Turbulent liquid crystals unveil universal fluctuation properties of growing interfaces

Kazumasa A. Takeuchi (Univ. of Tokyo)

Acknowledgment
Masaki Sano, Tomohiro Sasamoto, Herbert Spohn, Michael Prähöfer, Grégory Schehr
Interface Growth

Wide interest

- Ubiquitous.
  (e.g., coffee stain on a shirt, fabricating solid-state devices…)
- Obviously irreversible, thus out of equilibrium.
- Interesting pattern formation. (e.g., snowflakes, bacteria colony…) typically forming scale-invariant structures

Two types of mechanism

Non-local growth
- Metal dendrite
- Snowflake

Local growth
- Burning front
- Paper wetting
Interface Growth

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Two types of mechanism

- Non-local growth
  - Metal dendrite
  - Snowflake
- Local growth
  - Burning front
  - Paper wetting

Test-bed for universality out of equilibrium.
Roughening of Interfaces

Typically, local growth processes form **rough, self-affine interfaces**.

$h(x, t)$

**Paper wetting**
(and many other experiments)

**Eden model**
add a particle randomly onto the interface

**Ballistic deposition model**

**Self-affine:**
fluctuation properties are (statistically) invariant under $x \rightarrow ax, \ t \rightarrow a^z t, \ h \rightarrow a^\alpha h$
Characterizing Self-Affinity

“Interface width” quantifies the roughness of interfaces

\[ w(l, t) = \text{Standard deviation of } h(x, t) \text{ over length scale } l \]

\[ = \left\langle \sqrt{\langle [h(x, t) - \langle h \rangle_l]^2 \rangle_l} \right\rangle \]

Self-affinity of the interfaces implies: (Family-Vicsek scaling)

\[ w(l, t) \sim \begin{cases} 
  l^\alpha & (l \ll l_*) \\
  t^\beta & (l \gg l_*)
\end{cases} \]

\[ l_* \sim t^{1/z} \]

\[ \alpha : \text{roughness exponent} \quad z = \alpha/\beta \]

\[ \beta : \text{growth exponent} \quad : \text{dynamic exponent} \]
Basic Theory: KPZ Equation

- **Linear theory:** Edwards-Wilkinson eq.
  \[
  \frac{\partial}{\partial t} h(x, t) = v_0 + \nu \nabla^2 h + \xi(x, t)
  \]
  \[\langle \xi(x, t) \xi(x', t') \rangle = D \delta(x - x') \delta(t - t') \]

- **Kardar-Parisi-Zhang (KPZ) eq.**
  \[
  \frac{\partial}{\partial t} h(x, t) = v_0 + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \xi(x, t)
  \]
  ※by \( h \rightarrow h + v_0 t \), one can take \( v_0 = 0 \).

- In (1+1) dimensions, \( \alpha = 1/2, \beta = 1/3, z = 3/2 \)
- Exponents regularly seen in numerical models → **KPZ universality class**
- **Why \( \alpha = 1/2 \)?**
  1d EW/KPZ stationary interfaces = 1d Brownian motion
  \[ P[h] \sim \exp \left[ -\frac{\nu}{D} \int dx (\nabla h)^2 \right] \]
  \[ w \sim l^{1/2} \]
Situation in Experiments

Rough surfaces are ubiquitous, but KPZ is seen less frequently.

- Flow in porous media $\alpha = 0.81$, $\beta = 0.65$ [Horváth et al., 1991]
- Paper wetting $\alpha = 0.73$, $\beta = 0.60$ [Kobayashi et al., 2005]
- Growth of plant callus $\alpha = 0.86$, $\beta = 0.17$ [Galeano et al., 2003]
- Copper deposition $\alpha = 0.55$, $\beta > 1$ [Kahanda et al., 1992]
- Bacteria colony $\alpha = 0.78$ [Wakita et al., 1997]

Small, but growing # of experiments showing KPZ exponents

- Colony of mutant bacteria [Wakita et al., 1997]
- Slow combustion of paper [Maunuksela et al., 1997-]
- Turbulent liquid crystal [Takeuchi & Sano, 2010-]
- Tumor-like & tumor cells [Huergo et al., 2010-]
- Particle deposition on coffee ring [Yunker et al., 2013]

Advantages

- Simple growth mechanism
- Precise control
- Many experimental runs
- High statistical accuracy
Electroconvection

Nematic liquid crystal (e.g., MBBA)
- Rod-like molecule: \(\text{CH}_3\text{O} - \text{CH} = \text{N} - \text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3\)
- Strong anisotropy: \(\varepsilon_\parallel < \varepsilon_\perp, \sigma_\parallel > \sigma_\perp\)
  - Convection driven by electric field

\[
\begin{align*}
\text{phase diagram (MBBA; planar alignment)} \\
\text{vs. voltage amplitude (V)} \\
\text{frequency (Hz)} \\
\end{align*}
\]

