Modelling Proportionate Growth

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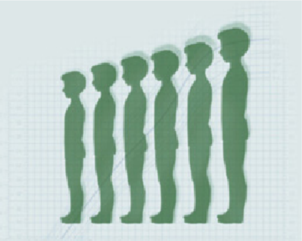
Outline

- Motivations
  - Proportionate growth
  - Pattern formation
  - Tropical polynomials and discrete analytic functions
- Growing sandpiles as models of proportionate growth
- Examples of sandpile patterns
- Characterizing the patterns
- Fast growing sandpiles
- Growing patterns on noisy backgrounds
- Affine growth
- Concluding remarks

Proportionate growth in animals



Different body parts in animals grow roughly at the same rate.

- Proportionate growth is typical in animal kingdom.
- Easier problem than development of animal from a single cell.
- Requires regulation, and communication between different parts.
- Same food becomes different tissues in different parts of the body.

The standard biological explanation of this growth and regulation involves identifying precise chemicals, growth factors, harmones, inhibitors that are turned and off by the masterplan encoded in the DNA.

This is similar to how patterns were generated in Chinese stadia by placards-carrying agents.

Does one need the full complexity of DNA to get proportionate growth?

Also, identifying chemicals is not the whole story.
It is like in a murder mystery, one says "the knife did it".

Our approach is like d'Arcy Thompson, emphasizes structure and growth, ignores chemical detail.

A 1940's picture of an organized placard display in China.

Growth models in Physics

Qualitatively different for previously studied models of growth by aggregation in physics, e.g.

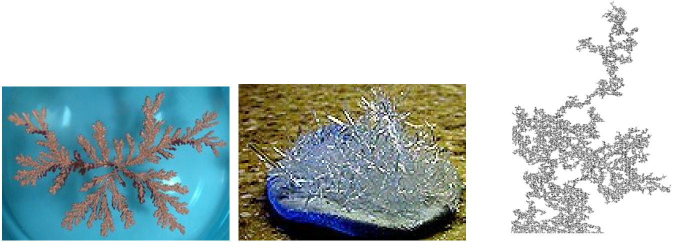


Figure: (a) A DLA cluster, (b) Epsom salt crystals grown from solution, (c) An invasion percolation cluster

In all these cases, growth occurs only in the outer regions.
Systems showing proportionate growth outside biology are hard to find.

Self-organization and sandpiles

In 1970's, Haken, Prigogine introduced the idea of living systems being 'self-organized'.

In 1987, Bak et al realized that many natural systems are self-organized to be at the edge of stability, and called these Self-Organized Critical.

They proposed a sandpile model as prototype model of SOC. Many earlier studies about the power-laws in distribution of avalanche sizes.

Our emphasis here is on pattern formation, and not on avalanche statistics in sandpile models.

Proportionate growth in patterns formed by growing sandpiles

Growing patterns formed in Abelian sandpiles show self-organization, and proportionate growth.

- A simple cellular automaton model of proportionate growth
- Complex but beautiful patterns
- Analytically tractable: Exact characterization of patterns
- Involves some interesting mathematics: discrete analytic functions, piece-wise linear functions

Proportionate growth in sandpile patterns

Basic facts from biology:

Food required for growth. Reaches different body parts.
Cell-division occurs only if the cell has enough nutrients.

A well-studied model of threshold dynamics is the Abelian Sandpile Model

Definition of ASM:

- Non-negative integer height z_i at sites i of a square lattice
- Add rule: $z_i \rightarrow z_i + 1$
- Relaxation rule : if $z_i > z_c = 3$, topple, and move one grain to each neighbor.

Rule for forming patterns:
Add N particles at one site on a periodic background, and relax.
Generalization to other lattices, higher dimensions

Sandpile Model: toppling rules

Start with a stable configuration, and add a particle :

0	0	0	0	0
0	0	0	0	0
0	0	4	0	0
0	0	0	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

$\longrightarrow$

0	0	0	0	0
0	0	3	0	0
0	3	4	3	0
0	0	3	0	0
0	0	0	0	0

Finally, we get stable configuration:

0	0	1	0	0
0	2	1	2	0
1	1	0	1	1
0	2	1	2	0
0	0	1	0	0

.

