Current Fluctuations in the Exclusion Process

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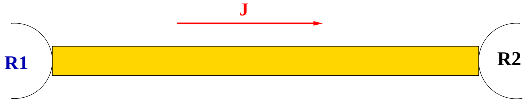
Classical Transport in 1d: ASEP

A paradigmatic picture of a non-equilibrium system



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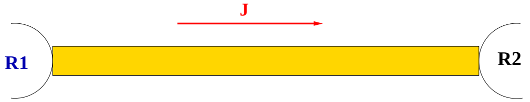
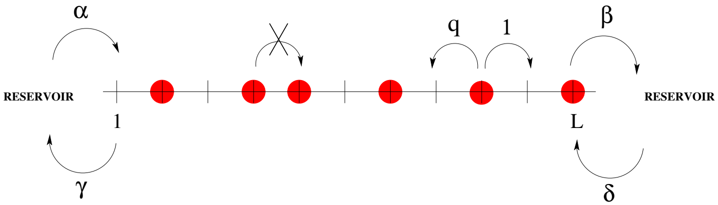
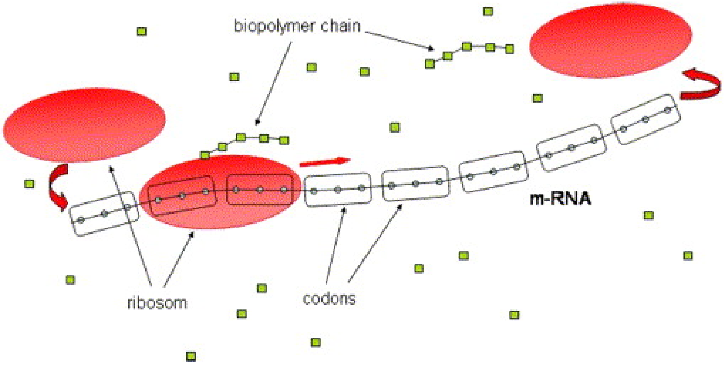


The asymmetric exclusion model with open boundaries



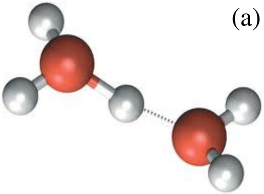
Our aim is to study the statistics of the current and its large deviations starting from this microscopic model.

Elementary Model for Protein Synthesis

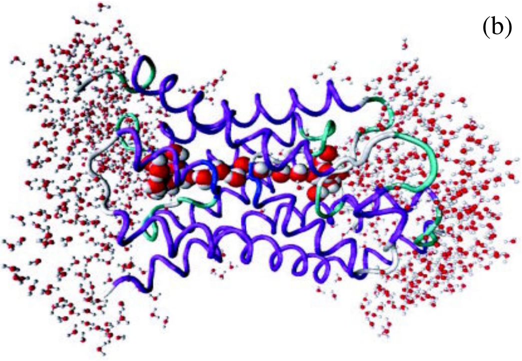


C. T. MacDonald, J. H. Gibbs and A.C. Pipkin, Kinetics of biopolymerization on nucleic acid templates, *Biopolymers* (1968).

The Grotthuss Mechanism for proton transfer



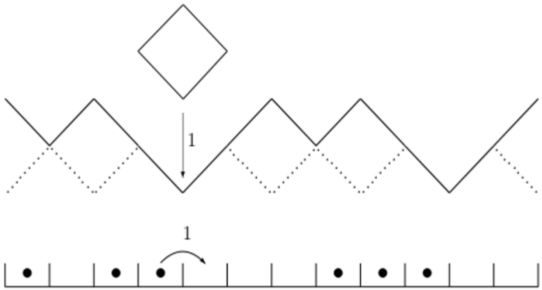
(a)



(b)

A proton hops along an oxygen backbone of a line of water molecules transiently converting each water molecule it visits into $\mathrm{H_3\,O^+}$.

