Synchronization in Ensembles of Oscillators: Theory of Collective Dynamics

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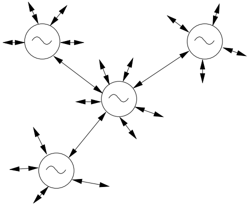
- **Beyond WS and OA: Kuramoto model with bi-harmonic coupling**

Ensembles of globally (all-to-all) couples oscillators

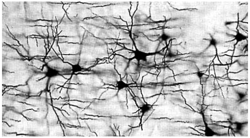
- Physics: arrays of Josephson junctions, multimode lasers,...
- Biology and neuroscience: cardiac pacemaker cells, population of fireflies, neuronal ensembles...
- Social behavior: applause in a large audience, pedestrians on a bridge,...



Main effect: Synchronization

Mutual coupling adjusts phases of indvidual systems, which start to keep pace with each other

Synchronization can be treated as a nonequilibrium phase transition!



Attempt of a general formulation

$$\dot{\vec{x}}_k = \vec{f}(\vec{x}_k, \vec{X}, \vec{Y}) \qquad \text{individual oscillators (microscopic)}$$

$$\vec{X} = \frac{1}{N} \sum_k \vec{g}(\vec{x}_k) \qquad \text{mean fields (generalizations possible)}$$

$$\dot{\vec{Y}} = \vec{h}(\vec{X}, \vec{Y}) \qquad \text{macroscopic global variables}$$

Typical setup for a synchronization problem:
$\vec{x}_k(t)$ – periodic or chaotic oscillators
$\vec{X}(t), \vec{Y}(t)$ periodic or chaotic $\Rightarrow$ collective synchronous rhythm
$\vec{X}(t), \vec{Y}(t)$ stationary $\Rightarrow$ desynchronization

Description in terms of macroscopic variables

The goal is to describe the ensemble in terms of macroscopic variables $\vec{W}$, which characterize the distribution of $\vec{x}_k$,

$$\dot{\vec{W}} = \vec{q}(\vec{W}, \vec{Y}) \qquad \text{generalized mean fields}$$

$$\dot{\vec{Y}} = \vec{h}(\vec{X}(\vec{W}), \vec{Y}) \qquad \text{global variables}$$

as a possibly low-dimensional dynamical system

Below: how this program works for phase oscillators by virtue of Watanabe-Strogatz and Ott-Antonsen approaches

Kuramoto model: coupled phase oscillators

Phase oscillators ($\varphi_k \sim x_k$) with all-to-all pair-wise coupling

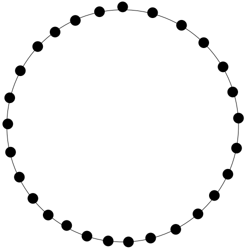
$$
\begin{aligned}
\dot{\varphi}_k &= \omega_k + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k) \\
&= \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \sin \varphi_j \right] \cos \varphi_k - \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \cos \varphi_j \right] \sin \varphi_k \\
&= \omega_k + \varepsilon R(t) \sin(\Theta(t) - \varphi_k) = \omega_k + \varepsilon \mathrm{Im}(Z e^{-i\varphi_k})
\end{aligned}
$$

System can be written as a mean-field coupling with the mean field (complex order parameter $Z \sim X$)

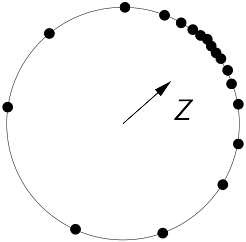
$$
Z = Re^{i\Theta} = \frac{1}{N} \sum_k e^{i\varphi_k}
$$

Synchronisation transition

ε_c $\sim$ width of distribution of frequecies $g(\omega)$ $\sim$ "temperature"



small ε: no synchronization, phases are distributed uniformly, mean field vanishes $Z = 0$



large ε: synchronization, distribution of phases is non-uniform, finite mean field $Z \neq 0$

Watanabe-Strogatz (WS) ansatz

[S. Watanabe and S. H. Strogatz, PRL 70 (2391) 1993; Physica D 74 (197) 1994]

Ensemble of **identical** oscillators driven by the same complex field $H(t)$

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

This equation also describes the dynamics of the rear wheel of a bicycle if the front one is driven
(Kohnhauser J.D.E., Velleman D., Wagon S., Which way did the bicycle go?)
If $\Gamma(x, y)$ is the given trajectory of the front wheel parametrized by its length r, $\kappa(r)$ is the curvature of this curve, and α is the angle between the curve and the bicycle, then

$$\frac{d\alpha}{dr} + \frac{\sin\alpha}{l} = \kappa(r)$$

"This track, as you perceive, was made by a rider who was going from the direction of the school."
"Or towards it?"
"No, no, my dear Watson... It was undoubtedly heading away from the school. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school"
Sherlock Holmes, during his visit to the Priory School

[As observed by Dennis Thron (Dartmouth Medical School), it is true that the rear wheel would obliterate the track of the front wheel at the crossings, but this would be true no matter which direction the bicyclist was going.]