Proportionate growth

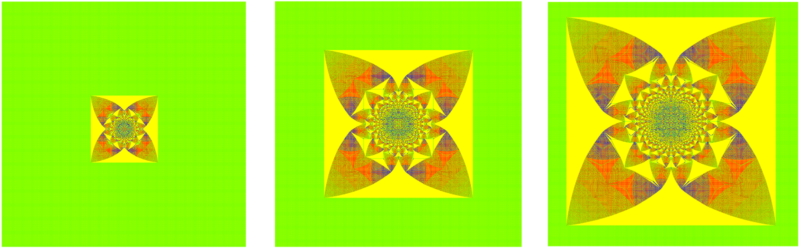


Figure: Patterns formed on a square lattice with initial height 2 at all sites. $N = (a) 4 \times 10^4$ (b) 2×10^5 (c) 4×10^5. Color code $0, 1, 2, 3 =$ R,B,G,Y

Diameter $\sim \sqrt{N}$.

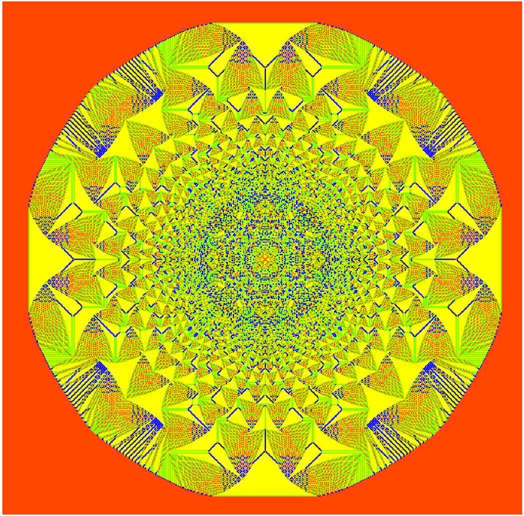


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0. Color code 0, 1, 2, 3 = R,B,G,Y

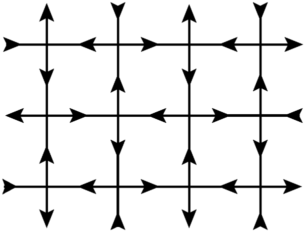


Figure: The F-lattice : A square lattice with directed bonds

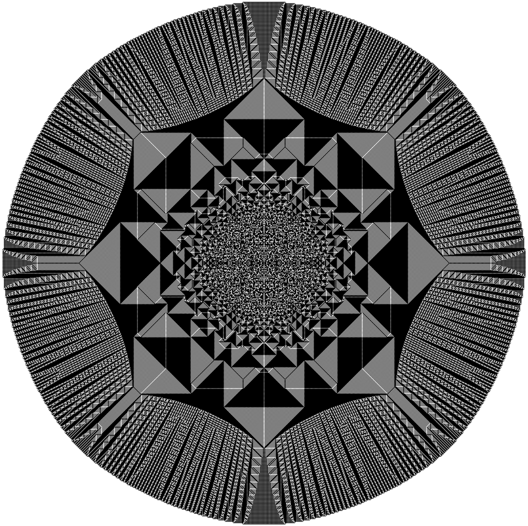


Figure: Pattern produced by adding 10^5 particles at the origin, on the F-lattice ASM, with initial state alternating columns of 1's and 0's. Color code: $B = 0, W = 1$

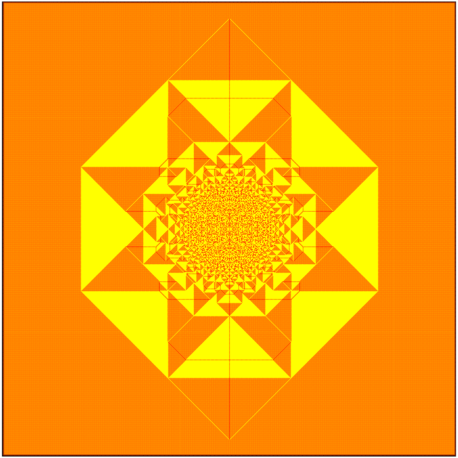
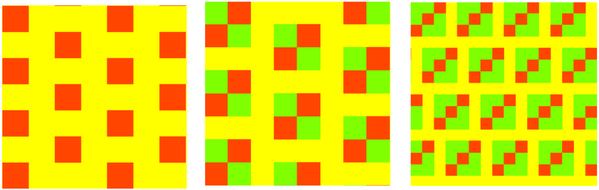


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard. Color code: 0 =R, 1=Y

The Key Observation
S. Ostojic (2003).

- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]

Examples of periodic patterns in patches



Characterizing the patterns

- We define reduced coordinates $\xi = X/\Lambda, \eta = Y/\Lambda$.
- Define coarse-grained density $\rho(\xi, \eta)$ for the asymptotic pattern, $\Lambda \to \infty$
- The asymptotic pattern is characterized by $\rho(\xi, \eta)$
- Identify 'patches' as regions of constant $\rho(\xi, \eta)$
- Specify the periodic pattern in each patch, and adjacency graph of patches
- Specify sizes of patches, and equations of boundary lines

Characterizing the patterns

Let $T_N(\vec{R})$ = the number of topplings at point $\vec{R}$.

Define reduced coordinate $\vec{r} = \vec{R}/\Lambda$, Λ = diameter

Proportionate growth $\Leftrightarrow$ Scaling $T_N(\vec{R}) \sim \Lambda^a \phi(\vec{r})$.

A non-trivial $\phi(\vec{r})$ defines the asymptotic pattern.

The excess density of grains $\nabla^2\phi(\vec{r})$ is bounded, for $\vec{r} \neq \vec{0} \implies a \leq 2$.

In addition, we have $N \sim \Lambda^b$.

The Main Result

In each patch with a periodic height pattern, we can only have

$a = 2$, and $\phi(x, y)$ is a quadratic function of x and y,
Or
$a = 1$, and $\phi(x, y)$ is a linear function of x and y.

Proof:
Expand $\phi(x_0 + \Delta x, y_0 + \Delta y)$ in a Taylor series:
$\phi(x_0 + \Delta x, y_0 + \Delta y) = \phi(x_0, y_0) + A\Delta x + B\Delta y + .. + K(\Delta x)^3 + ..$
Equivalently,
$T_N(X, Y) = \ldots + K(\Delta X)^3/\Lambda^{3-a}$
For finite ΔX integer, T is also integer, and no proliferation of defect lines $\Rightarrow K = 0$.
Same is true for all higher powers.
For a non-trivial dependence on x, y, if quadratic term is not zero, $a = 2$. Else, $a = 1$. Independent of dimension.

Characterizing the F-lattice pattern

- Only two types of patches: densities $1/2$ and 1.
- All boundaries are straight lines: slopes $0, \pm 1$, or ∞
- Each patch is 3- or 4- sided polygon
- The pattern may be viewed as a tiling of plane by squares of different sizes

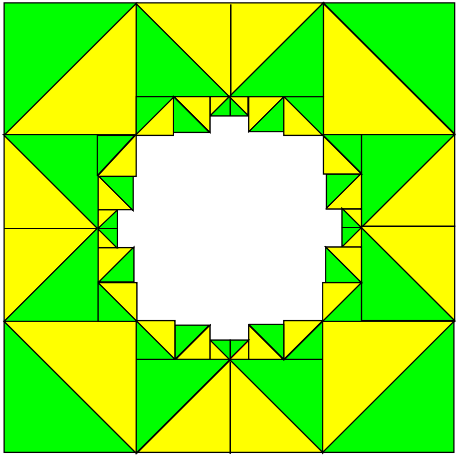


Figure: Tiling with square tiles

Characterizing the F-lattice pattern (continued)

The main simplifcation that allows the pattern on F-lattice exactly is the observation that the adjacency graph of the pattern is actually very simple.

This is not immediately obvious, by looking at the graph, but becomes so by applying a $1/z^2$ transformation to the picture.

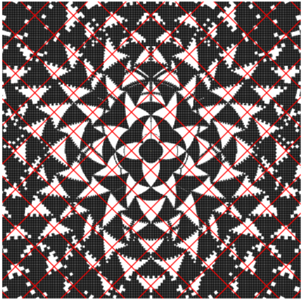


Figure: The $1/z^2$ transform of the pattern

Thus, we see that the 'light' patches can be labelled by integer coordinates (m, n), and the adjacency graph of patches in the original pattern is a square grid discretization of a two-sheeted Riemann surface.