The Kardar-Parisi-Zhang equation in 1d

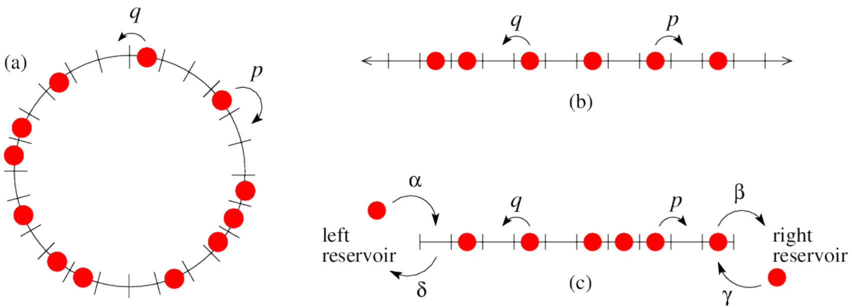


The height of an interface $h(x, t)$ satisfies the generic KPZ equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \xi(x, t)$$

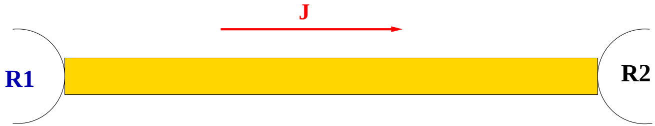
The ASEP is a discrete version of the KPZ equation in one-dimension.

Various Boundary Conditions for the ASEP



The pure ASEP can be studied on a periodic chain (a), on the infinite lattice (b) or on a finite lattice connected to two reservoirs (c).

Statistics of the Integrated Current



Let Y_t be the total charge transported through (a bond of) the system (Integrated or total current) between time 0 and time t.

In the stationary state: a non-vanishing mean-current $\frac{Y_t}{t} \to J$

The fluctuations of Y_t obey a Large Deviation Principle:

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

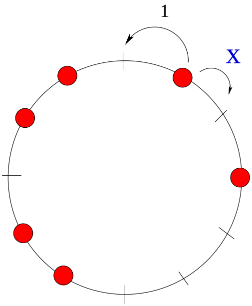
$\Phi(j)$ being the *large deviation function* of the integrated current

Equivalently, use the generating function: $\left\langle \mathrm{e}^{\mu Y_t} \right\rangle \simeq \mathrm{e}^{E(\mu)t} \text{for t} \to \infty,$

They are related by Legendre transform: $\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$

Steady State Properties of the ASEP

Steady-State of the PERIODIC ASEP



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

x asymmetry parameter

In the stationary state all configurations have the same probability.
If Y_t denotes the total number of particles having crossed any bond, then

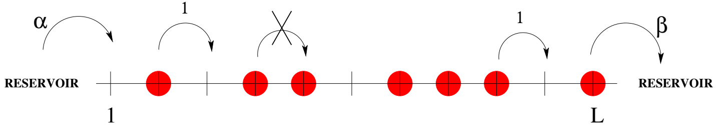
$$\frac{Y_t}{t} \to J = (1-x)\frac{N(L-N)}{L(L-1)}$$

where J is the mean-current in the steady state.

What are the current fluctuations?

Steady-State of the OPEN-BOUNDARY ASEP

Consider first the TASEP on a finite lattice with open boundaries.



In a system of size L, there are 2^L configurations. Each configuration can be represented by a binary string.

In the steady state, the configurations appear with some stationary probability: how are these weights be calculated?

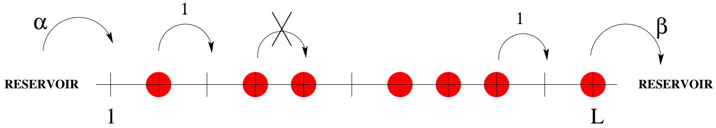
Exact Solution (1993)

The weights of the system satisfy recursion relations: the probability of a configuration of size L can be written as a linear combination of (at most 2) weights of configurations of size $L-1$: there is combinatorial structure between systems of different sizes.

These recursions can be encoded using generating functions (DDM, 1992: $\alpha = \beta = 1$; Schütz and Domany, 1993: General case).