Möbius transformation

Rewrite equation as

$$\frac{d}{dt}e^{\mathrm{i}\varphi_k} = \mathrm{i}\omega_k(t)e^{\mathrm{i}\varphi_k} + \frac{1}{2}H(t) - \frac{e^{\mathrm{i}2\varphi_k}}{2}H^*(t)$$

Möbius transformation from N variables φ_k to complex $z(t)$, $|z| \leq 1$, and N new angles $\psi_k(t)$, according to

$$e^{\mathrm{i}\varphi_k} = \frac{z + e^{\mathrm{i}\psi_k}}{1 + z^* e^{\mathrm{i}\psi_k}}$$

Since the system is over-determined, we require $N^{-1}\sum_{k=1}^{N} e^{\mathrm{i}\psi_k} = \langle e^{\mathrm{i}\psi_k}\rangle = 0$, what yields the condition $N^{-1}\sum_{k=1}^{N} \dot{\psi}_k e^{\mathrm{i}\psi_k} = \langle \dot{\psi}_k e^{\mathrm{i}\psi_k}\rangle = 0$.

Direct substitution allows one (1 page calculation) to get WS equations

$$\dot{z} = \mathrm{i}\omega z + \frac{H}{2} - \frac{H^*}{2} z^2$$
$$\dot{\psi}_k = \omega + \mathrm{Im}(z^* H)$$

Remarkably: dynamics of ψ_k does not depend on k, thus introducing $\psi_k = \alpha(t) + \tilde{\psi}_k$ we get constants $\tilde{\psi}_k$ and 3 WS equations

$$\frac{dz}{dt} = i\omega z + \frac{1}{2}(H - z^2 H^*) \qquad \frac{d\alpha}{dt} = \omega + \mathrm{Im}(z^* H)$$

Interpretation of WS variables

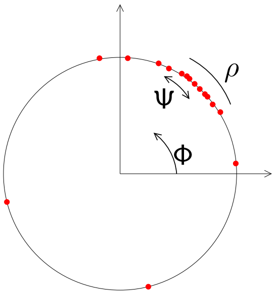
We write $z = \rho e^{\mathrm{i}\Phi}$, then

$$e^{i\varphi_k} = e^{i\Phi(t)} \frac{\rho(t) + e^{i(\tilde{\psi}_k + \alpha(t) - \Phi(t))}}{\rho(t) e^{i(\tilde{\psi}_k + \alpha(t) - \Phi(t))} + 1}$$

ρ measures the width of the bunch:
$\rho = 0$ if the mean field $Z = \sum_k e^{i\varphi_k}$ vanishes
$\rho = 1$ if the oscillators are fully synchronized and $|Z| = 1$

Φ is the phase of the bunch

$\Psi = \alpha - \Phi$ measures positions of individual oscillators with respect to the bunch



Summary of WS transformations

- Works for a large class of initial conditions [does not work if the condition $\langle e^{\mathrm{i}\psi_k}\rangle = 0$ cannot be satisfied, eg if large clusters exist]
- Applies for any N, allows a thermodynamic limit where distribution of $\tilde{\psi}_k$ is constant in time, and only $z(t), \alpha(t)$ evolve
- Applies only if the r.h.s. of the phase dynamics contains 1st harmonics $\sin\varphi$, $\cos\varphi$
- Applies only if the oscillators are identical and identically driven

Synchronization of uncoupled oscillators by external forces

Ensemble of **identical** oscillators driven by the same complex field $H(t)$

$$\frac{d\varphi_k}{dt} = \omega(t) + \operatorname{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

What happens to the WS variable ρ?
$\rho \to 1$: synchronization
$\rho \to 0$: desynchronization
Two basic examples oscillators and Jusephson junctions:

$$\dot{\varphi}_k = \omega - \sigma\xi(t)\sin\varphi_k \qquad\qquad \frac{\hbar}{2eR}\frac{d\varphi_k}{dt} + I_c\sin\varphi_k = I(t)$$

Hamiltonian reduction

$$\dot{\rho} = \frac{1-\rho^2}{2}\mathrm{Re}(H(t)e^{-i\Phi}) \,,$$
$$\dot{\Phi} = \Omega(t) + \frac{1+\rho^2}{2\rho}\mathrm{Im}(H(t)e^{-i\Phi}) \,.$$

in variables

$$q = \frac{\rho\cos\Phi}{\sqrt{1-\rho^2}} \,, \qquad p = -\frac{\rho\sin\Phi}{\sqrt{1-\rho^2}} \,,$$

reduces to a Hamiltonian system with Hamiltonian,

$$\mathcal{H}(q,p,t) = \Omega(t)\frac{p^2+q^2}{2} + H(t)\frac{p\sqrt{1+p^2+q^2}}{2}$$

Action-angle variables

$$J = \frac{\rho^2}{2(1-\rho^2)}, \qquad \Phi$$

Hamiltonian reads

$$\mathcal{H}(J, \Phi, t) = \Omega(t) J - H(t) \frac{\sqrt{2J(2J+1)}}{2} \sin\Phi$$

Synchrony: $\mathcal{H}, J \to \infty$
Asynchrony: $\mathcal{H}, J \to 0$
For general noise: "Energy" grows $\Rightarrow$ synchronization by common noise

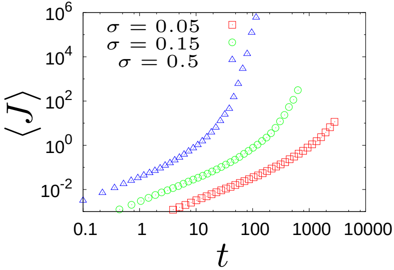
Analytic solution for the initial stage

Close to asynchrony: energy is small, equations can be linearized
$\Rightarrow$ exact solution (σ^2 is the noise intensity)

