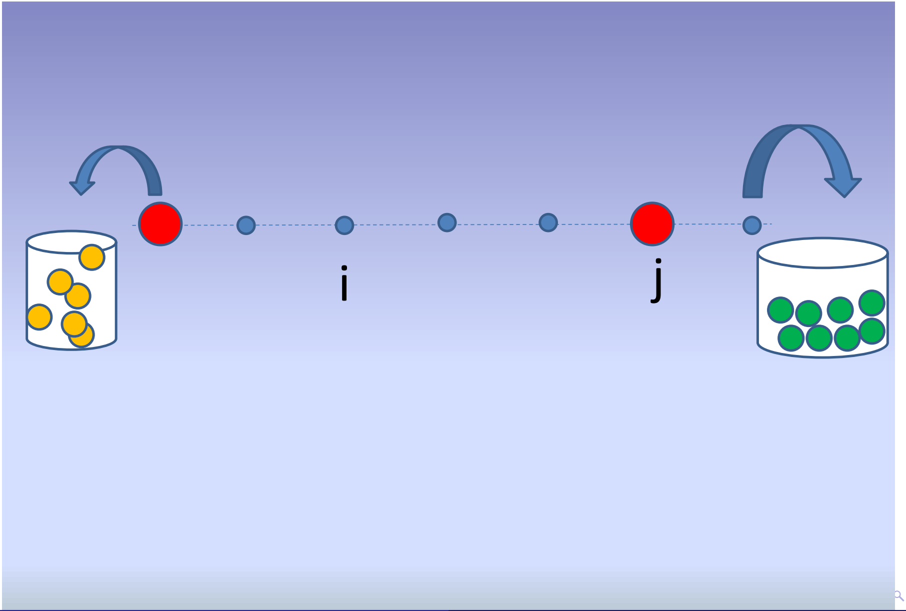
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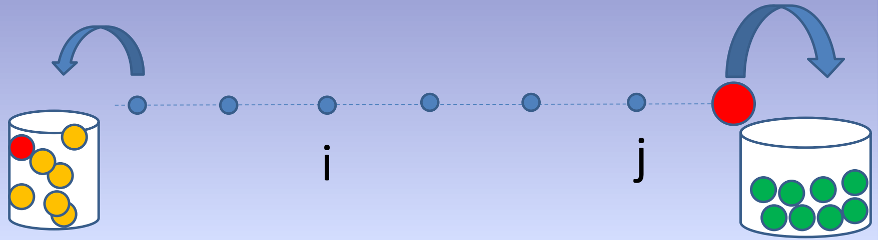
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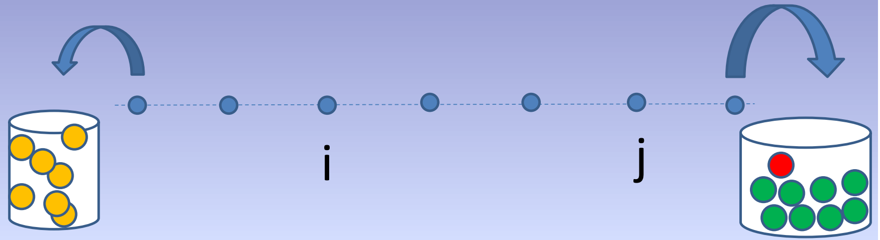
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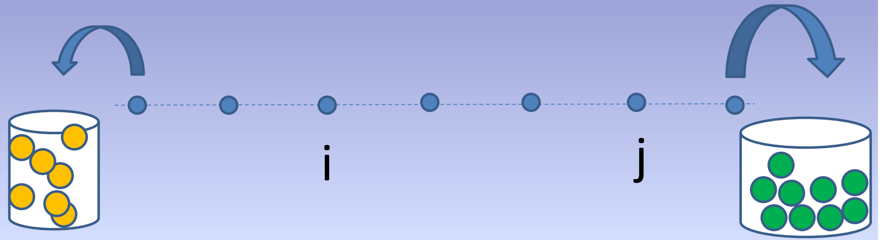
j



$$\mathbf{E}\left(x_i^2 x_j^2\right) = T_L^2\,\mathrm{P}\begin{pmatrix}{\color{orange}\bullet}\\{\color{orange}\bullet}\end{pmatrix} + T_R^2\,\mathrm{P}\begin{pmatrix}{\color{green}\bullet}\\{\color{green}\bullet}\end{pmatrix} + T_L T_R\left(\mathrm{P}\begin{pmatrix}{\color{orange}\bullet}\\{\color{green}\bullet}\end{pmatrix} + \mathrm{P}\begin{pmatrix}{\color{green}\bullet}\\{\color{orange}\bullet}\end{pmatrix}\right)$$

Energy covariance

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \geq 0$$

Remark: up to a sign, covariance is the same in the boundary driven Exclusion Process.

A larger picture & redistribution models

(i). Brownian Energy Process $BEP(m)$

(ii). Instantaneous thermalization

(iii). Symmetric exclusion (SEP(n))

(i) Brownian Energy Process: BEP

The energies of the Brownian Momentum Process

$$z_i(t) = x_i^2(t)$$

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evolve with

Generator

$$L^{BEP} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^2 - \frac{1}{2}(z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)$$

Generalized Brownian Energy Process: BEP(m)

$$L^{BMP(m)} = \sum_{(i,j)\in E} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2$$

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The energies $z_i(t) = \sum_{\alpha=1}^{m} x_{i,\alpha}^2(t)$ evolve with

Generator

$$L^{BEP(m)} = \sum_{(i,j)\in E} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - \frac{m}{2}(z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)$$

Stationary measures: product $\mathrm{Gamma}(\frac{m}{2}, \theta)$

Adding-up $SU(1,1)$ spins

$$\mathcal{L}^{(m)} = \sum_{(i,j)\in E} \left(\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^o \mathcal{K}_j^o + \frac{m^2}{8} \right)$$

$$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i\in V} \qquad \text{satisfy } SU(1,1)$$

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$$\{\mathcal{K}_i^+, \mathcal{K}_i^-, \mathcal{K}_i^o\}_{i\in V} \qquad \text{satisfy } SU(1,1)$$

$$\begin{cases} \mathcal{K}_i^+ = z_i \\ \mathcal{K}_i^- = z_i\,\partial_{z_i}^2 + \frac{m}{2}\partial_{z_i} \\ \mathcal{K}_i^0 = z_i\,\partial_{z_i} + \frac{m}{4} \end{cases} \qquad \begin{cases} \mathcal{K}_i^+|\xi_i\rangle = \left(\xi_i + \frac{m}{2}\right)|\xi_i + 1\rangle \\ \mathcal{K}_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle \\ \mathcal{K}_i^o|\xi_i\rangle = (\xi_i + m)\,|\xi_i\rangle \end{cases}$$

Generalized Symmetric Inclusion Process: SIP(m)

Generator

$$L^{SIP(m)} f(\xi) = \sum_{(i,j)\in E} \xi_i \left(\xi_j + \frac{m}{2}\right) [f(\xi^{i,j}) - f(\xi)] + \xi_j \left(\xi_i + \frac{m}{2}\right) [f(\xi^{j,i}) - f(\xi)]$$

Duality between BEP(m) and SIP(m)

Theorem 4

The process $\{z(t)\}_{t\geq 0}$ with generator $L^{BEP(m)}$ and the process $\{\xi(t)\}_{t\geq 0}$ with generator $L^{SIP(m)}$ are dual on

$$D(z,\xi) = \prod_{i\in V} z_i^{\xi_i} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}$$

(ii) Redistribution models

Generator

$$L^{KMP}f(z) =$$
$$\sum_i \int_0^1 dp[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z)]$$

KMP model is an instantaneous thermalization limit of BEP(2).

Instantaneous thermalization limit

$L^{IT}_{i,j} f(z_i, z_j)$

Instantaneous thermalization limit

$$L_{i,j}^{IT} f(z_i, z_j) := \lim_{t \to \infty} \left(e^{tL_{i,j}^{BEP(m)}} - 1 \right) f(z_i, z_j)$$

Instantaneous thermalization limit

$$
\begin{aligned}
L^{IT}_{i,j} f(z_i, z_j) &:= \lim_{t\to\infty} \left(e^{tL^{BEP(m)}_{i,j}} - 1 \right) f(z_i, z_j) \\
&= \int dz_i' dz_j' \; \rho^{(m)}(z_i', z_j' \mid z_i' + z_j' = z_i + z_j)[f(z_i', z_j') - f(z_i, z_j)]
\end{aligned}
$$

Instantaneous thermalization limit

$$
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&= \int_0^1 dp \; \nu^{(m)}(p) \; [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]
\end{aligned}
$$

Instantaneous thermalization limit

$$L^{IT}_{i,j} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{tL^{BEP(m)}_{i,j}} - 1 \right) f(z_i, z_j)$$
$$= \int dz'_i dz'_j \; \rho^{(m)}(z'_i, z'_j \mid z'_i + z'_j = z_i + z_j)[f(z'_i, z'_j) - f(z_i, z_j)]$$
$$= \int_0^1 dp \; \nu^{(m)}(p) \; [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

$$X, Y \sim \text{Gamma}\left(\frac{m}{2}, \theta\right) \quad \text{i.i.d.} \quad \Longrightarrow \quad P = \frac{X}{X+Y} \sim \text{Beta}\left(\frac{m}{2}, \frac{m}{2}\right)$$

For $m = 2$: uniform redistribution

(iii) Generalized Symmetric Exclusion Process, SEP(n) [Schütz]

Configuration $\xi = (\xi_1, \ldots, \xi_{|V|}) \in \{0, 1, 2, \ldots, n\}^{|V|}$

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$$L^{SEP(n)} f(\xi) = \sum_{(i,j)\in E} \xi_i (n - \xi_j)[f(\xi^{i,j}) - f(\xi)] + (n - \xi_i)\xi_j [f(\xi^{j,i}) - f(\xi)]$$

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Stationary measures: product with marginals Binomial(n,p)

Generalized Symmetric Exclusion Process: SEP(n)

$$\mathcal{L}^{(n)} = \sum_{(i,j)\in E} \left(J_i^+ J_j^- + J_i^- J_j^+ + 2J_i^o J_j^o - \frac{n^2}{2} \right)$$

$\{J_i^+, J_i^-, J_i^o\}$ satisfy $SU(2)$ commutation relations

$$[J_i^o, J_j^\pm] = \pm\delta_{i,j} J_i^\pm \qquad [J_i^-, J_j^+] = -2\delta_{i,j} J_i^o$$

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$$\begin{cases} J_i^+ |\xi_i\rangle = (n - \xi_i) |\xi_i + 1\rangle \\ \\ J_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle \\ \\ J_i^o |\xi_i\rangle = \left(\xi_i - \frac{n}{2}\right) |\xi_i\rangle \end{cases}$$

Self-duality of the SEP(n) process

Theorem 5

The process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma(n+1-\xi_i)}{\Gamma(n+1)}$$

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The process with generator $L^{SEP(n)}$ is self-dual on functions

$$D(\eta,\xi) = \prod_{i\in V} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma(n+1-\xi_i)}{\Gamma(n+1)}$$

Proof:

$$[L^{SEP(n)}, \sum_i J_i^o] = [L^{SEP(n)}, \sum_i J_i^+] = [L^{SEP(n)}, \sum_i J_i^-] = 0$$

Self-duality corresponds to the action of the symmetry $S = e^{\sum_i J_i^+}$

Summary of Self-duality

Theorem 6

The INCLUSION process is self-dual on

$$D(\eta,\xi) = \prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

Summary of Self-duality

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The INDEPENDENT WALKERS process is self-dual on

$$D(\eta,\xi) = \prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}$$

Summary of Self-duality

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$$D(\eta,\xi) = \prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}$$

The EXCLUSION process is self-dual on

$$D(\eta,\xi) = \prod_i \frac{\eta_i!}{(\eta_i-\xi_i)!}\frac{\Gamma\left(n+1-\xi_i\right)}{\Gamma\left(n+1\right)}$$

Perspectives

Energy/particle redistribution models [JSP (2013)]
Instantaneous thermalization limit of inclusion process, independent walkers, exclusion process

Mathematical population genetics: [arXiv:1302.3206]
e.g. Multi-type Wright Fisher diffusion, Moran model

Bulk-driven models: [work in progress]
Asymmetric processes and q-deformed algebras