The Matrix Ansatz (DEHP, 1993)

The totally asymmetric exclusion model with open boundaries



The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle \alpha | \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) | \beta \rangle \, .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant is $Z_L = \langle \alpha | (D + E)^L | \beta \rangle = \langle \alpha | C^L | \beta \rangle$

The operators D and E, the vectors $\langle \alpha |$ and $| \beta \rangle$ satisfy

$$D\,E = D + E$$

$$D \, | \beta \rangle = \frac{1}{\beta} | \beta \rangle \quad \text{and} \quad \langle \alpha | \, E = \frac{1}{\alpha} \langle \alpha |$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.
The matrices D and E commute whenever they are finite-dimensional:
$(D-1)(E-1) = 1$.

Infinite dimensional Representation:

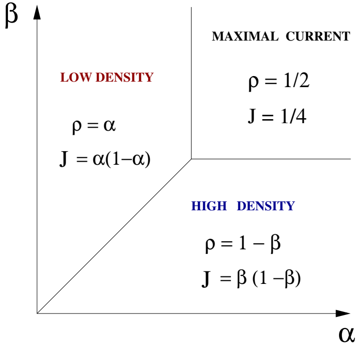
$$D = 1 + d \text{ where } d = \text{right-shift}.$$
$$E = 1 + e \text{ where } e = \text{left-shift}.$$

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad E = D^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ & & & \ddots & \ddots \end{pmatrix}$$

We also have $\langle\alpha| = (1, a, a^2, a^3 \dots)$ and $|\beta\rangle = (1, b, b^2, b^3 \dots)$ with
$a = (1-\alpha)/\alpha$ and $b = (1-\beta)/\beta$.

Phase Diagram of the TASEP

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations). In particular, the following Phase Diagram is found in the infinite size limit (DEHP, 1993; Schütz and Domany, 1993).



Equal-time Steady State Correlations

More generally, the Matrix Ansatz gives access to all equal time correlations in the steady-state.

Density Profile:

$$\rho_i = \langle \tau_i \rangle = \frac{\langle \alpha | C^{i-1}\, D\, C^{L-i} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle}$$

Average Stationary Current:

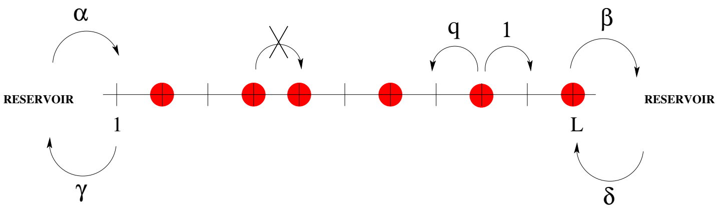
$$J = \langle \tau_i (1 - \tau_{i+1}) \rangle = \frac{\langle \alpha | C^{i-1}\, D\, E\, C^{L-i-1} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle} = \frac{\langle \alpha | C^{L-1} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle} = \frac{Z_{L-1}}{Z_L}$$

Explicit formulae either by using purely combinatorial/algebraic techniques or via a specific representation (e.g., C can be chosen as a discrete Laplacian):

$$\langle \alpha | C^L | \beta \rangle = \sum_{p=1}^{L} \frac{p\,(2L-1-p)!}{L!\,(L-p)!} \frac{\beta^{-p-1} - \alpha^{-p-1}}{\beta^{-1} - \alpha^{-1}}$$

For $\alpha = \beta = 1$, $\quad J = \frac{L+2}{2(2L+1)}$

The General Case (DEHP,1993)



The operators D and E, the vectors $\langle W|$ and $|V\rangle$ now satisfy

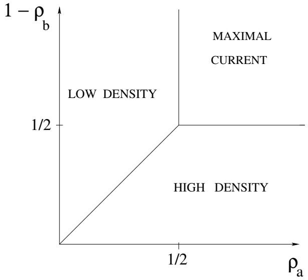
$$
\begin{aligned}
D\,E - qE\,D &= (1-q)(D+E) \\
(\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\
\langle W|(\alpha\,E - \gamma\,D) &= \langle W|
\end{aligned}
$$

Infinite dimensional Representations

The representations are now related to q-deformed oscillators.

$$D = \begin{pmatrix} 1 & \sqrt{1-q} & 0 & 0 & \dots \\ 0 & 1 & \sqrt{1-q^2} & 0 & \dots \\ 0 & 0 & 1 & \sqrt{1-q^3} & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad E = D^\dagger$$

The Phase Diagram of the open ASEP



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.
$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1-q-\alpha+\gamma) \pm \sqrt{(1-q-\alpha+\gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1-q-\beta+\delta) \pm \sqrt{(1-q-\beta+\delta)^2 + 4\beta\delta}}{2\beta}$$

The PASEP average current

The Matrix Ansatz again gives access to all equal time correlations in the steady-state of PASEP but the calculation are much harder.