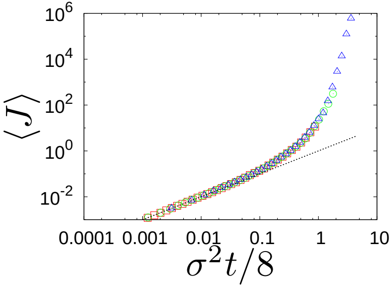
$$\mathcal{H}, J \sim A\sigma^2 t$$

Close to synchrony:

$$\mathcal{H}, J \sim \exp[\lambda t]$$



$\langle J \rangle$
t
$\sigma = 0.05$
$\sigma = 0.15$
$\sigma = 0.5$



$\langle J \rangle$
$\sigma^2 t/8$

Globally coupled ensembles

Kuramoto model with equal frequencies

$$\dot{\varphi}_k = \omega + \varepsilon \mathrm{Im}(Ze^{-i\varphi_k})$$

belongs to the WS-class

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

where H is the order parameter

$$Z = Re^{i\Theta} = \frac{1}{N}\sum_k e^{i\varphi_k}$$

Complex order parameters via WS variables

Complex order parameter can be represented via WS variables as

$$Z = \sum_k e^{i\varphi_k} = \rho e^{i\Phi}\gamma(\rho,\Psi) \qquad \gamma = 1 + (1-\rho^{-2}) \sum_{l=2}^{\infty} C_l(-\rho e^{-i\Psi})^l$$

where $C_l = N^{-1}\sum_k e^{il\psi_k}$ are Fourier harmonics of the distribution of constants ψ_k

Important simplifying case (adopted below):

Uniform distribution of constants ψ_k

$$C_l = 0 \quad \Rightarrow \quad \gamma = 1 \quad \Rightarrow \quad Z = \rho e^{i\Phi} = z$$

In this case WS variables yield the order parameter directly!

Closed equation for the order parameter for the Kuramoto-Sakaguchi model

Individual oscillators:

$$\dot{\varphi}_k = \omega + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k + \beta) = \omega + \varepsilon \mathrm{Im}(Z e^{i\beta} e^{-i\varphi_k})$$

Equation for the order parameter is just the WS equation:

$$\frac{dZ}{dt} = i\omega Z + \frac{\varepsilon}{2} e^{i\beta} Z - \frac{\varepsilon}{2} e^{-i\beta} |Z|^2 Z$$

Closed equation for the real order parameter $R = |Z|$:

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos\beta$$

Simple dynamics in the Kuramoto-Sakaguchi model

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos \beta$$

Attraction: $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ $\Longrightarrow$
Synchronization, all phases identical $\varphi_1 = \ldots = \varphi_N$, order parameter large $R = 1$
Repulsion: $-\pi < \beta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$ $\Longrightarrow$
Asynchrony, phases distributed uniformely, order parameter vanishes $R = 0$

Linear vs nonlinear coupling I

- Synchronization of a periodic autonomous oscillator is a nonlinear phenomenon
- it occurs already for infinitely small forcing
- because the unperturbed system is singular (zero Lyapunov exponent)

In the Kuramoto model "linearity" with respect to forcing is assumed

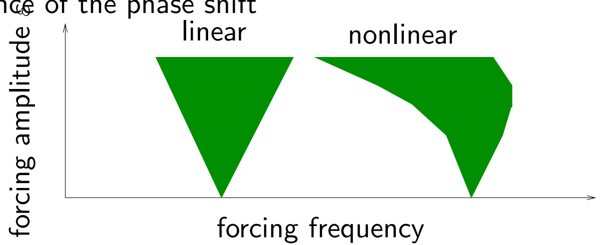
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon_1 \mathbf{f}_1(t) + \varepsilon_2 \mathbf{f}_2(t) + \cdots$$
$$\dot{\varphi} = \omega + \varepsilon_1 q_1(\varphi, t) + \varepsilon_2 q_2(\varphi, t) + \cdots$$

Linear vs nonlinear coupling II

Strong forcing leads to “nonlinear” dependence on the forcing amplitude

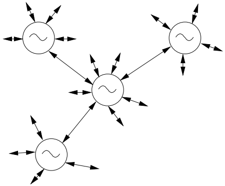
$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{f}(t) \\
\dot{\varphi} &= \omega + \varepsilon q^{(1)}(\varphi, t) + \varepsilon^2 q^{(2)}(\varphi, t) + \cdots
\end{aligned}
$$

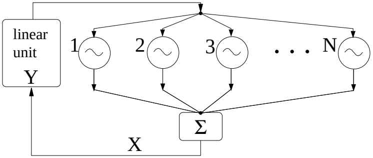
Nonlineraity of forcing manifests itself in the deformation/skeweness of the Arnold tongue and in the amplitude depnedence of the phase shift



Linear vs nonlinear coupling III

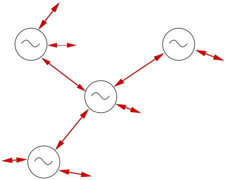
Small each-to-each coupling $\Longleftrightarrow$ coupling via linear mean field

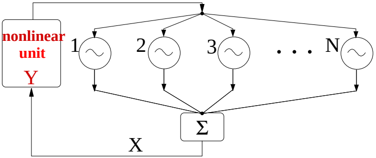




Strong each-to-each coupling $\Longleftrightarrow$ coupling via nonlinear mean field

[cf. Popovych, Hauptmann, Tass, Phys. Rev. Lett. 2005]





