Weak-noise limit of systems driven by non-Gaussian fluctuations

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Stochastic model for non-equilibrium systems

- Equation of motion:

$$\dot{q}(t) = F(q(t)) + \sqrt{D}\,\xi(t)$$

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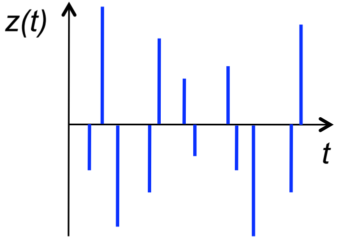
$$\dot{q}(t) = F(q(t)) + \sqrt{D}\,\xi(t) + z(t) - a$$

Poissonian shot noise (PSN)

$$z(t) = \sum_{i=1}^{N_t} A_i \delta(t - t_i)$$

- N_t Poisson distribution
- Times t_i uniform in $[0, t]$
- A_i are i.i.d. with density $\rho(A)$



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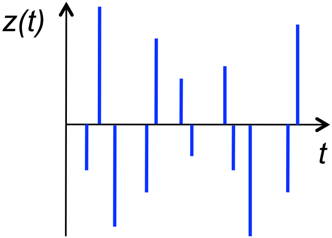

- *Lévy noise:* $\Gamma(t) = \sqrt{D}\xi(t) + z(t) - a, \qquad a = \langle z(t) \rangle$

Poissonian shot noise

- Average number of shots: $\langle N(t)\rangle = \lambda t$

$$\begin{aligned}\langle z(t)\rangle &= \lambda \langle A\rangle \\ \mathrm{Cov}(z(t), z(t')) &= \lambda \left\langle A^2\right\rangle \delta(t-t')\end{aligned}$$

- Infinite hierarchy of cumulants

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- Infinite hierarchy of cumulants
- *Non-local diffusion*

$$\begin{aligned}\frac{\partial}{\partial t}p(q,t) &= -\frac{\partial}{\partial q}(F(q)-a)p(q,t)+\frac{D}{2}\frac{\partial^2}{\partial q^2}p(q,t) \\ &\quad +\lambda\int_{-\infty}^{\infty}\mathrm{d}A\,p(q-A,t)\rho(A)-\lambda p(q,t)\end{aligned}$$

- *Weak-noise limit?*

Characteristic functional of PSN

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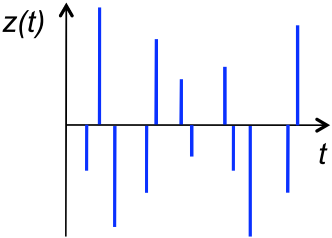

Calculate noise functional

$$G_z[g] = \left\langle \exp\left\{ i \int_0^t g(s) z(s) \mathrm{d}s \right\} \right\rangle = \exp\left\{ \lambda \int_0^t \left(\phi(g(s)) - 1\right) \mathrm{d}s \right\}$$

where $\phi(k) = \left\langle e^{iAk} \right\rangle$

Path-integral formalism

- Propagator given as path-integral over path weight $\mathcal{P}[q]$

$$\begin{aligned} f(q,t|q_0) &= \int_{(q_0,0)}^{(q,t)} \mathcal{D}q\,\mathcal{P}[q] \\ &= \int_{(q_0,0)}^{(q,t)} \mathcal{D}q \int \mathcal{D}g \exp\left\{-\int_0^t \mathcal{L}(q,g)\mathrm{d}s\right\} \end{aligned}$$

- Write $\mathcal{P}[q]$ as inverse functional FT

$$\mathcal{P}[q] = \int \mathcal{D}g \exp\left\{-i\int g(s)(\dot{q} - F_a(q))\mathrm{d}s\right\} G_\xi[g]\,G_z[g]$$

- Lagrangian:

$$\mathcal{L}(q,g) = ig(\dot{q} - F_a(q)) + \frac{1}{2}Dg^2 - \lambda(\phi(g) - 1)$$

- Conjugate momentum: $\partial\mathcal{L}/\partial\dot{q} = ig$

Path-integral formalism

- Lagrangian

$$\mathcal{L}(q,g) = ig(\dot{q} - F_a(q)) + \frac{1}{2}Dg^2 - \lambda(\phi(g) - 1)$$