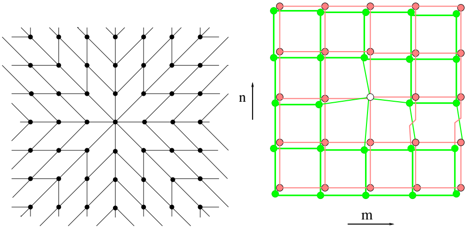


Figure: The Adjacency graph as (a) planar graph, (b) as a discretized two-sheeted Riemann surface

Given the adjacency graph of the pattern, the sizes of squares can be deduced using the Brooks -Smith-Stone-Tutte correspondence between tilings of plane with rectangles, and electical circuits.

The Quantitative characterization of the F-lattice pattern

The exact characterization involves four steps:

- Labelling patches using two integers (m, n). The adjacency graph is a discretized two-sheeted Riemann surface.
- Parameterize the potential in the (m, n) patch by

$$\phi_P(\xi, \eta) = \frac{1}{8}(m_P + 1)\xi^2 + \frac{1}{4} n_P \xi\eta + \frac{1}{8}(1 - m_P)\eta_P^2 + d_P \xi + e_P \eta + f_P$$

- Continuity of ϕ and derivatives implies that $d_{m,n}$ and $e_{m,n}$ both satisfy the equation

$$\psi_{m+1,n+1} + \psi_{m+1,n-1} + \psi_{m-1,n+1} + \psi_{m-1,n-1} - 4\psi_{m,n} = 0,$$

- Solve equations numerically on a large grid, to get the exact boundaries of patches

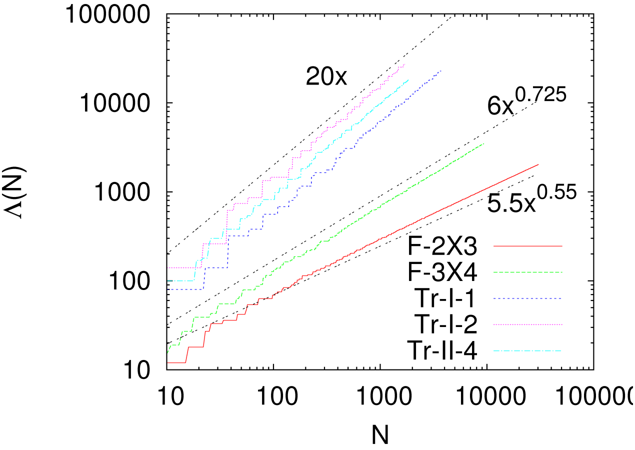
Dependence of the diameter Λ with N

This is much less constrained.

- If the initial background density is low enough everywhere,
  $$\Lambda \sim N^{1/d}$$
- If many sites have large heights
  $$\Lambda = \infty \quad \text{for finite N}$$
- For an in-between set of periodic backgrounds
  $$\Lambda \sim N^{\alpha} \quad \text{for } 1/\mathrm{d} < \alpha \leq 1$$

If $\Lambda \sim N^{\alpha}$, with $\alpha > 1/2$
We construct an infinite family of periodic backgrounds on the F-lattice that seem to have a different α for each member.