Nonlinear coupling: a minimal model

We take the standard Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \mathrm{Im}(He^{-i\varphi_k}) \qquad H \sim \varepsilon e^{-i\beta} Z \qquad Z = \frac{1}{N}\sum_j e^{i\varphi_j} = Re^{i\Theta}$$

and assume dependence of the acting force H on the "amplitude" of the mean field R:

$$\boxed{\dot{\varphi}_k = \omega + A(\varepsilon R)\varepsilon R \sin(\Theta - \varphi_k + \beta(\varepsilon R))}$$

E.g. attraction for small R vs repulsion for large R

WS equations for the nonlinearly coupled ensemble

$$\frac{dR}{dt} = \frac{1}{2}R(1 - R^2)\varepsilon A(\varepsilon R)\cos\beta(\varepsilon R)$$

$$\frac{d\Phi}{dt} = \omega + \frac{1}{2}(1 + R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

$$\frac{d\Psi}{dt} = \frac{1}{2}(1 - R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

Full vs partial synchrony

All regimes follow from the equation for the order parameter

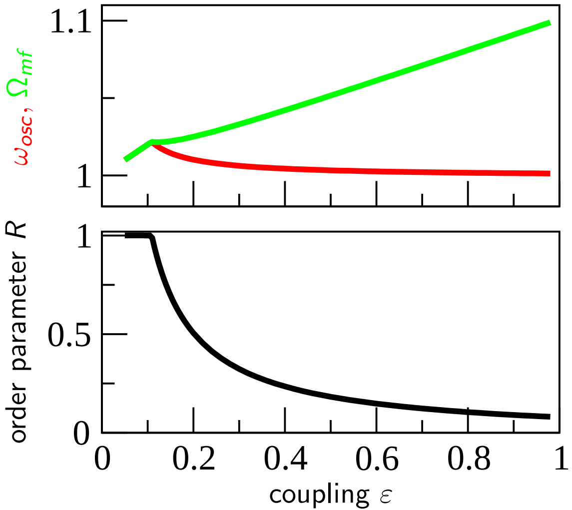
$$\frac{dR}{dt} = \frac{1}{2} R(1 - R^2)\varepsilon A(\varepsilon R) \cos \beta(\varepsilon R)$$

Fully synchronous state: $R = 1$, $\dot{\Phi} = \omega + \varepsilon A(\varepsilon) \sin \beta(\varepsilon)$
Asynchronous state: $R = 0$
Partially synchronous bunch state
$0 < R < 1$ from the condition $A(\varepsilon R) = 0$:
No rotations, frequency of the mean field = frequency of the oscillations
Partially synchronized quasiperiodic state
$0 < R < 1$ from the condition $\cos \beta(\varepsilon R) = 0$:
Frequency of the mean field $\Omega = \dot{\varphi} = \omega \pm A(\varepsilon R)(1 + R^2)/2$
Frequency of oscillators $\omega_{osc} = \omega \pm A(\varepsilon R) R^2$

Self-organized quasiperiodicity

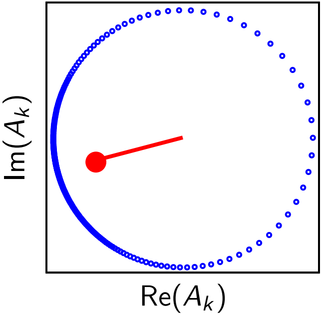
- frequencies Ω and ω_{osc} depend on ε in a smooth way
  $\Longrightarrow$ generally we observe a quasiperiodicity
- attraction for small mean field vs repulsion for large mean field
  $\Longrightarrow$ ensemble is always at the stabilty border
  $\beta(\varepsilon R) = \pm\pi/2$, i.e. in a
  self-organized critical state
- critical coupling for the transition from full to partial synchrony:
  $\beta(\varepsilon_q) = \pm\pi/2$
- transition at "zero temperature" like quantum phase transition

Simulation: loss of synchrony with increase of coupling



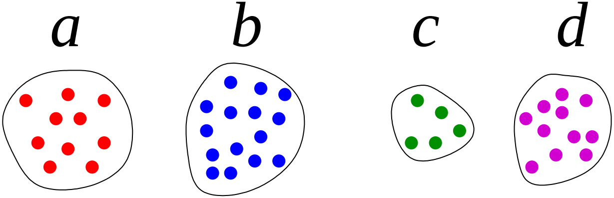
Simulation: snapshot of the ensemble

- non-uniform distribution of oscillator phases, here for $\varepsilon - \varepsilon_q = 0.2$
- different velocities of oscillators and of the mean field



Hierarchically organized populations of oscillators

We consider populations consisting of M identical subgroups (of different sizes)



Each subgroup is described by WS equations

$\Rightarrow$ system of $3M$ equations completely describes the ensemble

$$\frac{d\rho_a}{dt} = \frac{1-\rho_a^2}{2}\mathrm{Re}(H_a e^{-i\Phi_a}) \,,$$

$$\frac{d\Phi_a}{dt} = \omega_a + \frac{1+\rho_a^2}{2\rho_a}\mathrm{Im}(H_a e^{-i\Phi_a}) \,,$$

$$\frac{d\Psi_a}{dt} = \frac{1-\rho_a^2}{2\rho_a}\mathrm{Im}(H_a e^{-i\Phi_a}) \,.$$