- *Want:* $\mathcal{L} \to \tilde{\mathcal{L}}/D$. Introduce the *scaling:*

$$g \quad \to \quad \tilde{g}/D$$

Path-integral formalism

- Lagrangian

$$\mathcal{L}(q,g) = ig(\dot{q} - F(q)) + \frac{1}{2}Dg^2 + \lambda\left(\left\langle A^2\right\rangle \frac{g^2}{2!} + \left\langle A^3\right\rangle \frac{ig^3}{3!} + ...\right)$$

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Path-integral formalism

- Lagrangian

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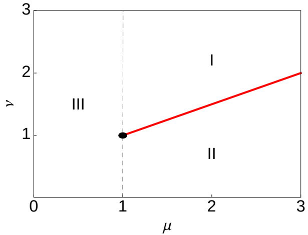

- *Gaussian weak-noise limit:*

$$\nu = \frac{1}{2}(\mu + 1), \qquad \mu > 1$$

- *PSN weak-noise limit:* $\mu = \nu = 1$

Euler-Lagrange equations

Saddle-point approximation for $D \longrightarrow 0$

$$f(q,t|q_0) = \psi(q^*,g^*)\exp\left\{-\frac{1}{D}\int_0^t \mathcal{L}(q^*,g^*)\mathrm{d}s\right\}(1+\mathcal{O}(D))$$

- Optimal paths determined by *coupled EL equations*

$$\begin{aligned}\dot{q} &= F_a(q) + ig - i\lambda\phi'(g)\\ \dot{g} &= -F_a'(q)g\end{aligned}$$

  with boundary conditions $q(0) = q_0$ and $q(t) = q_t$
- Prefactor $\psi(q^*,g^*)$ can be calculated by recursion relation

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- *Gaussian case* $(\lambda = 0)$

$$g = -i(\dot{q} - F(q)) \qquad \rightarrow \qquad \ddot{q} - F'(q)F(q) = 0$$

$$\rightarrow \qquad \mathcal{L} = \frac{1}{2}(\dot{q} - F(q))^2$$

Weak-noise limit of non-equilibrium systems

1. *Escape from metastable potential* $\longrightarrow$ asymptotic scaling of $\langle \tau_{\mathrm{ex}} \rangle$
2. Large deviations of *non-equilibrium observables*

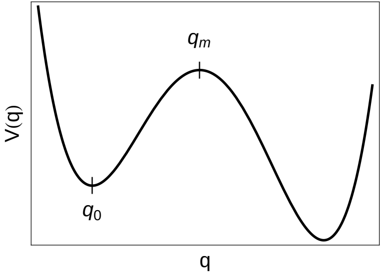
$$\begin{aligned} \Omega[q] &= \int_0^t U(\dot{q}, q)\mathrm{d}s \\ I(\omega) &= \lim_{D\to 0} D \log P_\Omega(\omega) \end{aligned}$$

3. *Piecewise linear transport model*
    - Simple model for noise induced transport
    - Stationary properties
    - Weak-noise approximation of finite time propagator

Escape from metastable potential

Kramer's rate

$$r = \frac{1}{\langle \tau_{\rm ex} \rangle} \propto e^{-\beta \Delta V}$$



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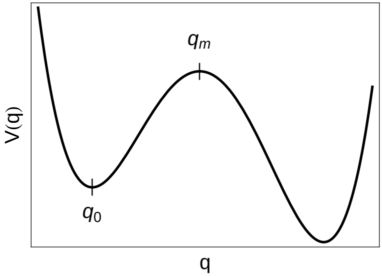

- Exact asymptotics of $\langle \tau_{\rm ex} \rangle$ (*Freidlin & Wentzell*):

$$\lim_{D \to 0} D \log \langle \tau_{\rm ex} \rangle = \inf_{t \geq 0} S(q_{\rm m}, t; q_0)$$

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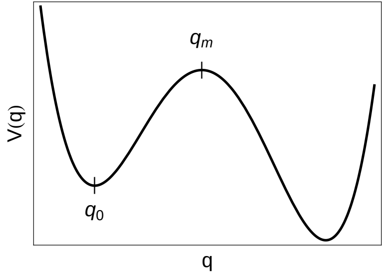

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$$\lim_{D \to 0} D \log \langle \tau_{\mathrm{ex}} \rangle = \inf_{t \geq 0} S(q_{\mathrm{m}}, t; q_0)$$