Average Stationary Current:

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = (1-q)\frac{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{z}}{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{(z+1)^2}}$$

(cf. T. Sasamoto, 1999.)

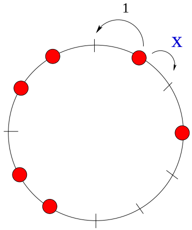
- The function $F(z)$ is the generating function of Askey-Wilson Polynomials:

$$\boxed{F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^kx)$ and $a_\pm$, $b_\pm$ depend on the boundary rates.

- The complex contour Γ encircles 0, $q^ka_+, q^ka_-, q^kb_+, q^kb_-$ for $k \geq 0$.

Current Fluctuations in the periodic ASEP



Total integrated current Y_t**,** *total distance covered by all the* N *particles, hopping on a ring of size* L*, between time 0 and time* t*.*

WHAT IS THE STATISTICS of Y_t**?**

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

into transitions for which $\Delta Y = 0, +1$ or -1.

One can prove that when $t \to \infty$:

$$\boxed{\left\langle \mathrm{e}^{\mu Y_t} \right\rangle \simeq \mathrm{e}^{E(\mu) t}}$$

The cumulant generating function $E(\mu)$ is the eigenvalue with maximal real part of the deformed operator

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

the Markov operator being splitted $M = M_0 + M_+ + M_-$ into positive, negative, null jumps.

The current statistics is reduced to an eigenvalue problem.

Bethe Ansatz for current statistics

The function $E(\mu)$ can be calculated by Bethe Ansatz, because the matrix $M(\mu)$ defines an integrable dynamics (related to XXZ spin-chain).

An Eigenvector ψ of $M(\mu)$ written as a linear combination of plane waves:

$$\psi(x_1, \ldots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The Bethe Equations quantify $z_1, \ldots z_N$

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_i + \mathrm{e}^{\mu}}{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_j + \mathrm{e}^{\mu}}$$

The eigenvalues of $M(\mu)$ are

$$E(\mu; z_1, z_2 \ldots z_N) = \mathrm{e}^{\mu} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\mu} \sum_{i=1}^{N} z_i - N(1+x) \,.$$

Totally Asymmetric Case (Derrida Lebowitz 1998)

The Bethe equations decouple for the special case $x = 0$.
The structure of the solution is given by a parametric representation of the cumulant generating function $E(\mu)$:

$$
\begin{aligned}
\mu &= -\frac{1}{L}\sum_{k=1}^{\infty} \frac{[kL]!}{[kN]!\,[k(L-N)]!}\,\frac{B^k}{k}\,, \\
E &= -\sum_{k=1}^{\infty} \frac{[kL-2]!}{[kN-1]!\,[k(L-N)-1]!}\,\frac{B^k}{k}\,.
\end{aligned}
$$

Mean Total current:

$$
J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \frac{N(L-N)}{L-1}
$$

Diffusion Constant:

$$
D = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)}\,\frac{C_{2L}^{2N}}{\left(C_L^N\right)^2}
$$

Exact expressions for the large deviation function.

General Case: Functional Bethe Ansatz

In the general case $x \neq 0$, there is NO DECOUPLING of the Bethe equations.

However, the problem can be solved by Functional Bethe Ansatz:

Find two polynomials $Q(T)$ and $R(T)$ such that

$$\boxed{Q(T)R(T) = e^{L\mu}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)}$$

where $Q(T)$ of degree N vanishes at the Bethe roots. (Baxter Equation)

This is a purely algebraic problem, that can be solved perturbatively w.r.t. μ. This provides us with an expansion of $E(\mu)$.

The General Case (K. M. and S. Prolhac, 2010)

For arbitrary asymmetry q on a ring, The function $E(\mu)$ is found by functional Bethe Ansatz, again in a parametric form:

$$\mu = -\sum_{k\geq 1} C_k \frac{B^k}{k} \quad \text{and} \quad E = -(1-x)\sum_{k\geq 1} D_k \frac{B^k}{k}$$

C_k and D_k are combinatorial factors enumerating some tree structures. There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$$

such that C_k and D_k are given by complex integrals along a small contour that encircles 0 :

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

The function $W_B(z)$ contains all information about the current statistics.

The function $W_B(z)$ is the solution of a functional Bethe equation:

$$\boxed{W_B(z) = -\ln\Big(1 - BF(z)e^{X[W_B](z)}\Big)}$$

where

$$\boxed{F(z) = \frac{(1+z)^L}{z^N}}$$

The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K(z_1, z_2)$$

with the kernel

$$\boxed{K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}}$$

Solving this Functional Bethe Ansatz equation to all orders enables us to calculate cumulant generating function. For $x = 0$, the TASEP result is readily retrieved.

The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.

From the Physics point of view, the solution allows one to

- Classify the different universality classes (KPZ, EW).
- Study the various scaling regimes.
- Investigate the hydrodynamic behaviour.

Cumulants of the Current

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

- Diffusion Constant: $D = (1-x)\frac{2L}{L-1}\sum_{k>0} k^2 \frac{C_L^{N+k}}{C_L^N}\frac{C_L^{N-k}}{C_L^N}\left(\frac{1+x^k}{1-x^k}\right)$

- Third cumulant (Skewness): $\rightarrow$ Non Gaussian fluctuations.

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

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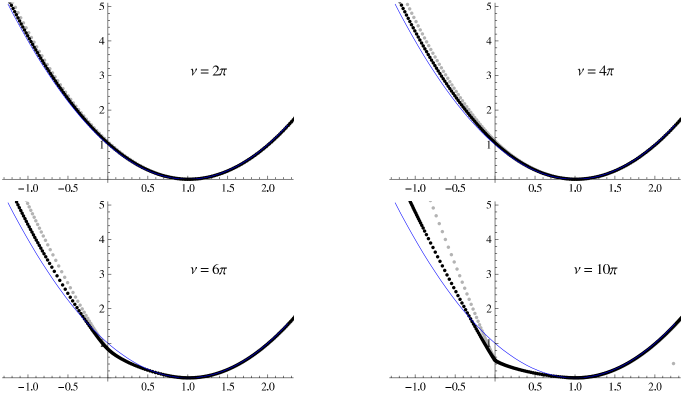
$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$\begin{aligned}
\frac{E_3}{6L^2} \quad = \quad & \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
- \quad & \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
- \quad & \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
- \quad & \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 + \frac{(1-x)N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
+ \quad & \frac{(1-x)N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} - \frac{(1-x)N(L-N)}{6(L-1)(3L-1)}\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}$$

Full large deviation function (weak asymmetry)



$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2+\mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2+\mu\nu)]$$

with $\quad \psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$

Current Fluctuations

in the open ASEP

Now, the observable Y_t counts the total number of particles exchanged between the system and the left reservoir between times 0 and t.

Hence, $Y_{t+dt} = Y_t + y$ with

- $y = +1$ if a particle enters at site 1 (at rate α),
- $y = -1$ if a particle exits from 1 (at rate γ)
- $y = 0$ if no particle exchange with the left reservoir has occurred during dt.

These three mutually exclusive types of transitions lead to a three parts decomposition of the Markov Matrix: $M = M_+ + M_- + M_0$.

The cumulant-generating function $E(\mu)$ when $t \to \infty$, $\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu)t}$, is the dominant eigenvalue of the deformed matrix

$$\boxed{M(\mu) = M_0 + e^{\mu} M_+ + e^{-\mu} M_-}$$

$E(\mu)$ can be calculated by using a Generalized Matrix Product Ansatz.