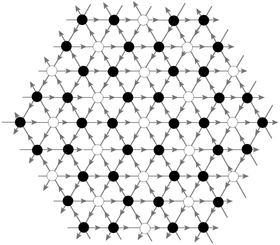
Graph of Diameter Λ vs N

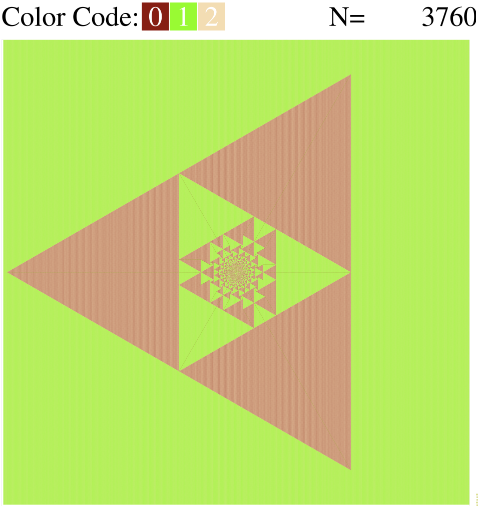


Patterns with fast-growing sandpiles

Directed traingular lattice with honeycomb background pattern

Diameter $\sim N$

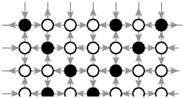




Examples of patterns with fast-growing sandpiles

The ‘Bat-pattern’ on F-lattice

Here $\Lambda \sim N^{\alpha}$, $\alpha \approx 0.55$



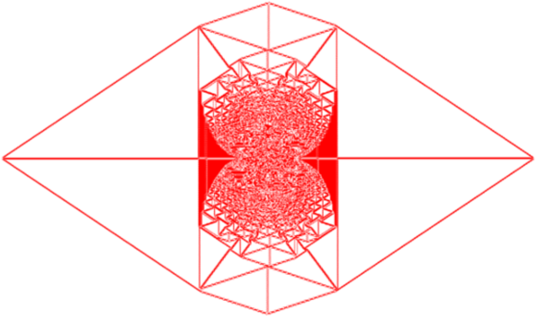


Figure: Only the boundaries of patches are shown.

Quantitative characterization

Consider the case with $a = 1$, say the triangular lattice pattern.
The exact characterization of the patterns here is easier:

- $\phi(\xi, \eta)$ is a piece-wise linear function, with rational slopes. Parameterize as $\phi_P(\xi, \eta) = a_P x + b_P y + c_P$
- The allowed values of (a_P, b_P) for different patches form a periodic hexagonal lattice.
- The condition that three patches meet at a point implies that c_P satisfies a Laplace equation on the adjacency graph of patches.
- Exact solution of these equations gives the exact boundaries of patches

Robustness of the pattern

The arguments only depend on the existence of only two types of patches, and straight line boundaries.

These can be found (by trial and error) in other cases also.
Then the asymptotic pattern is identical.
Some examples:

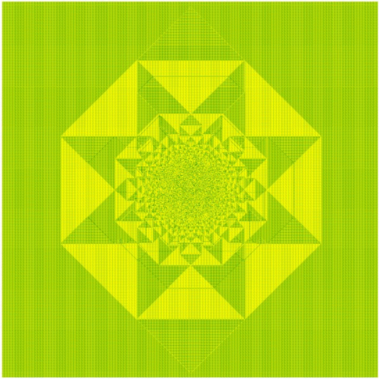


Figure: F-lattice with background density 5/8

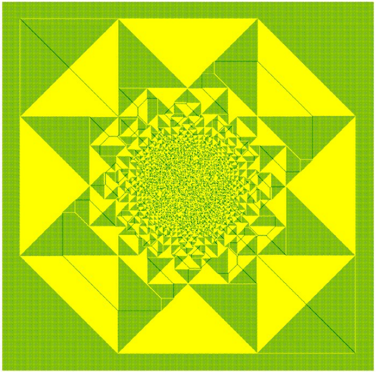


Figure: Manhattan lattice, with initial density $1/2$, and $120,000$ particles

Pattern formation in a noisy background

In presence of noise, the function ϕ is no longer polynomial, but the proprtionate growth still holds.

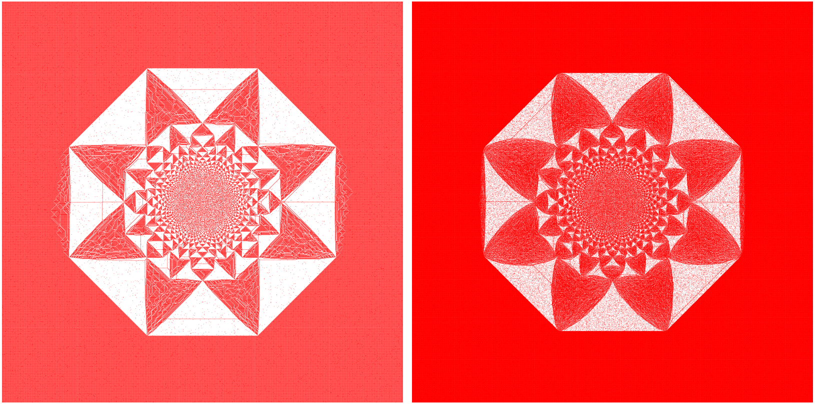


Figure: Pattern grown on the F-lattice with some heights 1 replaced by 0's. (a) 1% sites changed, N=228,000 , (b) 10% changed, N= 896,000.

If some 0's are also repalced by 1's, the effect is more dramatic.

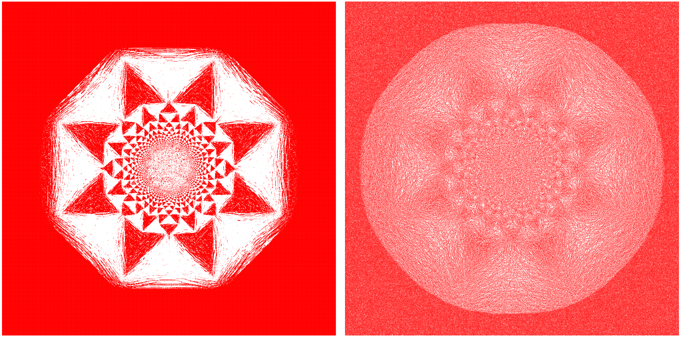


Figure: Pattern grown on the F-lattice with some heights flipped. (a) 1% sites changed (b) 10% sites changed

At higher noise level, the details of the pattern are not easy to see, but averaging over different realizations of noise brings out the pattern clearly.

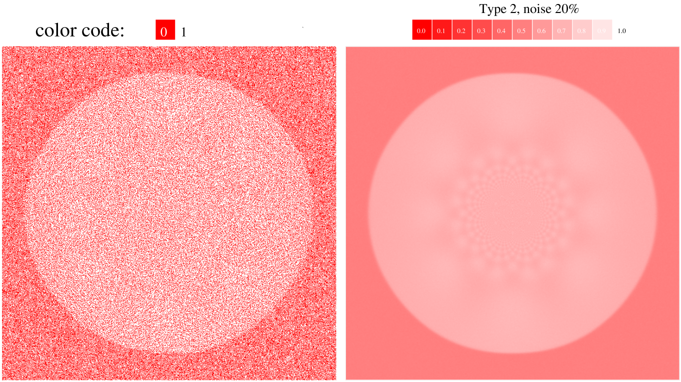


Figure: F-lattice, checkerboard with 20% sites flipped. $N = 57000$. (a) single realization (b) averaged height over 10^5 realizations.

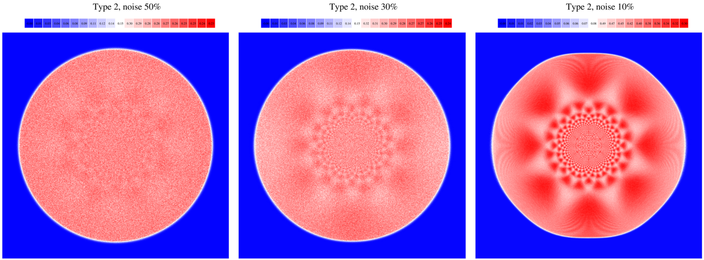


Figure: Averaged change in height with decreasing noise strength $50\%, 30\%, 10\%$ The color code for each pattern representing the height values are shown in the colorbar.

If we apply a $z \to 1/z^2$ transformation to these figures, we get

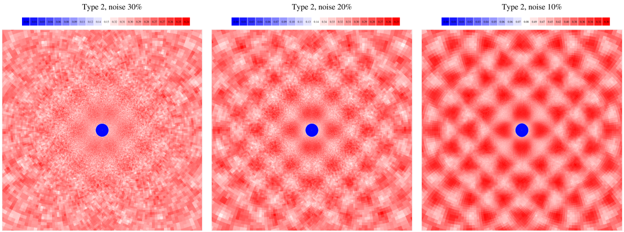


Figure: Result of applying $1/z^2$ transformation. Note the nearly gridlike pattern

This suggests that we can write the change in density as

$$\Delta\rho(x,y) = \left[A + B\epsilon^2 g(x,y) + \mathcal{O}(\epsilon^4)\right] f(x,y), \qquad (1)$$

Where ϵ is a measure of difference of noise strength from 50%

This suggests that the simplest perturbation to the density field in the high noise limit is a periodic perturbation in the z'-coordinates.

$$g(x, y) = -\cos\frac{\pi x'}{2}\cos\frac{\pi y'}{2}, \tag{2}$$

where $x' = \frac{2xy}{(x^2+y^2)^2}$, and $y' = \frac{x^2-y^2}{(x^2+y^2)^2}$.
A pictorial representation of this function is given below.

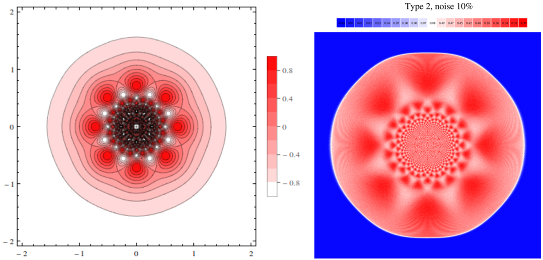
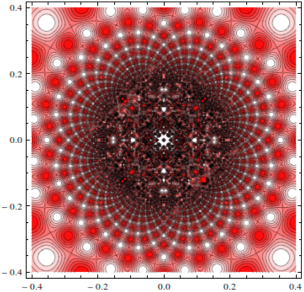


Figure: Density pattern using the function $g(x, y)$, compared to actual pattern. The black lines are contours of constant density.



A zoom-in on the theoretical lowest -mode density perturbation $g(x, y)$.

The pattern selection problem

For a given background, how is the pattern selected amongst the many possible pattern?

An important principle to look for is a variational principle. This allows approximate calculations, if exact calculation is not possible.

In our case, there is a very nice, and elementary variational principle, that I have called the lazy man's Principle of least 'action': Don't do anything unless you have to.
i.e. the total number of topplings required to get a stable pattern from the original configuration is the minimum for the actual pattern.
The proof is immediate for abelian sandpiles.

Reminiscent of minimum entropy production principle.

Unexpectedly generate features at several length scales, having a striking similarity to the natural ones

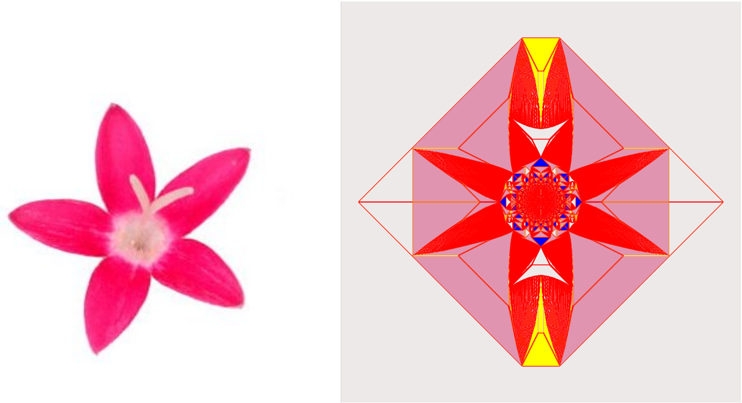


Figure: (a) A flower. (b) pattern produced by adding 256k particles on the F-lattice, with tilted squares backgound with spacing 4. Different colours denote different densities of particles, averaged over the unit cell of the background pattern

Directed growth patterns

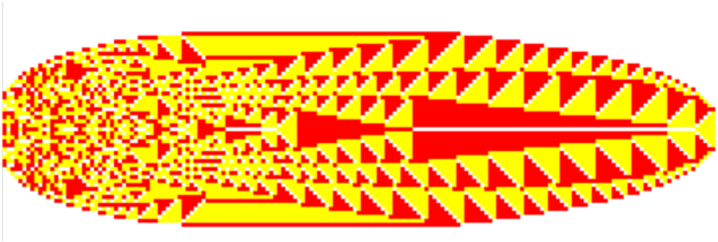


Figure: A 'larva' pattern. Produced on square lattice, with particle transfer on toppling only to up, down, right neighbors. Here $N = 10^4$. Particles are added at the left column center. Color code: 0=white, 1= red, 2=yellow.

A 'larva' pattern formed on a directed cubic lattice

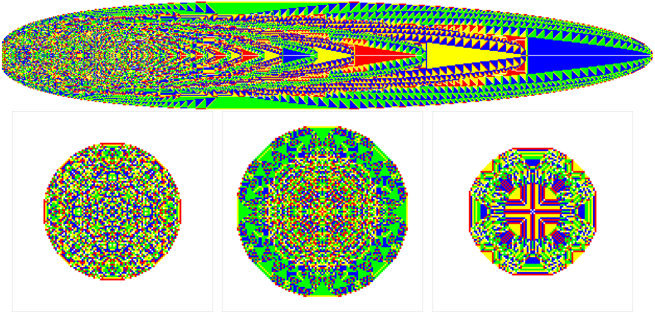


Figure: $N = 10^7$ grains added on an initial background of all heights zero. The particles were added at the central point of the left end. The first shows the mid-saggital section, and the next three show different transverse sections. Colour code : $0, 1, 2, 3, 4 =$ white, red, yellow, blue and green.