General force acting on subgroup a:

$$H_a = \sum_{b=1}^{M} n_b E_{a,b} Z_b + F_{ext,a}(t)$$

n_b: relative subgroup size
$E_{a,b}$: coupling between subgroups a and b

Thermodynamic limit

If the number of subgroups M is very large, one can consider a as a continuous parameter and get a system

$$\frac{\partial \rho(a,t)}{\partial t} = \frac{1-\rho^2}{2}\mathrm{Re}(H(a,t)e^{-i\Phi})$$

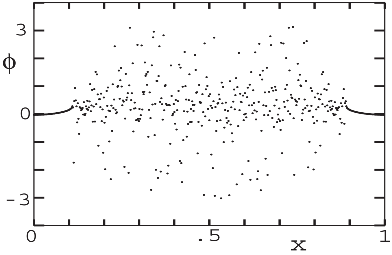
$$\frac{\partial \Phi(a,t)}{\partial t} = \omega(a) + \frac{1+\rho^2}{2\rho}\mathrm{Im}(H(a,t)e^{-i\Phi})$$

$$\frac{\partial \Psi(a,t)}{\partial t} = \frac{1-\rho^2}{2\rho}\mathrm{Im}(H(a,t)e^{-i\Phi})$$

$$H(a,t) = F_{ext}(a,t) + \int db\, E(a,b)n(b)Z(b)$$

Chimera states

Y. Kuramoto and D. Battogtokh observed in 2002 a symmetry breaking in non-locally coupled oscillators
$H(x) = \int dx' \exp[x' - x] Z(x')$



This regime was called “chimera” by Abrams and Strogatz

Chimera in two subpopulations

Model by Abrams et al:

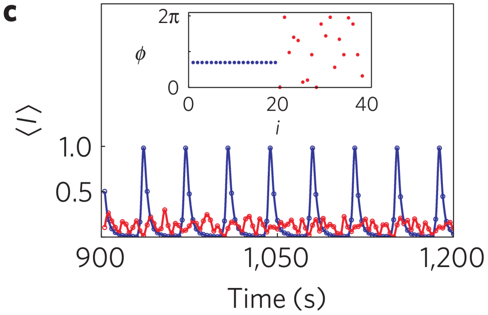
$$\dot\varphi_k^a = \omega + \mu \frac{1}{N}\sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^a + \alpha) + (1-\mu)\sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^a + \alpha)$$

$$\dot\varphi_k^b = \omega + \mu \frac{1}{N}\sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^b + \alpha) + (1-\mu)\sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^b + \alpha)$$

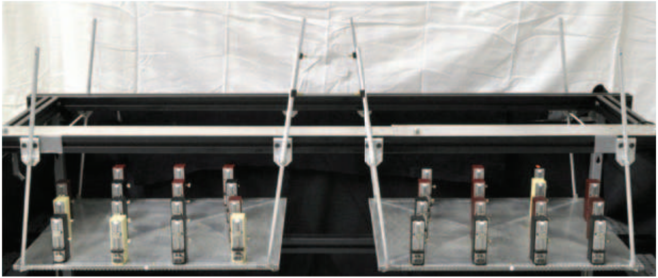
Two coupled sets of WS equations: $\rho^a = 1$ and $\rho^b(t)$ quasiperiodic are observed

Chimera in experiments I

Tinsley et al: two populations of chemical oscillators



Chimera in experiments II



Erik A. Martens,
MPI für Dynamik und Selbstorganisation

Ott-Antonsen ansatz

[E. Ott and T. M. Antonsen, CHAOS 18 (037113) 2008]

Consider the same system

$$\frac{d\varphi_k}{dt} = \omega(t) + \operatorname{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \dots, N$$

in the thermodynamic limit $N \to \infty$ and write equation for the probability density $w(\varphi, t)$:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}\left(\omega + \frac{1}{2\mathrm{i}}(He^{-\mathrm{i}\varphi} - H^* e^{\mathrm{i}\varphi})\right) = 0$$

Expanding density in Fourier modes $w = (2\pi)^{-1} \sum W_k(t)e^{-\mathrm{i}k\varphi}$ yields an infinite system

$$\frac{dW_k}{dt} = \mathrm{i}k\omega W_k + \frac{k}{2}(HW_{k-1} - H^* W_{k+1})$$

$$\frac{dW_1}{dt} = \mathrm{i}\omega W_1 + \frac{1}{2}(H - H^* W_2)$$

$$\frac{dW_k}{dt} = \mathrm{i}k\omega W_k + \frac{k}{2}(HW_{k-1} - H^* W_{k+1})$$

With an ansatz $W_k = (W_1)^k$ we get for $k \geq 2$

$$\frac{dW_k}{dt} = kW_1^{k-1}\left[\mathrm{i}\omega W_1 + \frac{1}{2}(H - H^* W_1^2)\right]$$

ie all the infite system is reduced to one equation.
Because $W_1 = \langle e^{\mathrm{i}\varphi}\rangle = Z$ we get the Ott-Antonsen equation

$$\frac{dZ}{dt} = \mathrm{i}\omega Z + \frac{1}{2}(H - H^* Z^2)$$

Relation WS $\leftrightarrow$ OA

- OA is the same as WS for $N \to \infty$ and for the uniform distribution of constants ψ_k
- A special familly of distributions satisfying $W_k = (W_1)^k$ is called OA manifold, it corresponds to all possible Möbius transformation of the uniform density of constants
- OA is formulated directly in terms of the Kuramoto order parameter