Generalized Matrix Ansatz

We have proved that the dominant eigenvector of the deformed matrix $M(\mu)$ is given by the following matrix product representation:

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} \left(\tau_i D_k + (1-\tau_i) E_k\right) | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are the same as above

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

The boundary vectors $\langle W_k|$ and $|V_k\rangle$ are constructed recursively:

$$|V_k\rangle = |\beta\rangle |\tilde{V}\rangle |V_{k-1}\rangle \quad \text{and} \quad \langle W_k| = \langle W^\mu | \langle \tilde{W}^\mu | \langle W_{k-1}|$$

$$[\beta(1-d) - \delta(1-e)]\, |\tilde{V}\rangle = 0$$

$$\langle W^\mu | [\alpha(1 + \mathrm{e}^{\mu}\, e) - \gamma(1 + \mathrm{e}^{-\mu}\, d)] = (1-q) \langle W^\mu |$$

$$\langle \tilde{W}^\mu | [\alpha(1 - \mathrm{e}^{\mu}\, e) - \gamma(1 - \mathrm{e}^{-\mu}\, d)] = 0$$

Structure of the solution I

For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\begin{aligned} \mu &= -\sum_{k=1}^{\infty} C_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k} \\ E &= -\sum_{k=1}^{\infty} D_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k} \end{aligned}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\, i\, \pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\, i\, \pi} \frac{\phi_k(z)}{(z+1)^2}$$

There exists an auxiliary function

$$W_B(z) = \sum_{k \geq 1} \phi_k(z) \frac{B^k}{k}$$

that contains the full information about the statistics of the current.

Structure of the solution II

This auxiliary function $W_B(z)$ solves a functional Bethe equation:

$$\boxed{W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)}$$

- The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K\left(\frac{z_1}{z_2}\right)$$

$$\boxed{\text{with kernel} \quad K(z) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \left\{z^k + z^{-k}\right\}}$$

- The function $F(z)$ is given by

$$\boxed{F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$ and $a_\pm$, $b_\pm$ depend on the boundary rates.

- The complex contour $\mathcal{C}$ encircles 0, $q^k a_+, q^k a_-, q^k b_+, q^k b_-$ for $k \geq 0$.

Discussion

- These results are of *combinatorial nature: valid for arbitrary values of the parameters and for any system sizes with no restrictions.*
- *Average-Current:*

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = (1-q)\frac{D_1}{C_1} = (1-q)\frac{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{z}}{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{(z+1)^2}}$$

(cf. T. Sasamoto, 1999.)

- *Diffusion Constant:*

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = (1-q)\frac{D_1 C_2 - D_2 C_1}{2 C_1^3}$$

where C_2 and D_2 are obtained using

$$\phi_1(z) = \frac{F(z)}{2} \quad \text{and} \quad \phi_2(z) = \frac{F(z)}{2}\left(F(z) + \oint_\Gamma \frac{dz_2 F(z_2) K(z/z_2)}{2\imath\pi z_2}\right)$$

(TASEP case solved in B. Derrida, M. R. Evans, K. M., 1995)

Asymptotic behaviour in the Phase Diagram

- Maximal Current Phase:

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k$$

$$\mathcal{E} - \frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k$$

- Low Density (and High Density) Phases:
  Dominant singularity at a_+: $\phi_k(z) \sim F^k(z)$. By Lagrange Inversion:

$$E(\mu) = (1-q)(1-\rho_a)\frac{\mathrm{e}^{\mu} - 1}{\mathrm{e}^{\mu} + (1-\rho_a)/\rho_a}$$

(de Gier and Essler, 2011).

Current Large Deviation Function:

$$\boxed{\Phi(j) = (1-q)\left\{\rho_a - r + r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}}$$

where the current j is parametrized as $j = (1-q)r(1-r)$.

Matches the predictions of Macroscopic Fluctuation Theory in the Weak Asymmetry Limit, as obtained by T. Bodineau and B. Derrida.

The TASEP case

Here $q = \gamma = \delta = 0$ and (α, β) are arbitrary.
The parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha,\beta) \frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(\alpha,\beta) \frac{B^k}{2k}$$

with

$$C_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}, \qquad a = \frac{1-\alpha}{\alpha}, \qquad b = \frac{1-\beta}{\beta}$$

A special TASEP case

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

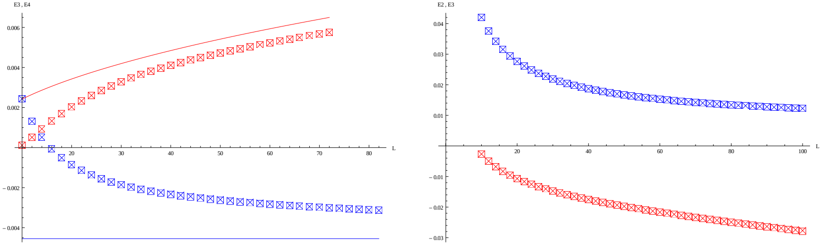
$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} \,,$$

$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} \,.$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368}\pi \sim -0.0090978...$

Numerical results (DMRG)



Left: **Max. Current** ($q = 0.5$, $a_+ = b_+ = 0.65$, $a_- = b_- = 0.6$), Third and Fourth cumulant.

Right: **High Density** ($q = 0.5$, $a_+ = 0.28$, $b_+ = 1.15$, $a_- = -0.48$ and $b_- = -0.27$), Second and Third cumulant.

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Large Deviations at the Hydrodynamic Level

What is the probability to observe an atypical current $j(x,t)$ and the corresponding density profile $\rho(x,t)$ during $0 \leq s \leq L^2\, T$?

$$\mathrm{Pr}\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

where the Large-Deviation functional is given by macroscopic fluctuation theory (Jona-Lasinio et al.)

$$\mathcal{I}(j,\rho) = \int_0^T dt \int_0^1 dx \frac{\left(j - \nu\sigma(\rho) + \frac{1}{2}\nabla\rho\right)^2}{\sigma(\rho)}$$

with the constraint: $\partial_t \rho = -\nabla . j$

This leads to a variational procedure to calculate deviations of the density and of the associated current: an optimal path problem. From $\mathcal{I}(j,\rho)$ one can deduce the LDF of the current or the profile. For example $\Phi(j) = \min_\rho\{\mathcal{I}(j,\rho)\}$

MFT Equations

Mathematically, one has to solve the corresponding Euler-Lagrange equations. After some transformations, one obtains a set of coupled PDE's (here, we take $\nu = 0$):

$$\begin{aligned} \partial_t q &= \partial_x[D(q)\partial_x q] - \partial_x[\sigma(q)\partial_x p] \\ \partial_t p &= -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2 \end{aligned}$$

where $q(x,t)$ is the density-field and $p(x,t)$ is a conjugate field.
The physical content is encoded in the 'transport coefficients' $D(q)$ and $\sigma(q)$ that contain the information of the microscopic dynamics relevant at the macroscopic scale.
Note that these equations have a Hamiltonian structure.

- A general framework but these non-linear MFT equations are very difficult to solve in general.
- For a finite external field (that does not vanish with the system size), the M. F. T. framework has to be extended (Jensen-Varadhan Large Deviation Theory).

Conclusion

Large deviation functions (LDF) play a crucial role in non-equilibrium physics: they are studied through experimental, mathematical or computational techniques. The formulae presented here are one of very few exact analytically exact formulae known for Large Deviation Functions, valid for systems with arbitrary finite size.

- What is the crossover between finite-size system statistics and the KPZ statistics in the infinite system?
- Could we derive current-fluctuations directly from the MFT without having to use combinatorics/Bethe Ansatz?
- The tensor Matrix Ansatz provides us with a formal representation for the optimal profile that solves the MFT equation for ASEP. Could these equations be integrable in the ASEP case?
- Tagged particle Large Deviation Function? (see Tridib's talk).

These results were obtained in collaboration with A. Lazarescu and S. Prolhac.

Florence et Stendhal

"J'étais dans une sorte d'extase par l'idée d'être à Florence. Absorbé dans la contemplation de la beauté sublime, je la voyais de près, je la touchais pour ainsi dire..."