- *Action for PSN:*

$$S(q_{\mathrm{m}}, t; q_0) = \int_0^t \mathcal{L}(q^*, g^*) \mathrm{d}s$$

Escape path

- *Gaussian case* $(\lambda = 0)$

$$\ddot{q} - F'(q)F(q) = 0 \qquad \rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}(\dot{q}^2 - F(q)^2) = 0$$

Escape path

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Optimal paths:

$\dot{q} = F(q)$	$\dot{q} = -F(q)$
Relaxation: zero action	Excitation: non-zero action

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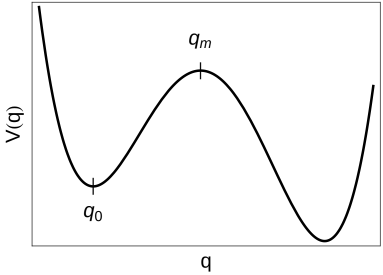
Relaxation: zero action — Excitation: non-zero action

- Escape path is the *time-reverse* of a deterministic relaxation path. Action:

$$S = \frac{1}{2}\int_0^t (\dot{q} - F(q))^2 \mathrm{d}s = 2\Delta V$$

Escape path



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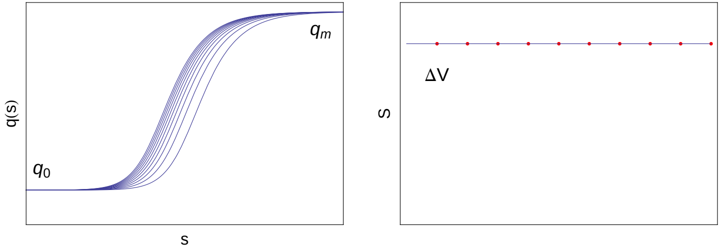

Escape path

- *PSN case* ($\lambda \neq 0$)

$$\begin{aligned} \dot{q} &= F_a(q) + ig - i\lambda\phi'(g) \\ \dot{g} &= -F_a'(q)g \end{aligned}$$

with boundary conditions $q(0) = q_0$ and $q(t) = q_{\rm m}$

- Action:

$$S(q_{\rm m}, t; q_0) = \int_0^t \mathcal{L}(q^*, g^*)\mathrm{d}s$$

- Noise-free deterministic relaxation:

$$g = 0 \qquad \longrightarrow \qquad S = 0$$

Escape path

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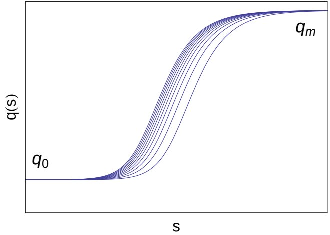

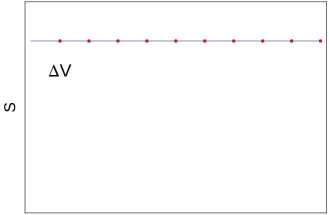

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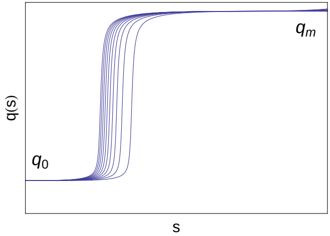
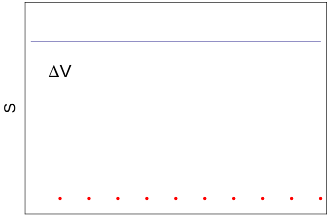
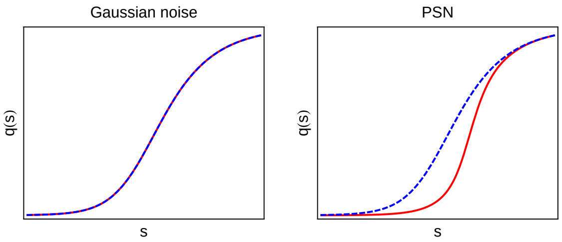




Time-reversal symmetry

- Optimal paths break time-reversal symmetry



- Relation with *fluctuation theorems*

Ratio of path probabilities

$$\log \frac{p[q(s)|q_0]}{p[\tilde{q}(s)|\tilde{q}_t]} = \begin{cases} \beta\Delta E & \textit{thermal noise} \\ \beta\Delta\mathcal{S} & \textit{driving} \\ ? & \textit{PSN} \end{cases}$$