Application to nonidentical oscillators

Assuming a distribution of natural frequencies $g(\omega)$, one introduces $Z(\omega) = \rho(\omega)e^{i\Phi(\omega)}$ and obtains the Ott-Antonsen integral equations

$$\frac{\partial Z(\omega,t)}{\partial t} = i\omega Z + \frac{1}{2}Y - \frac{Z^2}{2}Y^*$$

$$Y = \langle e^{\mathrm{i}\varphi} \rangle = \int d\omega\, g(\omega) Z(\omega)$$

OA equations for Lorentzian distribution of frequencies

If

$$g(\omega) = \frac{\Delta}{\pi((\omega - \omega_0)^2 + \Delta^2)}$$

and Z has no poles in the upper half-plane, then the integral $Y = \int d\omega\, g(\omega) Z(\omega)$ can be calculated via residues as $Y = Z(\omega_0 + i\Delta)$

This yields an ordinary differential equation for the order parameter Y

$$\frac{dY}{dt} = (i\omega_0 - \Delta)Y + \frac{1}{2}\varepsilon(1 - |Y|^2)Y$$

Hopf normal form / Landau-Stuart equation/ Poincaré oscillator

$$\frac{dY}{dt} = (a + ib - (c + id)|Y|^2)Y$$

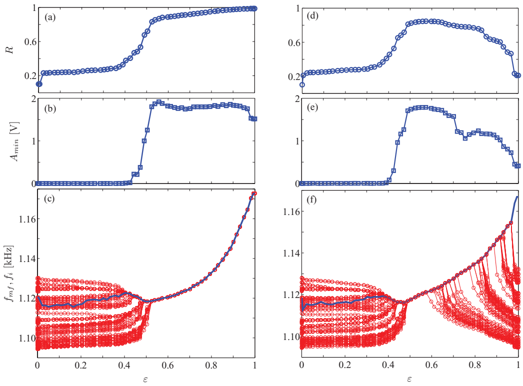
Nonidentical oscillators with nonlinear coupling

$$\frac{dY}{dt} = (i\omega_0 - \Delta)Y + \frac{1}{2}\varepsilon A(\varepsilon R)(e^{i\beta(\varepsilon R)} - e^{-i\beta(\varepsilon R)}|Y|^2)Y$$

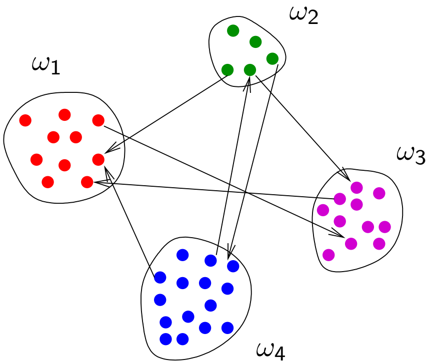
Lorentzian distribution of natural frequencies $g(\omega)$
$\Rightarrow$ standard "finite temperature" Kuramoto model of globally coupled oscillators with **nonlinear** coupling
(attraction for a small force, repulsion for a large force)
Novel effect: Multistability
Different partially synchronized states coexist for the same parameter range

Experiment

[Temirbayev et al, PRE, 2013]



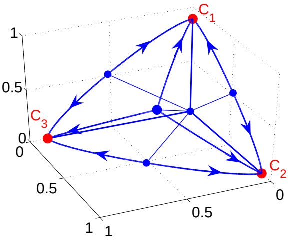
Non-resonantly interacting ensembles (with M. Komarov)



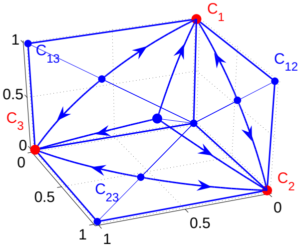
Frequencies are different – all interactions are non-resonant (only amplitudes of the order parameters involved)

$$\dot{\rho}_l = (-\Delta_l - \Gamma_{lm}\rho_m^2)\rho_l + (a_l + A_{lm}\rho_m^2)(1 - \rho_l^2)\rho_l, \qquad l = 1, \ldots, L$$

Competition for synchrony



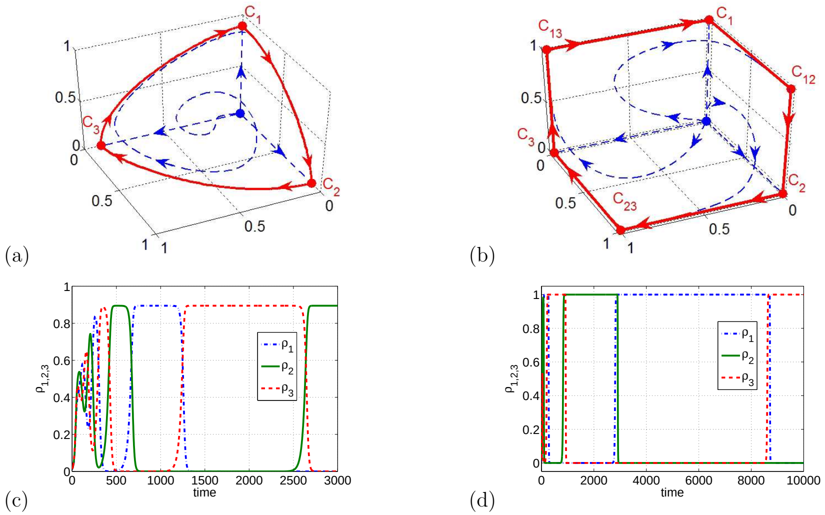
(a)



(b)