- DSM2 nucleation (\(V \gg V_c\))
- Dynamic Scattering Mode 2 (DSM2)
- Dynamic Scattering Mode 1 (DSM1)
- Grid pattern
- Williams domain

100 \(\mu\text{m}\)
Two Turbulent States: DSM1 & DSM2

DSM1

nucleation if \( V \gg V_c \)

DSM2

0V \( \rightarrow \) 72V \( \rightarrow \) 0V \((V_c \approx 30\ V, \text{speed \times 3})\)

DSM2 = topological-defect turbulence (analogy with "quantum turbulence"?)

We focus on DSM1-DSM2 interfaces and study their fluctuations
Experimental Setup

- Quasi-2d cell: $16 \text{ mm} \times 16 \text{ mm} \times 12 \mu\text{m}$
- Nematic liquid crystal: MBBA
- Homeotropic alignment (to work with isotropic growth)
- Temperature control: $T = 25 ^\circ \text{C}$
- Nucleation of DSM2 by UV pulse laser

26V, 250Hz  Speed x2, $\rightarrow 200 \mu\text{m}$

Rough interface appears

(schematic)
Scaling Exponents

interfaces at $t = 2, 7, 12, \cdots, 27\text{sec}$

$\text{interface width } w(l, t) = \text{standard deviation of } h(x, t) \text{ over length } l$

$= \left\langle \sqrt{\langle h(x, t) - \langle h l \rangle \rangle^2}\right\rangle$

Family-Vicsek scaling

$w(l, t) \sim t^\beta F(l t^{-1/z}) \sim \begin{cases} l^\alpha & (l \ll l_*) \\ t^\beta & (l \gg l_*) \end{cases}$

$l_* \sim t^{1/z}, z = \alpha/\beta$

$w(l, t)$ vs $l$

slope $\alpha_{\text{KPZ}} = 1/2$

$w(l, t)$ vs time $t$

slope $\beta_{\text{KPZ}} = 1/3$

data collapse

slope $\alpha_{\text{KPZ}} = 1/2$

Both exponents $(\alpha, \beta)$ agree with the KPZ class
Deeper Look at Height Fluctuations

**Key quantity:** $n$th-order cumulant $\langle h^n \rangle_c$

\[
\begin{align*}
\langle h^2 \rangle_c &\equiv \langle \delta h^2 \rangle \sim t^{2/3} \quad (\delta h \equiv h(x, t) - \langle h \rangle) \\
\langle h^3 \rangle_c &\equiv \langle \delta h^3 \rangle \\
\langle h^4 \rangle_c &\equiv \langle \delta h^4 \rangle - 3 \langle \delta h^2 \rangle^2
\end{align*}
\]

This suggests $h(t) \simeq v_\infty t + (\Gamma t)^{1/3} \chi$ ($\chi$: non-Gaussian random variable)

$\chi$ obeys the largest-eigenvalue distribution [Tracy-Widom (TW) dist.] of GUE random matrices!
Tracy-Widom Distribution

describes the largest-eigenvalue distribution of Gaussian random matrices

e.g.) Gaussian Unitary Ensemble (GUE)

complex Hermite matrix \( A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \ldots & A_{NN} \end{pmatrix} \)

\( A_{ij} = \bar{A}_{ji} = a_{ij} + ib_{ij} \)

\( A_{ii} = a_{ii} \)

prob. density for all eigenvalues (Wigner’s semicircle law)

\( \lambda_{max} \approx 2N + N^{1/3} \chi_{\text{GUE}} \)

Gaussian mean 0 variance \( N/2 \)

mean 0 variance \( N \)

Experiment:

height fluctuations

\( h(t) \approx v_\infty t + (\Gamma t)^{1/3} \chi \)

apparent correspondence

\( t \leftrightarrow N \)

\( \chi \leftrightarrow \chi_{\text{GUE}} \)

pdf(\( \chi_{\text{GUE}} \)) \equiv \text{GUE Tracy-Widom dist.}
Universal Distribution!