Large deviations of non-equilibrium observables

- Consider functionals of $q(s)$

$$\Omega[q] = \int_0^t U(\dot{q}, q)\mathrm{d}s$$

- We are interested in large deviations

$$I(\omega) = \lim_{D\to 0} -D \log P_\Omega(\omega)$$

- Consider *scaled cumulant generating function*

$$\Lambda(\alpha) = \lim_{D\to 0} D \log \left\langle e^{\alpha \int_0^t U(\dot{q},q)\mathrm{d}s} \right\rangle$$

- Legendre transform

$$I(\omega) = \sup_\alpha(\alpha\omega - \Lambda(\alpha))$$

Large deviations of non-equilibrium observables

- Obtain from path-integral

$$\Lambda(\alpha) = -\inf_{q_t} \tilde{S}(q_t, t; q_0)$$

- Modified Lagrangian

$$\tilde{\mathcal{L}}(q^*, g^*) = \mathcal{L}(q^*, g^*) - \alpha U(\dot{q}^*, q^*)$$

Large deviations of non-equilibrium observables

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- Euler-Lagrange equations

$$\begin{aligned} \dot{q} &= F_a(q) + ig - i\lambda\phi'(g) \\ \dot{g} &= -F_a'(q)g - i\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}} - \frac{\partial U}{\partial q}\right) \end{aligned}$$

Exact solution for linear force

- Consider linear force and linear functional (*dragged particle model*)

$$\begin{aligned} F(q) &= -\gamma q + f \\ U(\dot{q}, q) &= q \end{aligned}$$

- EL equations with boundary conditions $q(0) = q_0$ and $q(t) = q_t$

$$\begin{aligned} \dot{q} &= -\gamma q + f - a + ig - i\lambda\phi'(g) \\ \dot{g} &= \mu g + i\alpha \end{aligned}$$

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- Action: $\tilde{S}(q_t, t; q_0; g_0)$. *Integration constant g_0*

$$\frac{\partial}{\partial g_0}\tilde{S}(q_t, t; q_0; g_0) = 0$$

Exact solution for linear force

- Scaled cumulant generating function

$$\Lambda(\alpha) = -\inf_{q_t} \tilde{S}(q_t, t; q_0; g_0) = -\tilde{S}(q_t^*, t; q_0; g_0)$$

with $\frac{\partial}{\partial q_t^*}\tilde{S}(q_t^*, t; q_0; g_0) = 0$. Solve for $g_0(q_t^*)$.

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- Long time limit

$$\lim_{t\to\infty} \frac{1}{t}\Lambda(\alpha) = \frac{\alpha^2}{2\mu^2} - \frac{\alpha}{\mu}(f-a) + \lambda\left(\phi\left(\frac{i\alpha}{\mu}\right) - 1\right)$$

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- Result previously obtained for particular ϕ and *arbitrary D*

Baule & Cohen, PRE (2009)

- Weak-noise approximation yields exact solution for linear systems

Stochastic model for noise-induced transport

Equation of motion:

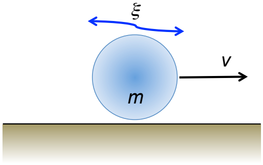
$$\dot{v}(t) = F(v) + z(t) - a$$

with

$$F(v) = \begin{cases} F_+(v), & v > 0 \\ F_-(v), & v < 0 \end{cases}$$

- *Piecewise-linear* force (dry friction) and *PSN*
- Granular Brownian motors

Directed motion due to interplay of friction and noise



- Brownian motion:

$$m\,\dot{v}(t) = -\gamma v(t) + \xi(t)$$

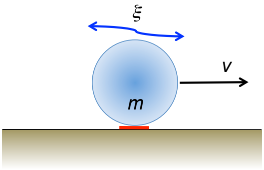
  - *linear friction*
  - average velocity:

$$\langle v\rangle = \frac{1}{\gamma}\langle\xi(t)\rangle = 0 \qquad \rightarrow \textit{no directed motion}$$

  - fluctuations do not exert a net force:

$$\langle\xi(t)\rangle = 0$$

Directed motion due to interplay of friction and noise



- Stochastic equation of motion (*diffusion process*):

$$m\,\dot{v}(t) = -\gamma v(t) - m\Delta f(v) + \xi(t)$$

  - *nonlinear friction*
  - average velocity:

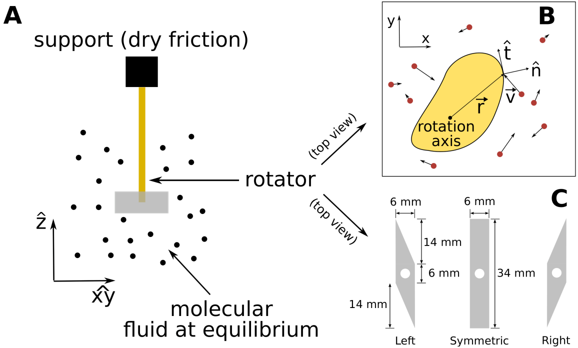
$$\langle v\rangle = -\Delta\tau\langle f(v)\rangle \neq 0, \qquad \text{for} \quad \langle\xi(t)\rangle = 0$$

    - ★ inertia
    - ★ nonlinear response
    - ★ asymmetric $p(v) \longrightarrow$ asymmetric $\xi(t)$

Granular Brownian motors

Equation of motion:

$$\dot{\omega}(t) = -\gamma\omega(t) - \sigma\left[\omega(t)\right]\Delta + \eta_{coll}(t)$$



Gnoli *et al*, PRL (2013)

Rare and frequent collision limits

- Consider parameter

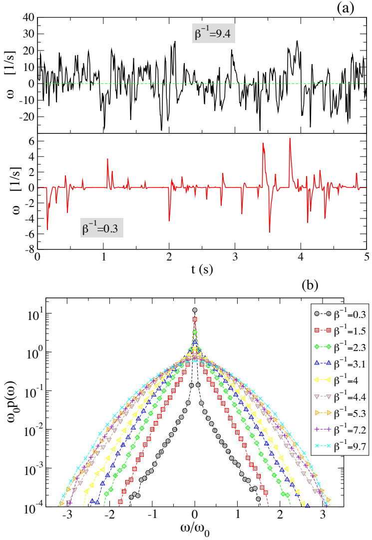
$$\beta = \frac{\tau_c}{\tau_\Delta}$$

- Angular velocity PDF exhibits *delta-peak* for

$$\beta \to \infty$$

Rare collision limit

Gnoli, Puglisi, Touchette, EPL (2013)



Formal mapping of collision process to PSN

Master equation (low density gas): Cleuren & Eichhorn, JSTAT (2008)

$$\frac{\partial}{\partial t}p(\omega,t)+\frac{\partial}{\partial\omega}F(\omega)p(\omega,t)=\int \mathrm{d}\omega'\left[W(\omega|\omega')p(\omega',t)-W(\omega'|\omega)p(\omega,t)\right]$$

$$=\lambda(\omega)\left(\int p(\omega-A,t)\rho(\omega,A)\mathrm{d}A-p(\omega,t)\right)$$

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$$=\lambda(\omega)\left(\int p(\omega-A,t)\rho(\omega,A)\mathrm{d}A-p(\omega,t)\right)$$

Approximate in the *rare collision regime*

$$\lambda(\omega) \approx \int \mathrm{d}\omega\lambda(\omega)p(\omega)\approx\lambda(0)$$

$$\rho(\omega,A) \approx \int \mathrm{d}\omega\rho(\omega,A)p(\omega)\approx\rho(0,A)$$

$\rightarrow$ PSN with frequency λ and amplitude distribution ρ

Stationary solution

- Density $p(v,t)$ satisfies (KF equation)

$$\frac{\partial}{\partial t}p(v,t) + \frac{\partial}{\partial v}F(v)p(v,t) = \lambda \int_{-\infty}^{\infty} \mathrm{d}A\, p(v-A,t)\rho(A) - \lambda p(v,t)$$

- Diffusion part is *non-local*

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- Diffusion part is *non-local*
- Stationarity condition

$$F(v)p(v) = \int_{-\infty}^{\infty} \mathrm{d}v'\, G(v-v')p(v')$$

Around $v=0$: $\quad F(0^+)p(0^+) = F(0^-)p(0^-)$

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- Stationarity condition

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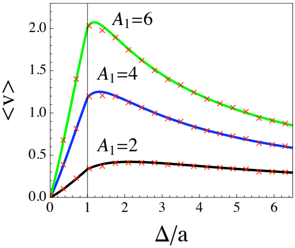
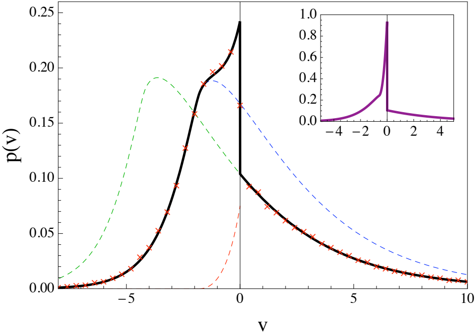
Around $v=0$: $$F(0^+)p(0^+) = F(0^-)p(0^-)$$