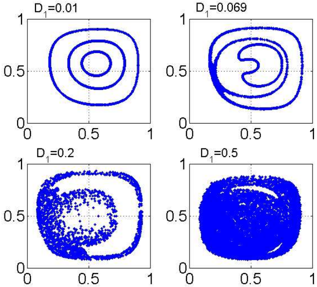
Only one ensemble is synchronous – depending on initial conditions

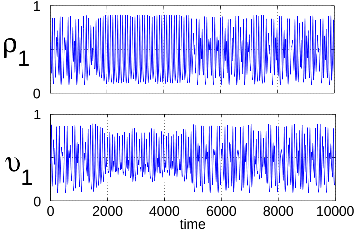
Heteroclinic synchrony cycles



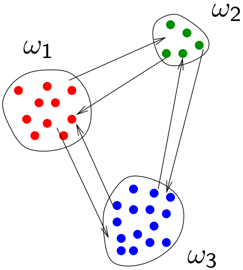
Chaotic synchrony cycles

Order parameters demonstrate chaotic oscillations





Resonantly interacting ensembles (with M. Komarov)



Most elementary nontrivial resonance $\omega_1 + \omega_2 = \omega_3$
Triple interactions:

$$\dot{\phi}_k = \ldots + \Gamma_1 \sum\nolimits_{m,l} \sin(\theta_m - \psi_l - \phi_k + \beta_1)$$

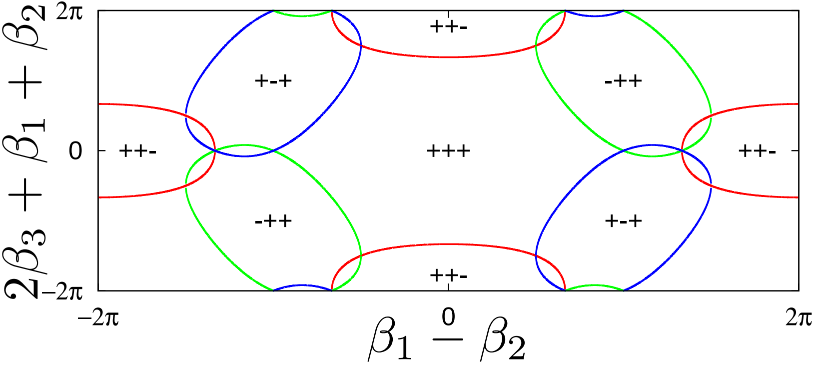
$$\dot{\psi}_k = \ldots + \Gamma_2 \sum\nolimits_{m,l} \sin(\theta_m - \phi_l - \psi_k + \beta_2)$$

$$\dot{\theta}_k = \ldots + \Gamma_3 \sum\nolimits_{m,l} \sin(\phi_m + \psi_l - \theta_k + \beta_3)$$

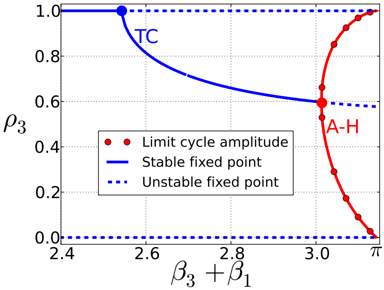
Set of three OA equations

$$\dot{z}_1 = z_1(i\omega_1 - \delta_1) + (\epsilon_1 z_1 + \gamma_1 z_2^* z_3 - z_1^2(\epsilon_1^* z_1^* + \gamma_1^* z_2 z_3^*))$$
$$\dot{z}_2 = z_2(i\omega_2 - \delta_2) + (\epsilon_2 z_2 + \gamma_2 z_1^* z_3 - z_2^2(\epsilon_2^* z_2^* + \gamma_2^* z_1 z_3^*))$$
$$\dot{z}_3 = z_3(i\omega_3 - \delta_3) + (\epsilon_3 z_3 + \gamma_3 z_1 z_2 - z_3^2(\epsilon_3^* z_3^* + \gamma_3^* z_1^* z_2^*))$$

Regions of synchronizing and desynchronizing effect from triple coupling



Bifurcations in dependence on phase constants



Beyond WS and OA theory: bi-harmonic coupling (with M. Komarov)

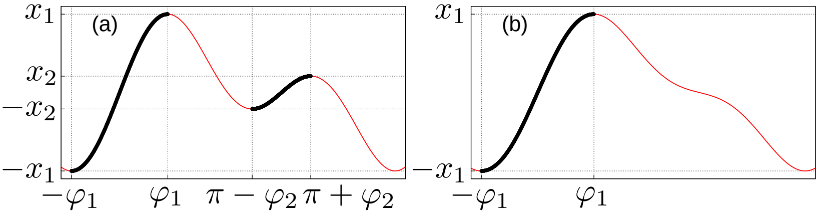
$$\dot{\varphi}_k = \omega_k + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\phi_j - \phi_k) \qquad \Gamma(\psi) = \varepsilon\sin(\psi) + \gamma\sin(2\psi)$$

Corresponds to XY-model with nematic coupling

$$H = J_1 \sum_{ij}\cos(\theta_i - \theta_j) + J_2 \sum_{ij}\cos(2\theta_i - 2\theta_j)$$

Multi-branch entrainment

$$\dot{\varphi} = \omega - \varepsilon R_1 \sin(\varphi) - \gamma R_2 \sin(2\varphi)$$