Define the rescaled height

\[ q \equiv (h - v_\infty t)/(\Gamma t)^{1/3} \approx \chi \]

Difference from GUE-TW distribution

Interface fluctuations precisely agree with the GUE-TW distribution up to the 4th order cumulant! Finite-time effect \( \sim t^{-1/3} \) for the mean

GUE-TW statistics was first found in solvable models [Johansson 2000; Prähofer & Spohn 2000] and recently in an exact solution of KPZ eq. [Sasamoto & Spohn, Amir et al., 2010]
In case of the PNG (= polynuclear growth) model [Prähofer & Spohn, PRL 2000]

Time evolution: (1) stochastic nucleations
(2) deterministic lateral expansion

For circular interfaces, first nucleation at \((x,t) = (0,0)\)

\[ h(0,t) = \text{# of lines to pass when moving from } (0,0) \text{ to } (0,t) \]
\[ = \text{max # of dots passed by directed polymer between } (0,0) \text{ and } (0,t) \]
\[ = \text{length of longest increasing subsequences} \]
\[ = \ldots \text{(Young tableau)} \ldots = \text{asymptotically, GUE-TW dist.} \]

\( (\text{curved}) \text{ PNG fluctuations obey the GUE-TW dist.} \)
\[ h(0,t) \approx \sqrt{2}t + \left( \frac{t}{\sqrt{2}} \right)^{1/3} \chi_{\text{GUE}} \]

Experiment implies universality of the GUE-TW distribution

related to random matrix, combinatorics, disordered systems, etc.
Geometry-Dependent Universality

Flat interfaces can also be created by shooting line-shaped laser pulses

26V, 250Hz  Speed x5,  — 200μm
Same KPZ exponents are found.

however
measuring
the distribution.

circular : $h(t) \approx v_\infty t + (\Gamma t)^{1/3} \chi_{\text{GUE}}$
flat : $h(t) \approx v_\infty t + (\Gamma t)^{1/3} \chi_{\text{GOE}}$

Same results in solvable models [Prähofer & Spohn 2000]

Same exponents, but different distributions!!

KPZ class splits into (at least) two universality sub-classes:
“curved KPZ sub-class” & “flat KPZ sub-class”
Why Different Distributions?

Quick answer: Because of different space-time symmetry

For the PNG model

- Circular  Consider a square connecting (0,0) and (0,t)  GUE
- Flat Consider a triangle connecting $t=0$ and (0,t)  GOE

Different initial conditions (curved or not) lead to different symmetries and to different universal sub-classes! [GUE-TW (curved) & GOE-TW (flat)]
Extreme-Value Statistics (circular)

Max heights of circular interfaces obey the GOE-TW dist.!!

Max radius $\max R$ : Gumbel dist.

cdf($x$) = $\exp(-e^{\alpha(x-u)})$

radius $R$ : GUE-TW

$\max\left(R\sin\theta\right)$ : GOE-TW distribution!!

Max heights of circular interfaces obey the GOE-TW dist.!!
Why GOE-TW for the Max Heights?

For the PNG model

- **Circular** Consider a **square** connecting (0,0) and (0,t) ➔ GUE
- **Flat** Consider a **triangle** connecting \( t = 0 \) and (0,t) ➔ GOE
- **Max height of droplet** ➔ **triangle** connecting (0,0) and \( t = t \) ➔ GOE!

Max-height dist. for circular interfaces has the same symmetry as the one-point dist. for flat interfaces ➔ GOE-TW dist.!

[proof: Johansson 2003]
Spatial Correlation Function

Predictions for solvable models:

\[
C_s(l, t) \equiv \langle h(x + l, t)h(x, t) \rangle - \langle h \rangle^2 \\
\approx (\Gamma t)^{2/3} g_i(\zeta)
\]

with \( i = 1 \) (flat), \( i = 2 \) (circular),

\[
\zeta \equiv l \sqrt{I/2v_\infty(\Gamma t)^{-2/3}} \\
g_i(\zeta) \equiv \langle \mathcal{A}_i(t + \zeta)\mathcal{A}_i(t) \rangle - \langle \mathcal{A}_i(t) \rangle^2
\]

\( \mathcal{A}_i(t) \): Airy\(_i\) process (cf. Airy\(_2\) = largest-eigenvalue dynamics in Dyson’s Brownian motion of GUE matrices)

Two-point correlation function

Correlation of flat / circular interfaces is governed by the Airy\(_1\) / Airy\(_2\) process

Qualitatively different decay

\[
\begin{cases} 
  g_2(u) \sim u^{-2} \quad \text{(circular)} \\
  g_1(u) \text{: faster than exponential} \quad \text{(flat)}
\end{cases}
\]
Spatial Persistence

Spatial Persistence probability $P_{\pm}^{(s)}(l; t)$

$= \text{joint probability that } \delta h \equiv h(x, t) - \langle h \rangle \text{ keeps the same sign over length } l \text{ in space at fixed time } t$

• Exponential decay $P_{\pm}^{(s)} \sim \exp(-\kappa_{\pm}^{(s)} u)$ with symmetric coefficients $\kappa_{+}^{(s)} = \kappa_{-}^{(s)}$

• $\kappa_{\pm}^{(s)}$ expected to be universal $\kappa_{+}^{(s)} \approx 2.0 \text{ (flat)}$ $\kappa_{+}^{(s)} \approx 0.9 \text{ (circular)}$ [cf. Ferrari&Frings 2013]

• Extension of the Newell-Rosenblatt theorem for Airy$_2$ process?
NR theorem: for stat. Gaussian processes, $P^{(s)} \sim \exp(-\kappa t)$ if $\langle \mathcal{A}(t)\mathcal{A}(0) \rangle \sim t^{-\mu}$ ($\mu > 1$)
Temporal Correlation Function

\[ C_t(t, t_0) \equiv \langle h(x, t)h(x, t_0) \rangle - \langle h(x, t) \rangle \langle h(x, t_0) \rangle \]

analytically unsolved yet

- Natural scaling ansatz works

\[ C_t(t, t_0) \simeq (\Gamma^2 t_0 t)^{1/3} F_t(t/t_0) \]

- In particular,

\[ F_t(t/t_0) \sim (t/t_0)^{-\tilde{\lambda}} \text{ with } \tilde{\lambda} = 1 \]

cf. Kallabis-Krug conjecture \( \tilde{\lambda} = \beta + d/z = 1 \)

- The natural scaling does not seem to work as well.

- In particular,

\[ C_t(t, t_0) > 0 \quad (t \to \infty) \quad (!) \]
Temporal Persistence (Flat Case)

**Persistence probability** $P_{\pm}(t, t_0)\) = joint probability that $\delta h \equiv h(x, t) - \langle h \rangle$ at a fixed position $x$ is positive (negative) at time $t_0$ and keeps the same sign until time $t$.

typically decay with a power law $P_{\pm}(t, t_0) \sim t^{-\theta_{\pm}}$

\[
\begin{cases}
\theta_+ = 1.35(5) \\
\theta_- = 1.85(10)
\end{cases}
\]

$\theta_+ < \theta_-$ (flat)

because of the KPZ nonlinearity $\frac{\partial h}{\partial t} = v\nabla^2 h + \frac{1}{2}(\nabla h)^2 + \xi$
Temporal Persistence (Circular Case)

Persistence probability \( P_{\pm}(t, t_0) \)

= joint probability that \( \delta h \equiv h(x, t) - \langle h \rangle \) at a fixed position \( x \)
  is positive (negative) at time \( t_0 \) and keeps the same sign until time \( t \)

typically decay with a power law \( P_{\pm}(t, t_0) \sim t^{-\theta_{\pm}} \)

\[
\begin{align*}
\theta_+ &= 0.81(2) \\
\theta_- &= 0.80(2)
\end{align*}
\]

Asymmetry in persistence exponents is cancelled for the circular interfaces!
3 Important Sub-classes

Circular (curved) interfaces

- Init. cond.: point or curved line
- Asymptotics: GUE Tracy-Widom dist., Airy$_2$ process

Flat interfaces

- Init. cond.: straight line
- Asymptotics: GOE Tracy-Widom dist., Airy$_1$ process
- Proved for: PNG [Prähofer-Spohn PRL 2000], TASEP [Sasamoto JPA 2005], KPZ eq.

Stationary interfaces

- Init. cond.: stationary interface (= trajectory of 1d-Brownian motion)
- Asymptotics: Baik-Rains $F_0$ distribution, Airy$_{stat}$ process
- Proved for: PNG [Baik-Rains JSP 2000], TASEP, KPZ eq. [Imamura-Sasamoto PRL 2012]

※ Scaling exponents are the same.
※ Other sub-classes are also argued.
Toward the Stationary Subclass

Truly stationary state is never attained unless it is taken as an initial condition, but, approach, or crossover to the stationary subclass can be studied.

\[ h(x, t_0 + \Delta t) \]

\[ \Delta h \approx v_\infty \Delta t + (\Gamma \Delta t)^{1/3} \chi \]

\[ h(x, t_0) \]

rescaled height difference

\[ \Delta q \equiv \frac{\Delta h - v_\infty \Delta t}{(\Gamma \Delta t)^{1/3}} \approx \chi \]

PNG model (simulation)

- \( t_0 \geq 1 \)
- \( t_0 = 0.1 \)

experiment

- Scaling functions \( \langle \Delta q^n \rangle_c \approx G_n(\Delta t/t_0) \) describing flat-stationary crossover is found.
- Experiment seems to indicate the same scaling functions, so universal!
Summary

Evidence for KPZ geometry-dependent universal fluctuations in growing interfaces of liquid-crystal turbulence (DSM2)

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<th>Circular interfaces</th>
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<td>remains strictly positive</td>
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*deep & direct link between quantitative experiment and exactly solvable problems*

Deadline for contributed talks: June 13th
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