Conjectures

The theoretical understanding of these patterns is very limited.

- In general, we expect that $\Lambda_{longitudinal} \sim \Lambda^2_{transverse}$.
- Patterns do not show proportionate growth, but there are easily identified sub-structures, independent of N.
- For large N, $T_N(X, Y) \sim \Lambda^2 \phi(X/\Lambda, Y/\Lambda^2)$, where $\phi(\xi, \eta)$ defines the asymptotic pattern.
- In each periodic patch, $\phi(\xi, \eta)$ is a polynomial function of ξ and η, of degree 2 in ξ, and degree 1 in η.
- A repeated motif here is a layer of square patches of slowly varying sizes.

Detailed characterization ?

Connection to Tropical Mathematics

Define

$$a \oplus b = Max[a, b] \tag{3}$$

$$a \otimes b = a + b \tag{4}$$

Then standard properties of usual addition and multiplication (commutative, identity, distributive ..) contiue to hold.

Example: $3 \oplus 5 \oplus 2 = 5$

$3 \otimes 4 = 7$

Tropical polynomials: $a \otimes x \otimes x \oplus b \otimes x \oplus c$

Example: $x \otimes x \oplus 2 \otimes x \oplus 5 = Max[2x, x + 2, 5]$.

Fundamental theorem of tropical algebra.

A piecewise -linear convex function can be represented as a tropical polynomial.

Hence useful for describing the toppling function function in growing sandpiles where toppling function is piece-wise linear. Say, for the linearly growing triangular pattern,

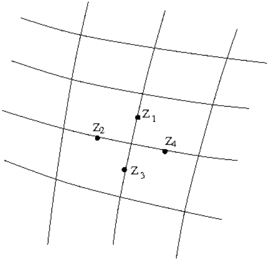
$$\phi(\xi, \eta) = \oplus_{l,m=0}^{\infty} \;\; F_{l.m} \otimes \xi^{l} \otimes \eta^{m}$$

Crumpled paper may be described by piece-wise linear functions

picture from www.myjanee.com/tuts/crumpled

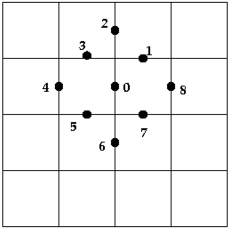
Discrete Analytic Functions

Functions defined only on discrete points in the complex $z-$ plane.



Discrete Cauchy-Riemann conditions:

$$\frac{F(z_1) - F(z_3)}{z_1 - z_3} = \frac{F(z_2) - F(z_4)}{z_2 - z_4}$$



On a square grid :

$$\Delta F_{13} + \Delta F_{35} + \Delta F_{57} + \delta F_{71} = 0$$

is equivalent to

$$\Delta F_{02} + \Delta F_{04} + \Delta F_{06} + \Delta F_{08} = 0$$

Discrete Laplace Equation.
Sum, but not product, of discrete analytic functions is also DA

We find that the coefficients of the linear terms in the toppling function define a discrete analytic function $d + ie$ of the complex variable $m + in$, where (m, n) is the patch label.
In fact the discrete analytic function is defined by the condions that $D(m + in)$ is a discrete analytic function, with D(0)=0, and $D(z)/z^{1/2}$ tends to 1 for large $|z|$.
These conditions determine the function $D(z)$ and hence the pattern, completely.
For growth near an edge, the function becomes discrete approximant to the function $z^{1/3}$

Acknowledgements

- S. Ostojic (2002)
- S. B. Singha
- S. Chandra
- T. Sadhu *
- R. Kapri
- R. Dandekar

References

T. Sadhu and DD, Current Sci., 10 Sep.2012 issue. D. Dhar and T. Sadhu, J. Stat. Mech. (2013) P110006.[arXiv:1310.1359]

Thank You.