Self-consistent theory in the thermodynamic limit

Two relevant order parameters $R_m e^{i\Theta_m} = N^{-1} \sum_k e^{im\phi_k}$ for $m = 1, 2$ Dynamics of oscillators (due to symmetry $\Theta_{1,2} = 0$)

$$\dot{\varphi} = \omega - \varepsilon R_1 \sin(\varphi) - \gamma R_2 \sin(2\varphi)$$

yields a stationary distribution function $\rho(\varphi|\omega)$ which allows one to calculate the order parameters

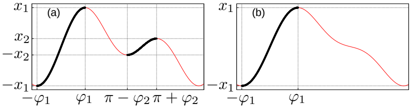
$$R_m = \iint d\varphi d\omega \; g(\omega)\rho(\varphi|\omega) \cos m\varphi, \qquad m = 1, 2$$

Where $g(\omega)$ is the distribution of natural frequencies

Multiplicity at multi-branch locking

Three shapes of phase distribution

$$\rho(\varphi|\omega) = \begin{cases} (1 - S(\omega))\delta(\varphi - \Phi_1(\omega)) + \\ \ + S(\omega)\delta(\varphi - \Phi_2(\omega)) & \text{for two branches} \\ \delta(\varphi - \Phi_1(\omega)) & \text{for one locked bracnch} \\ \frac{C}{|\dot{\varphi}|} & \text{for non-locked} \end{cases}$$



$0 \leq S(\omega) \leq 1$ is an **arbitrary** indicator function

Explicit (parametric) solution of the self-consistent eqs

We introduce

$$\cos\theta = \gamma R_2/R, \quad \sin\theta = \varepsilon R_1/R, \quad R = \sqrt{\gamma^2 R_2^2 + \varepsilon^2 R_1^2}, \quad x = \omega/R$$

so that the equation for the locked phases is

$$x = y(\theta,\varphi) = \sin\theta \sin\varphi + \cos\theta \sin 2\varphi$$

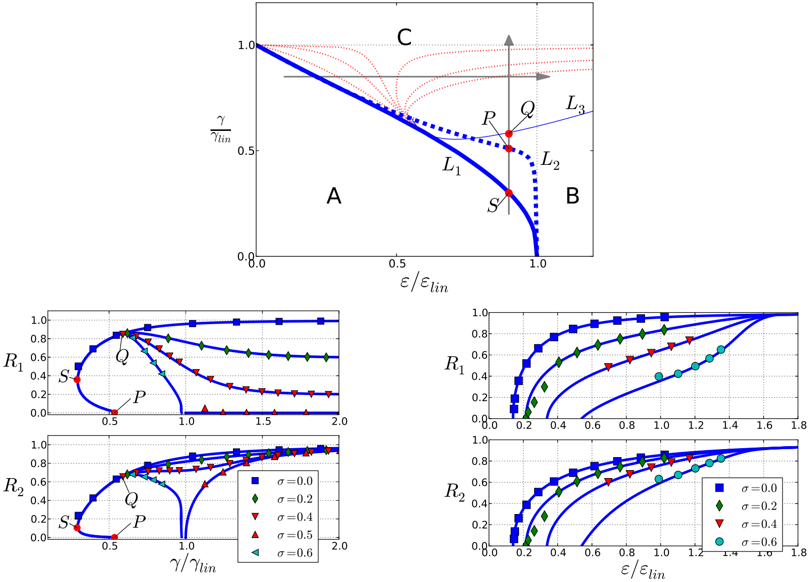
Then by calculating two integrals

$$F_m(R,\theta) = \int_{-\pi}^{\pi} d\varphi \cos m\varphi \left[A(\varphi) g(Ry) \frac{\partial y}{\partial \varphi} + \int_{|x|>x_1} dx \, \frac{C(x,\theta)}{|x - y(\theta,\varphi)|} \right]$$

we obtain a solution

$$R_{1,2} = R F_{1,2}(R,\theta), \quad \varepsilon = \frac{\sin\theta}{F_1(R,\theta)}, \quad \gamma = \frac{\cos\theta}{F_2(R,\theta)}$$

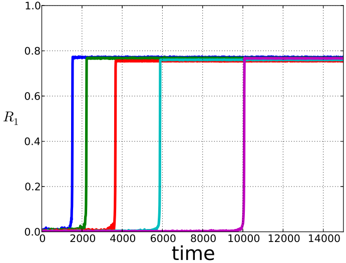
Phase diagram of solutions

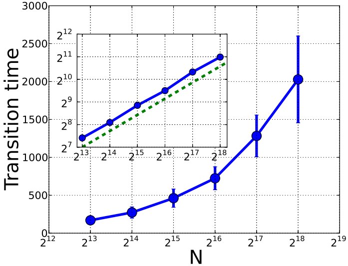


Stability issues

We cannot analyze stability of the solutions analytically (due to signularity of the states), but can perform simulations of finite ensembles

Nontrivial solution coexist with netrally stable asynchronous state





$N = 5 \cdot 10^4, 10^5, 2 \cdot 10^5, 5 \cdot 10^5, 10^6$

$T \propto N^{0.72}$

Effect of noise

With noise

$$\dot{\varphi} = \omega - \varepsilon R_1 \sin(\varphi) - \gamma R_2 \sin(2\varphi) + \sqrt{D}\xi(t)$$

the phase density $\rho(\varphi|\omega)$ is the stationary solution of the Fokker-Planck equation $\Rightarrow$ no multiplicity of states, transition to synchrony is a usual bifurcation