  - $p(v)$ is *discontinuous* at $v=0$
  - $p(v)$ contains *delta peak* at $v=0$ for $F(0^-)F(0^+) < 0$

Stationary regime

- *Non-monotonic transport* for increased friction
- Superposition of integrable and *non-integrable solutions*





Baule & Sollich (EPL, 2011); (PRE, 2012)

Finite time propagator in the weak-noise limit

- Optimal paths determined by coupled EL equations

$$\begin{aligned} \dot{v} &= F_a(v) + ig - i\lambda\phi'(g) \\ \dot{g} &= -F_a'(v)g \end{aligned}$$

with boundary conditions $v(0) = v_0$ and $v(t) = v_t$

- For *piecewise-linear* force obtain solution $v_+(s)$ for $v > 0$ and $v_-(s)$ for $v < 0$

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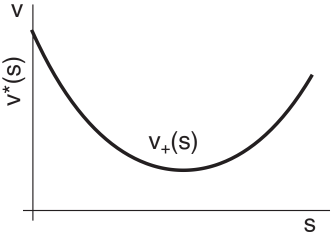
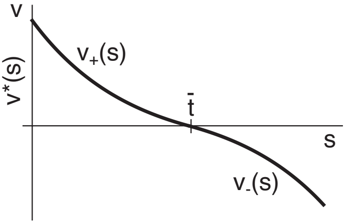
- Determine cross-over at $v = 0$ by *second action minimization:*

$$\inf_{\bar{t}}[S_+(0, \bar{t}; q_0, 0) + S_-(q_t, t; 0, \bar{t})]$$

Optimal paths in the velocity-time plane

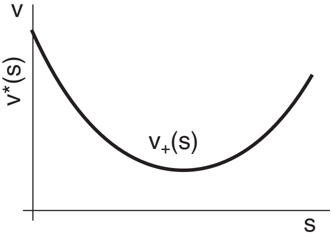
- *Direct paths:* pure slip motion

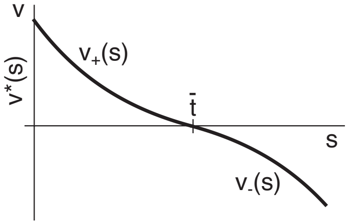




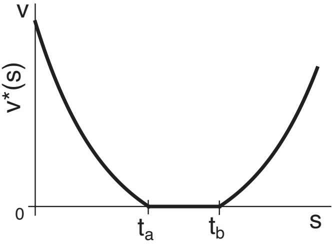
Optimal paths in the velocity-time plane

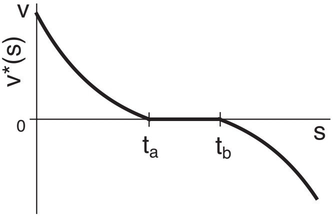
- *Direct paths:* pure slip motion





- *Indirect paths:* stick-slip motion

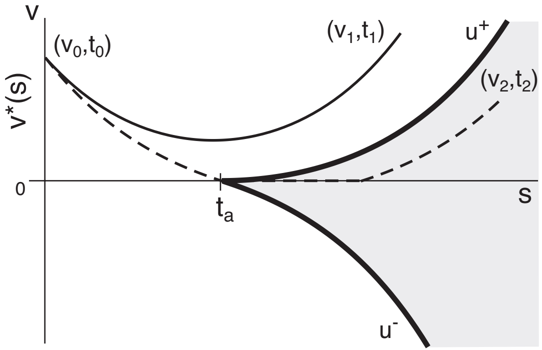




Structure of the optimal paths

Dynamical phase diagram

- Second action minimization distinguishes **direct (slip)** and **indirect (stick-slip)** paths



Baule, Cohen, Touchette, JPhysA (2011)

Result for the propagator

Pure PSN case: