Synchronization: Bringing Order to Chaos

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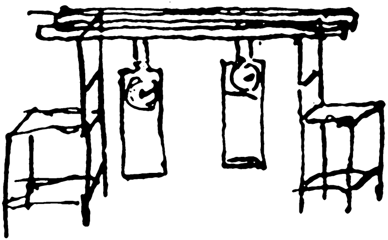
Historical introduction

Christiaan Huygens (1629-1695) first observed a synchronization of two pendulum clocks

He described:

> *"... It is quite worths noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."*



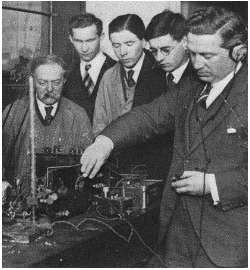
Lord Rayleigh described synchronization in acoustical systems:

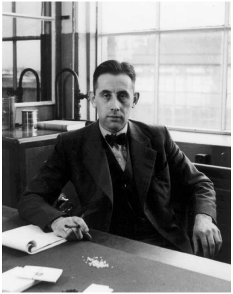
"When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak in absolute unison, in spite of inevitable small differences."

Modern Times

W. H. Eccles and J. H. Vincent applied for a British Patent confirming their discovery of the synchronization property of a triode generator

Edward Appleton and **Balthasar van der Pol** extended the experiments of Eccles and Vincent and made the first step in the theoretical study of this effect (1922-1927)





Biological observations

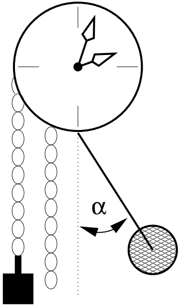
Jean-Jacques Dortous de Mairan reported in 1729 on his experiments with the haricot bean and found a circadian rhythm (24-hours-rhythm): motion of leaves continues even without variations of the illuminance

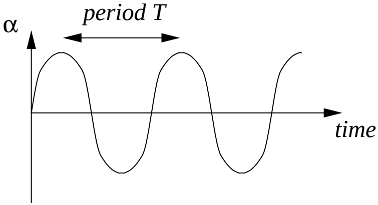
Engelbert Kaempfer wrote after his voyage to Siam in 1680:

> *"The glowworms . . . represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness . . . ".*

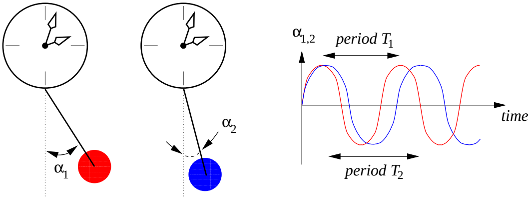
Basic effect

A pendulum clock generates a (nearly) periodic motion characterized by the period T and the frequency $\omega = \frac{2\pi}{T}$

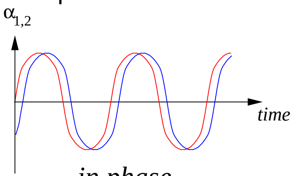




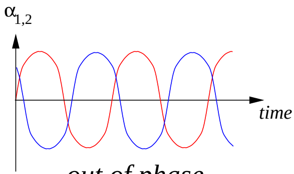
Two such clocks have different periods frequencies



Being coupled, they adjust their rhythms and have the same frequency $\omega_1 < \Omega < \omega_2$. There are different possibilities: in-phase and out-of-phase

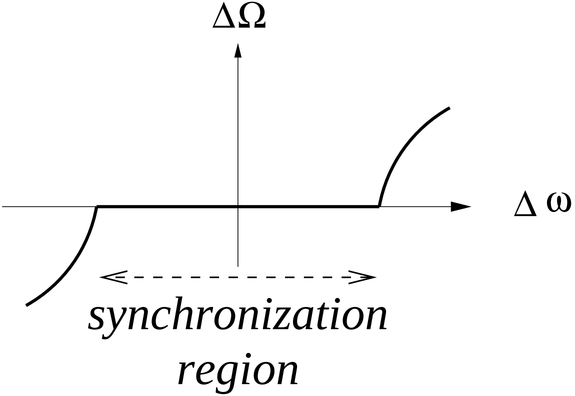


in phase



out of phase

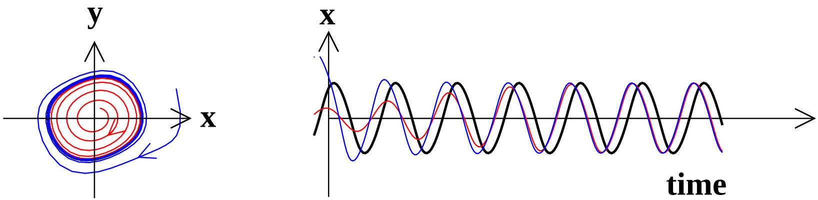
Synchronization occurs within a whole region of parameters



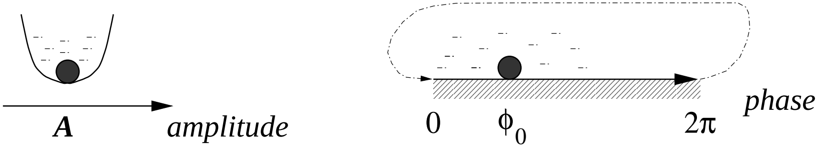
$\Delta\omega$ – frequency mismatch (difference of natural frequencies)
$\Delta\Omega$ – difference of observed frequencies

Self-sustained oscillations

Synchronization is possible for self-sustained oscillators only
Self-sustained oscillators
- generate periodic oscillations
- without periodic forces
- are active/dissipative nonlinear systems
- are described by autonomous ODEs
- are represented by a limit cycle on the phase plane (plane of all variables)

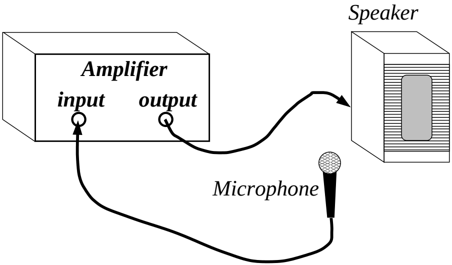


PHASE is the variable proportional to the fraction of the period
amplitude measures deviations from the cycle
- amplitude (form) of oscillations is fixed and stable
- **PHASE** of oscillations is free

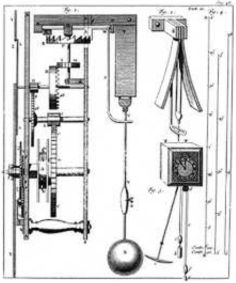


Examples:

amplifier with a feedback loop



clocks: pendulum, electronic,...

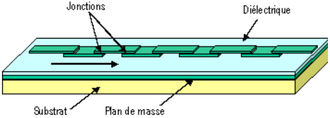


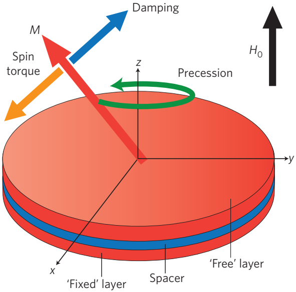
metronom

lasers, elsctronic generators, whistle,

Josephson junction,

spin-torque oscillators

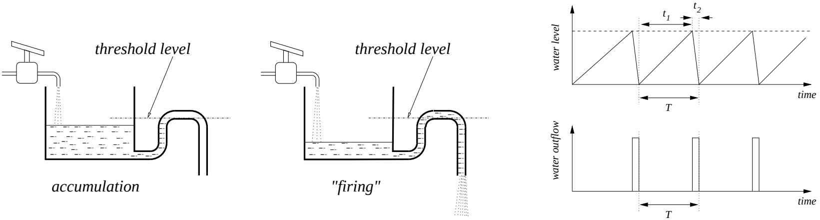




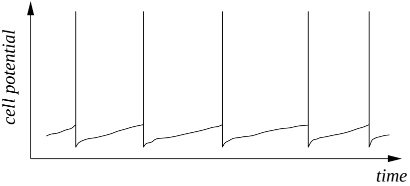
The concept can be extended to non-physical systems!

Ecosystems, predator-pray, rhythmic (e.g. circadian) processes in cells and organizms

relaxation integrate-and-fire oscillators

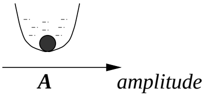


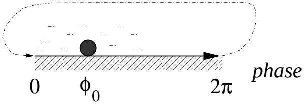
e.g., firing neuron

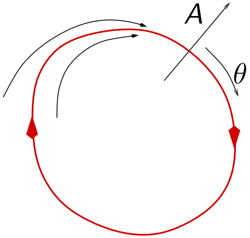


Autonomous oscillator

- amplitude (form) of oscillations is fixed and stable
- **PHASE** of oscillations is free







$\dot{\theta} = \omega_0$ (Lyapunov Exp. 0)

$\dot{A} = -\gamma(A - A_0)$ (LE $-\gamma$)

Forced oscillator

With small periodic external force (e.g. $\sim \varepsilon \sin \omega t$):
only the phase θ is affected

$$\frac{d\theta}{dt} = \omega_0 + \varepsilon G(\theta, \psi) \qquad \frac{d\psi}{dt} = \omega$$

$\psi = \omega t$ is the phase of the external force, $G(\cdot, \cdot)$ is 2π-periodic
If $\omega_0 \approx \omega$ then $\varphi = \theta(t) - \psi(t)$ is slow
$\Rightarrow$ perform averaging by keeping only slow terms (e.g. $\sim \sin(\theta - \psi)$)

$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin \varphi}$$

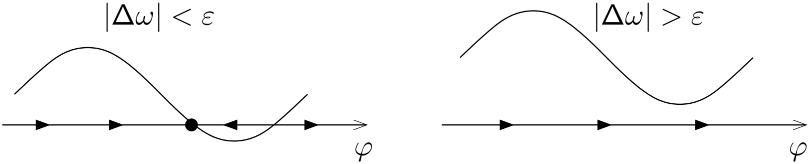
Parameters in the **Adler equation (1946)**:

$$\begin{array}{rl} \Delta\omega = \omega_0 - \omega & \text{detuning} \\ \varepsilon & \text{forcing strength} \end{array}$$

Solutions of the Adler equation

$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin\varphi}$$

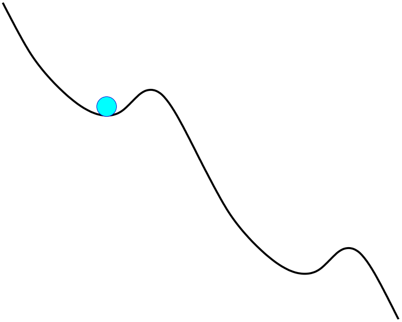
Fixed point for $|\Delta\omega| < \varepsilon$:
Frequency entrainment $\Omega = \langle\dot{\theta}\rangle = \omega$
Phase locking $\varphi = \theta - \psi = \text{const}$
Periodic orbit for $|\Delta\omega| > \varepsilon$: an asynchronous quasiperiodic motion



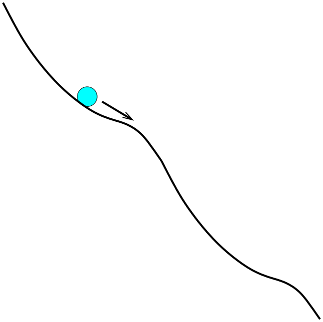
Phase dynamics as a motion of an overdamped particle in an inclined potential

$$\frac{d\varphi}{dt} = -\frac{dU(\varphi)}{d\varphi} \qquad U(\varphi) = -\Delta\omega \cdot \varphi + \varepsilon \cos\varphi$$

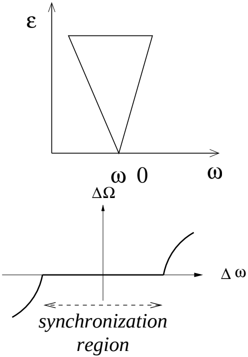
$|\Delta\omega| < \varepsilon$

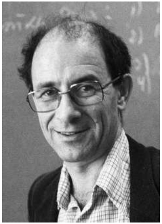


$|\Delta\omega| > \varepsilon$



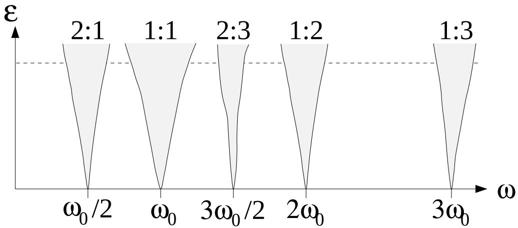
Synchronization region – Arnold tongue

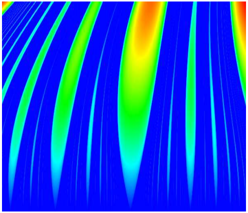


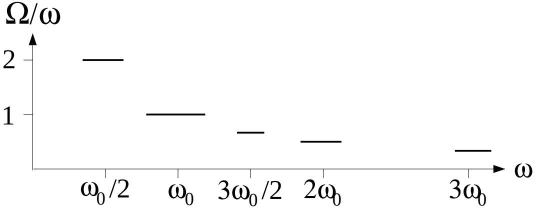


Unusual situation: synchronization occurs for very small force $\varepsilon \to 0$, but cannot be obtain with a simple perturbation method: the perturbation theory is singular due to a degeneracy (vanishing Lyapunov exponent)

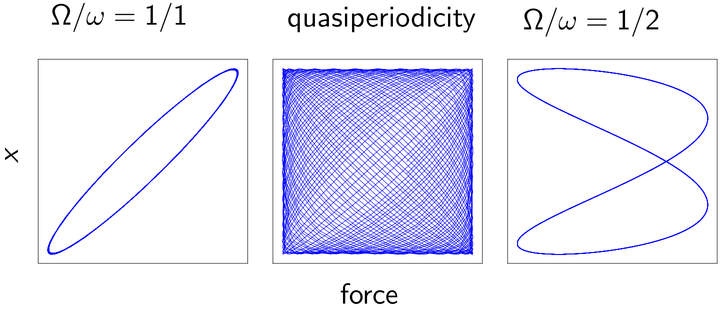
More generally: synchronization of higher order is possible, whith a relation $\frac{\Omega}{\omega} = \frac{m}{n}$



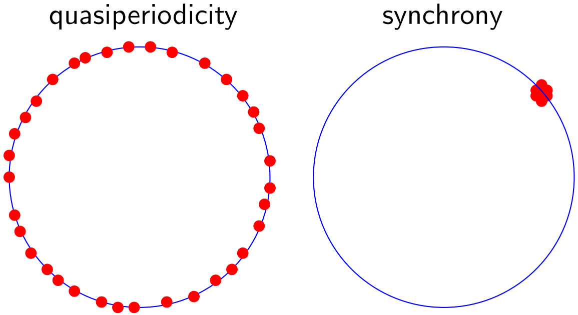




The simplest ways to observe synchronization:
Lissajous figure

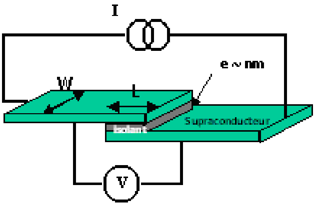


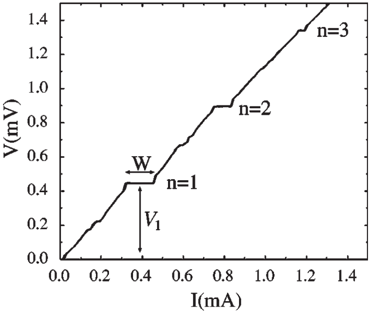
Stroboscopic observation:
Plot phase at each period of forcing



Example: Periodically driven Josephson junction

Synchronization regions – Shapiro steps, frequency $\sim$ voltage
$V = \frac{\hbar}{2e}\dot{\varphi}$





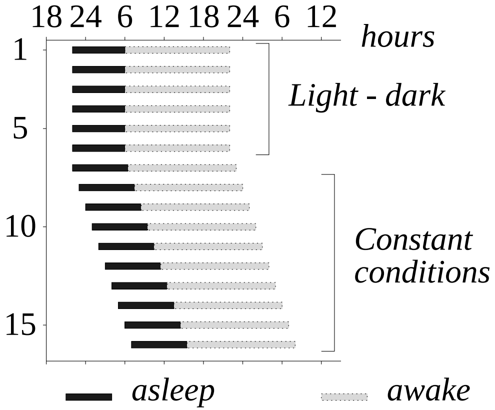
[Lab. Nat. de métrologie et d'essais]

Example: Radio-controlled clocks

Atomic clocks in the PTB, Braunschweig

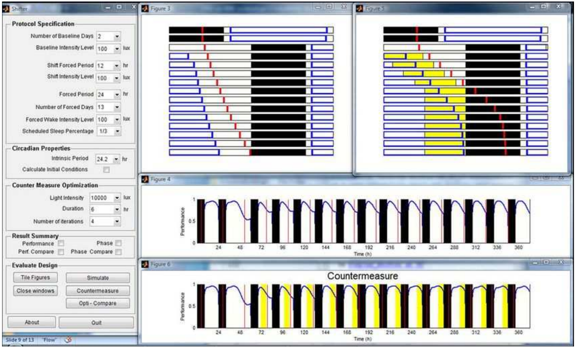
Radio-controlled clocks

Example: circadian rhythm



Jet-lag is the result of the phase difference shift – one needs to resynchronize

One can control re-synchronization (eg for shift-work in space)



[E. Klerman, Brigham and Women's Hospital, Boston]

Mutual synchronization

Two non-coupled self-sustained oscillators:

$$\frac{d\theta_1}{dt} = \omega_1 \qquad \frac{d\theta_2}{dt} = \omega_2$$

Two weakly coupled oscillators:

$$\frac{d\theta_1}{dt} = \omega_1 + \varepsilon G_1(\theta_1, \theta_2) \qquad \frac{d\theta_2}{dt} = \omega_2 + \varepsilon G_2(\theta_1, \theta_2)$$

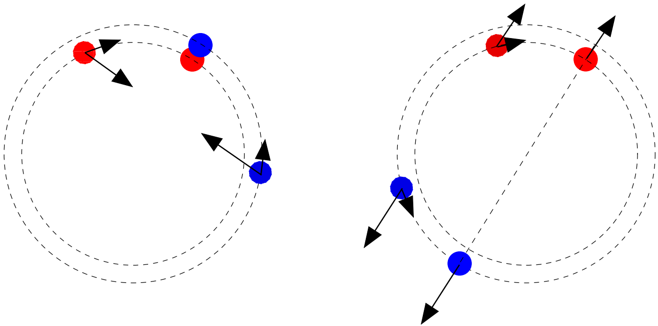
For $\omega_1 \approx \omega_2$ the phase difference $\varphi = \theta_1 - \theta_2$ is slow
$\Rightarrow$ averaging leads to the Adler equation

$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin\varphi}$$

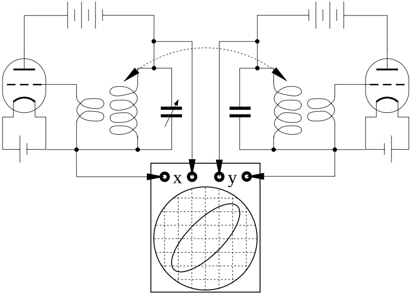
Parameters:

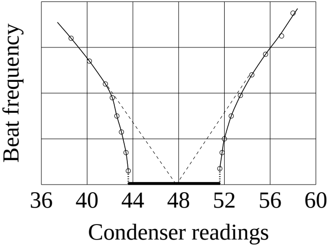
$$\begin{array}{rl} \Delta\omega = \omega_1 - \omega_2 & \text{detuning} \\ \varepsilon & \text{coupling strength} \end{array}$$

Interaction of two periodic oscillators may be attractive ore repulsive: one observes **in phase** or **out of phase** synchronization, correspondingly



Example: classical experiments by Appleton





Example: Metronoms

Attracting coupling: synchrony in phase

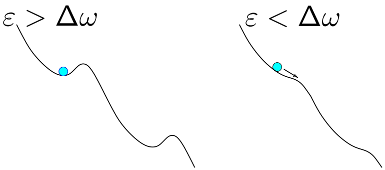
 $\Leftrightarrow$ 

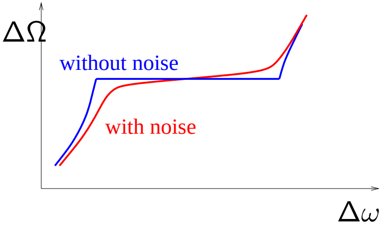
Repulsive coupling: synchrony out of phase

 $\Leftrightarrow$ 

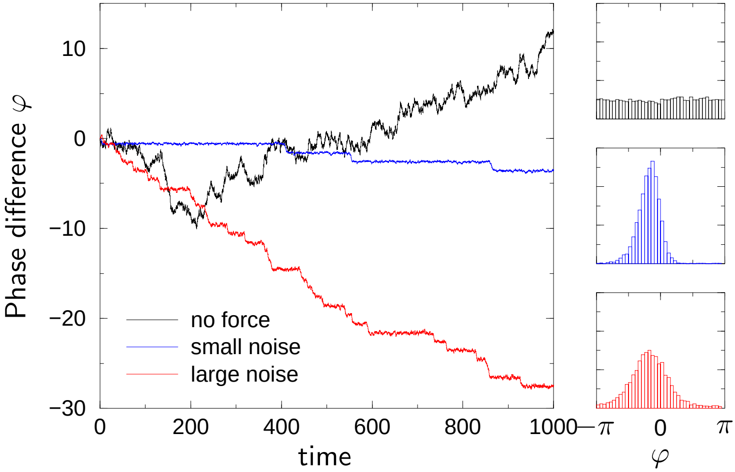
Effect of noise

The Langevin dynamics of the phase =
the Langevin dynamics of an overdamped particle
in an inclined ($\propto \Delta\omega$) potential
ρ = mean flow = smooth function of parameters





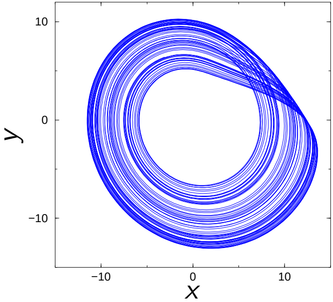
Large noise: no synchronization
Small noise: rare irregular phase slips and long phase-locked intervals



Phase of a chaotic oscillator

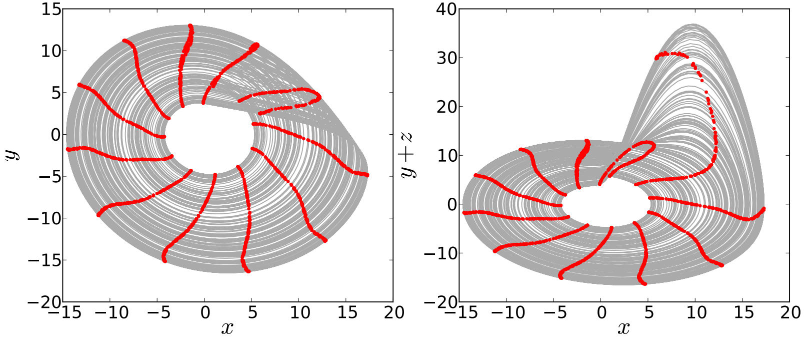
Rössler attractor:

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + 0.15y\\ \dot{z} &= 0.4 + z(x - 8.5)\end{aligned}$$



phase should correspond to the zero Lyapunov exponent!

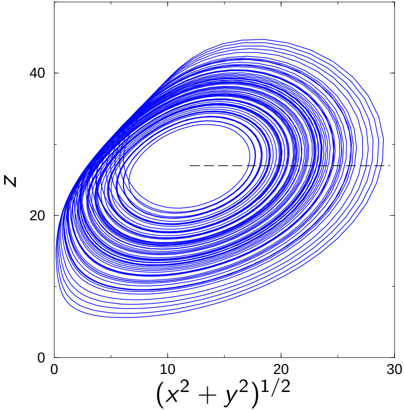
naive definition of the phase: $\theta = \arctan(y/x)$
More advanced: From the condition of maximally uniform rotation
[J. Schwabedal et al, PRE (2012)]



For the topologically simple attractors all definitions are good

Lorenz attractor:

$$\begin{aligned}\dot{x} &= 10(y - x)\\ \dot{y} &= 28x - y - xz\\ \dot{z} &= -8/3z + xy\end{aligned}$$



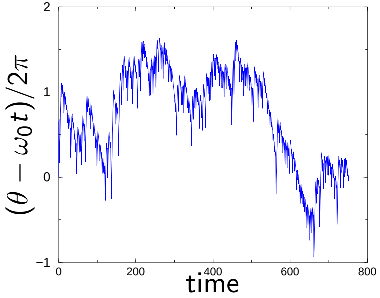
Phase dynamics in a chaotic oscillator

A model phase equation: $\frac{d\theta}{dt} = \omega_0 + F(A)$
(first return time to the surface of section depends on the coordinate on the surface)

A: chaotic $\Rightarrow$ phase diffusion $\Rightarrow$ broad spectrum

$$\langle(\theta(t) - \theta(0) - \omega_0 t)^2\rangle \propto D_p t$$

D_p **measures coherence of chaos**



$$\frac{d\theta}{dt} = \omega_0 + F(A)$$

$F(A)$ is like effective noise $\Rightarrow$

Synchronization of chaotic oscillators $\approx$
$\approx$ synchronization of noisy periodic oscillators $\Rightarrow$

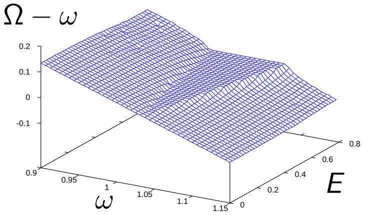
phase synchronization can be observed while the "amplitudes" remain chaotic

Synchronization of a chaotic oscillator by external force

If the phase is well-defined $\Rightarrow \Omega = \langle \frac{d\theta}{dt} \rangle$ is easy to calculate (e.g. $\Omega = 2\pi \lim_{t\to\infty} N_t/t$, N is a number of maxima)

Forced
Rössler oscillator:

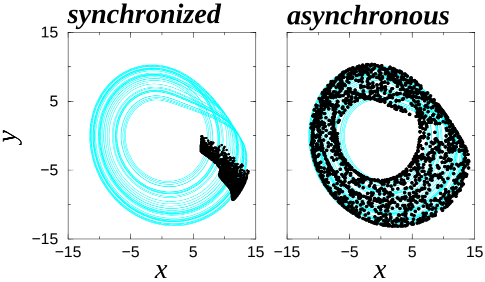
$$\begin{aligned} \dot{x} &= -y - z + E\cos(\omega t) \\ \dot{y} &= x + ay \\ \dot{z} &= 0.4 + z(x - 8.5) \end{aligned}$$



phase is locked, amplitude is chaotic

Stroboscopic observation

Autonomous chaotic oscillator: phases are distributed from 0 to 2π.
Under periodic forcing: if the phase is locked, then $W(\theta, t)$ has a sharp peak near $\theta = \omega t + const$.



Phase synchronization of chaotic gas discharge by periodic pacing

[Tacos et al, Phys. Rev. Lett. **85**, *2929 (2000)]*

Experimental setup:

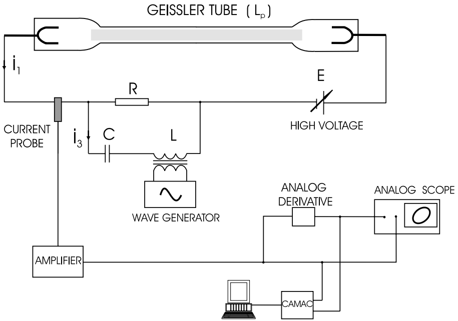
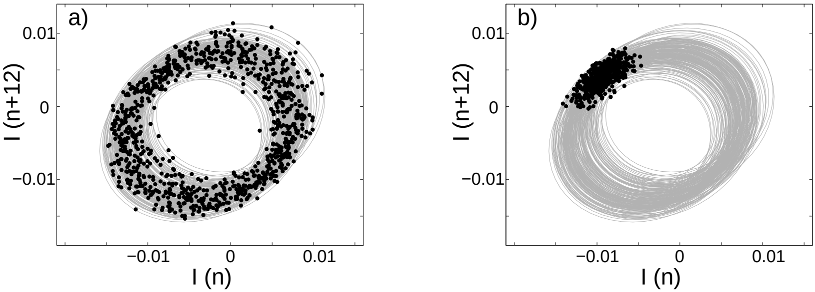
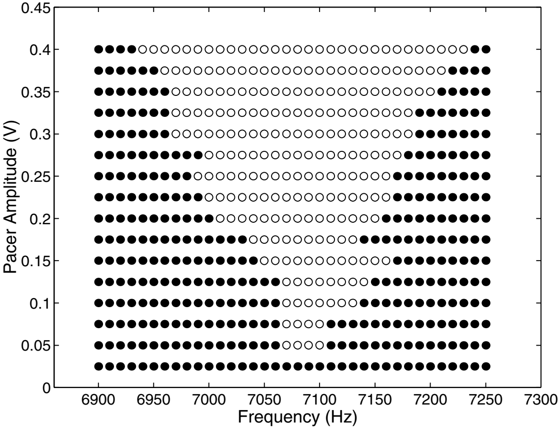


FIG. 2. Schematic representation of our experimental setup.

Phase plane projections in non-synchronized and synchronized

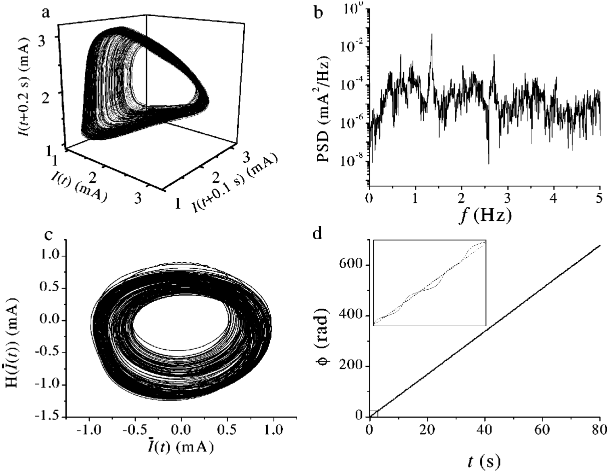


Synchronization region:

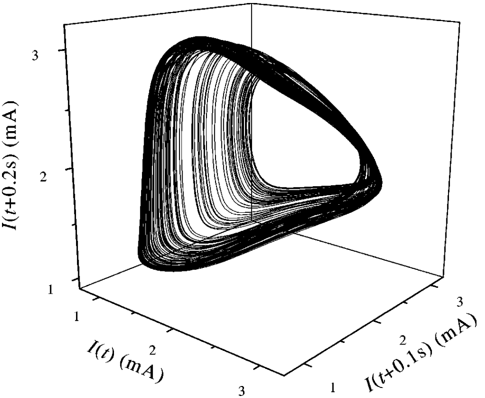


Electrochemical chaotic oscillator

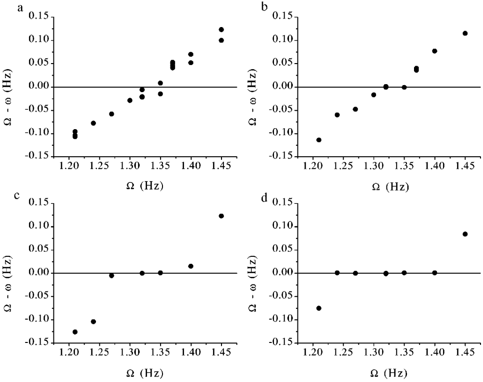
[Kiss and Hudson, Phys. Rev. E **64**, *046215 (2001)]*

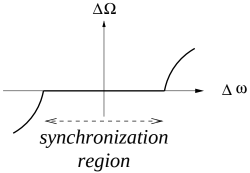


The synchronized oscillator remains chaotic:

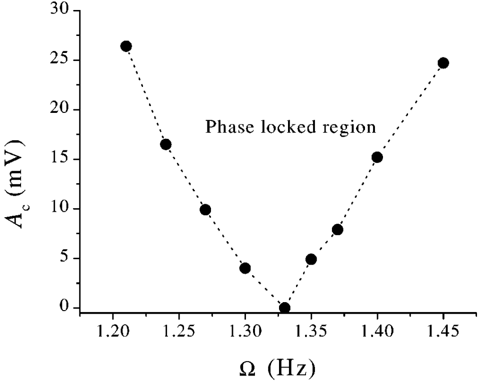


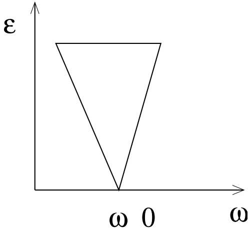
Frequency difference as a function of driving frequency for different amplitudes of forcing:



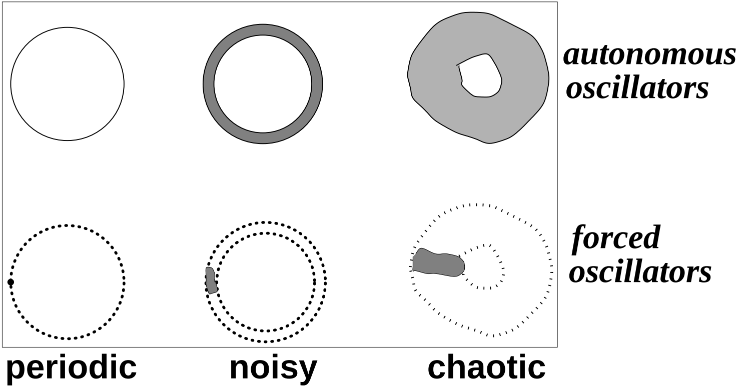


Synchronization region:





Unified description of regular, noisy, and chaotic oscillators

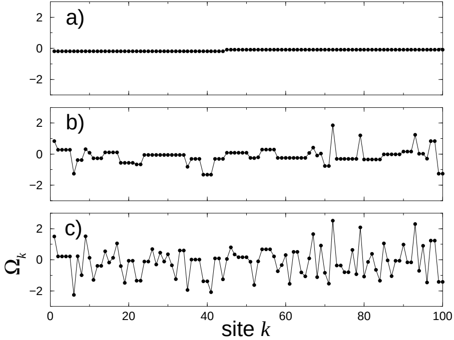


Many coupled oscillators

Typical setups:
Lattices
Networks
Global coupling

Lattice of phase oscillators

$$\frac{d\varphi_n}{dt} = \omega_n + \varepsilon q(\varphi_{n+1} - \varphi_n) + \varepsilon q(\varphi_{n-1} - \varphi_n)$$



Small ε: no synchrony
Intermediate ε: clusters
Large ε: full synchrony (one cluster)

Continuous phase profile

In a large system frequencies can be entrained but the phases widely distributed

For a continuous phase profile in the case of noisy/chaotic oscillators one obtains a KPZ equation

$$\frac{\partial \varphi}{\partial t} = \omega(x) + \varepsilon \nabla^2 \varphi + \beta (\nabla \varphi)^2 + \xi(x, t)$$

Roughening means lack of phase synchrony on large scales

Dissipative vs conservative coupling

$$\frac{d\varphi_n}{dt} = \omega + q(\varphi_{n+1} - \varphi_n) + q(\varphi_{n-1} - \varphi_n)$$

For the phase difference $v_n = \varphi_{n+1} - \varphi_n$ we get

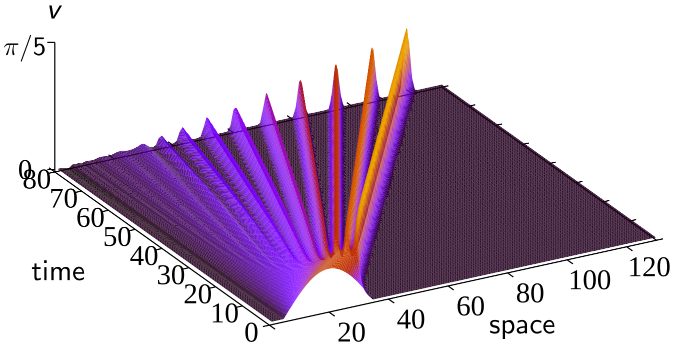
$$\frac{dv_n}{dt} = q(-v_n) + q(v_{n+1}) - q(-v_{n-1}) - q(v_n)$$

Odd and even parts of the coupling function:
$q(v) = \varepsilon \sin v + \alpha \cos v$:

$$\frac{dv_n}{dt} = \alpha \nabla_d \cos v + \varepsilon \Delta_d \sin v$$

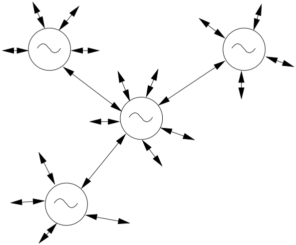
where ∇_d and Δ_d are discrete nabla and Laplacian operators purely conservative coupling, traveling waves (compactons)

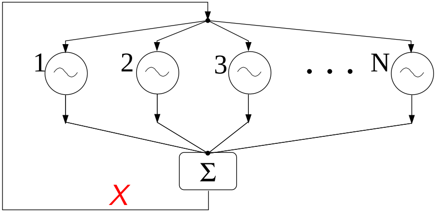
$$\frac{dv_n}{dt} = \nabla_d \cos v = \cos v_{n+1} - \cos v_{n-1}$$



Global coupling of oscillators

Each-to-each coupling $\Longleftrightarrow$ **Mean field** coupling





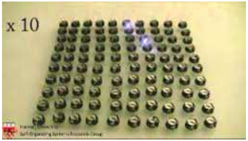
Physical systems with global coupling

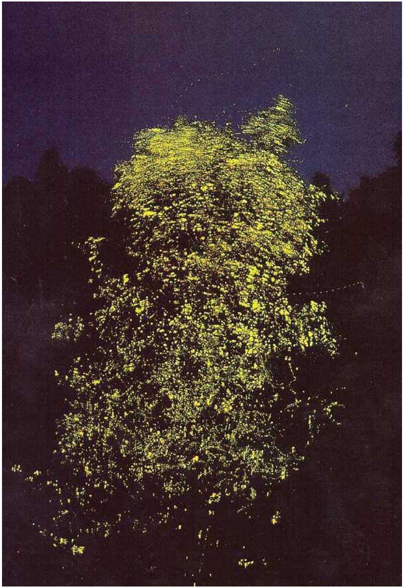
- Josephson junction arrays
- Generators (eg spin-torque oscillators, electrochemical oscillators,...) with a common load
- Multimode lasers

Example: Metronoms on a common plattform

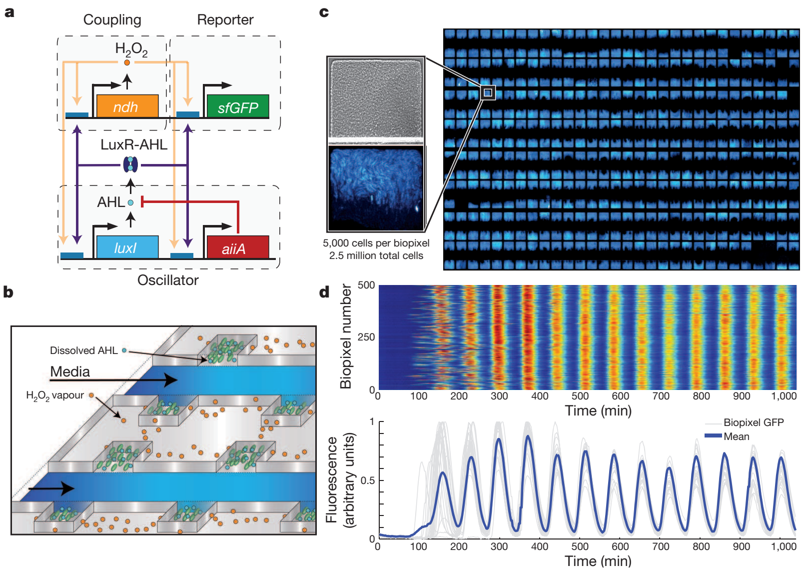
Example: blinking fireflies

Also electronic fireflies:



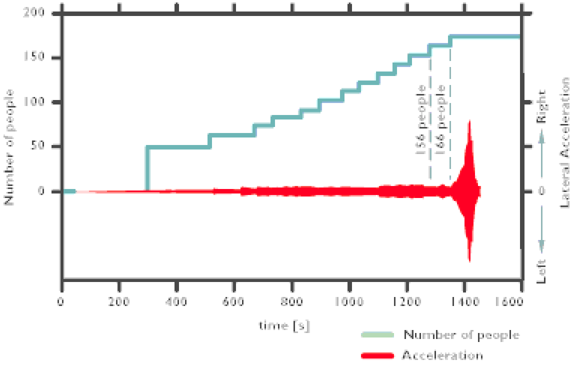


Genetic “fireflies” [Pridle et al, Nature (2011)]



A macroscopic example: Millenium Bridge

Experiment with Millenium Bridge



Kuramoto model: coupled phase oscillators

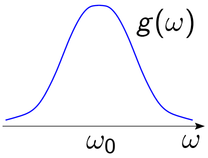
Generalize the Adler equation to an ensemble with all-to-all coupling

$$\dot{\phi}_i = \omega_i + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_i)$$

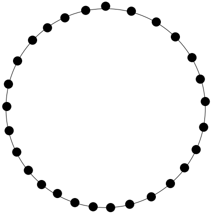
Can be written as a mean-field coupling

$$\dot{\phi}_i = \omega_i + \varepsilon(-X \sin\phi_i + Y \cos\phi_i) \qquad X + iY = M = \frac{1}{N}\sum_j e^{i\phi_j}$$

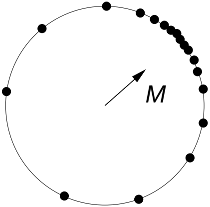
The natural frequencies are distributed around some mean frequency ω_0



Synchronisation transition



small ε: no synchronization, phases are distributed uniformly, mean field $= 0$

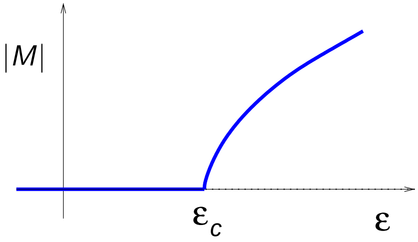


large ε: synchronization, distribution of phases is non-uniform, mean field $\neq 0$

Theory of transition

Like the mean-field theory of ferromagnetic transition:
a self-consistent equation for the mean field

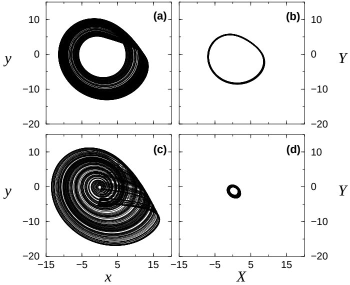
$$M = \int_0^{2\pi} n(\phi) e^{i\phi}\, d\phi = M\varepsilon \int_{-\pi/2}^{\pi/2} g(M\varepsilon \sin\phi) \cos\phi \; e^{i\phi}\, d\phi$$



Globally coupled chaotic oscillators

Each chaotic oscillator is like a noisy phase oscillator
$\Rightarrow$ A regular mean field appears at the critical coupling ε_{cr}.
Example: Rössler oscillators with Gaussian distribution of frequencies

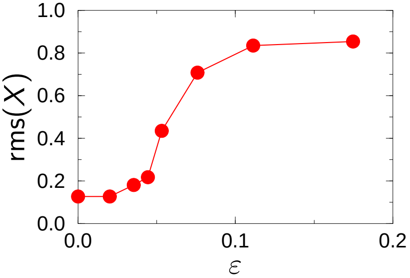
$$\begin{aligned}
\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon X, \\
\dot{y}_i &= \omega_i x_i + a y_i, \\
\dot{z}_i &= 0.4 + z_i(x_i - 8.5),
\end{aligned}$$



Experiment

Experimental example: synchronization transition
in ensemble of 64 chaotic electrochemical oscillators
Kiss, Zhai, and Hudson, Science, 2002

Finite size of the ensemble yields fluctuations of the mean field $\sim \frac{1}{N}$

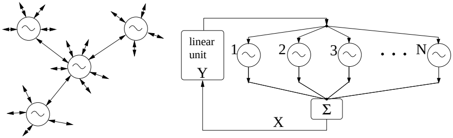


Synchronisation transition at zero temperature

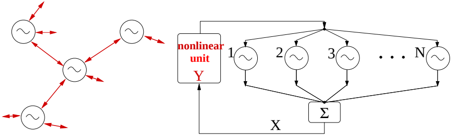
[M. Rosenblum, A. P., PRL (2007)]
Identical oscillators = "zero temerature"

Attraction: synchronization Repulsion: desynchronization

Small each-to-each coupling $\Longleftrightarrow$ coupling via linear mean field



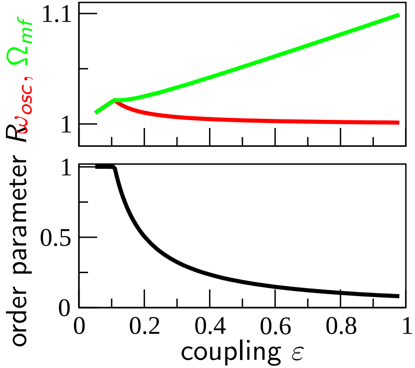
Strong each-to-each coupling $\Longleftrightarrow$ coupling via nonlinear mean field



Loss of synchrony with increase of coupling

Attraction for small coupling Repulsion at large coupling

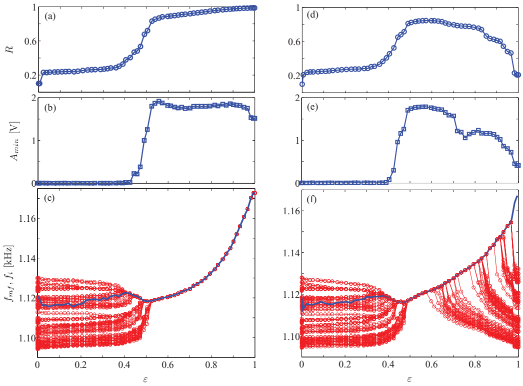
A state on the border, where the mean field is finite but the oscillators are not locked (quasiperiodicity!) establishes



Experiment

[Temirbayev et al, PRE, 2013]

Linear coupling Nonlinear coupling



Other topics

- Synchronization by common noise [Braun et al, EPL (2012)]
- Control of synchrony (eg for suppression of pathological neural synchrony at Parkinson [Montaseri et al, Chaos (2013)])
- Synchrony in multifrequency populations (eg resonances $\omega_1 + \omega_2 = \omega_3$, [Komarov an A.P., PRL (2013)])
- Inverse problems: infer coupling function from the signals (eg ECG + respiration signals yield coupling Heart-Respiration, [Kralemann et al, Nat. Com. (2013)])