A brief review of quantum annealing

Hidetoshi Nishimori

Tokyo Institute of Technology

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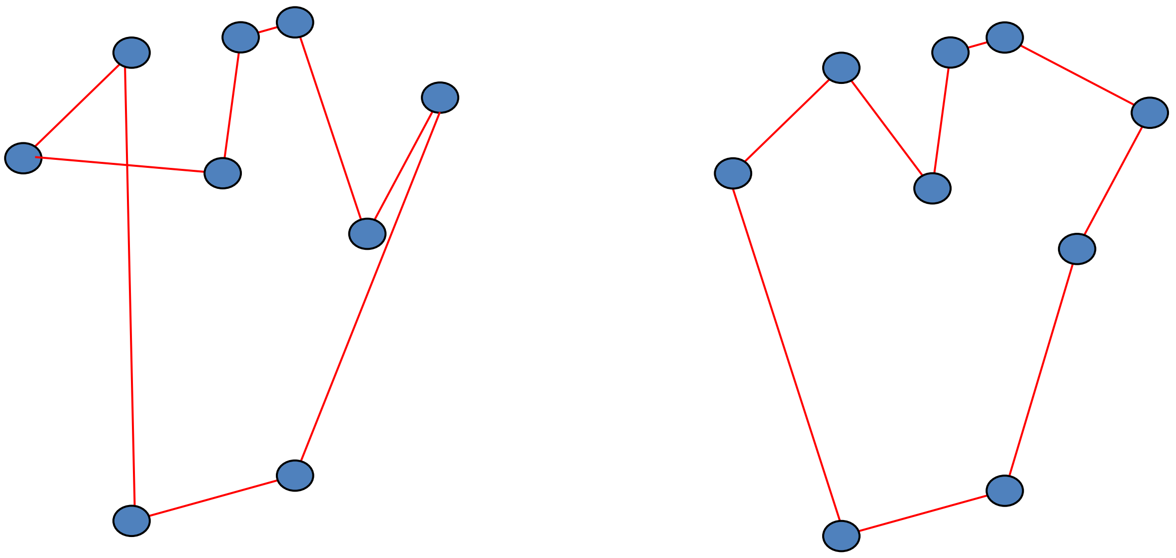
Pursuing Excellence

Combinatorial optimization

Traveling salesman problem



Minimize the cost function (tour length)

Combinatorial optimization
Formulation

Minimize the cost function (a function of discrete variables)

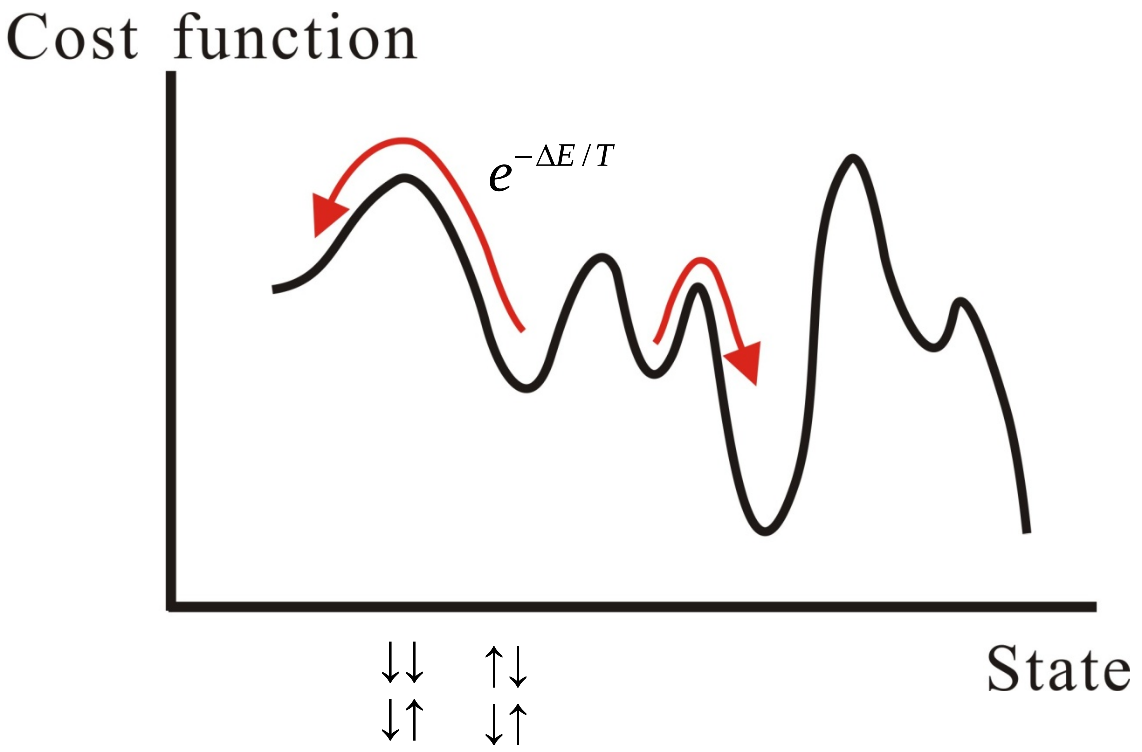
Ising model

Cost function,, (classical) Hamiltonian

$$H_0 = -\sum J_{ij}\sigma_i\sigma_j \quad (\sigma_i = \pm 1)$$

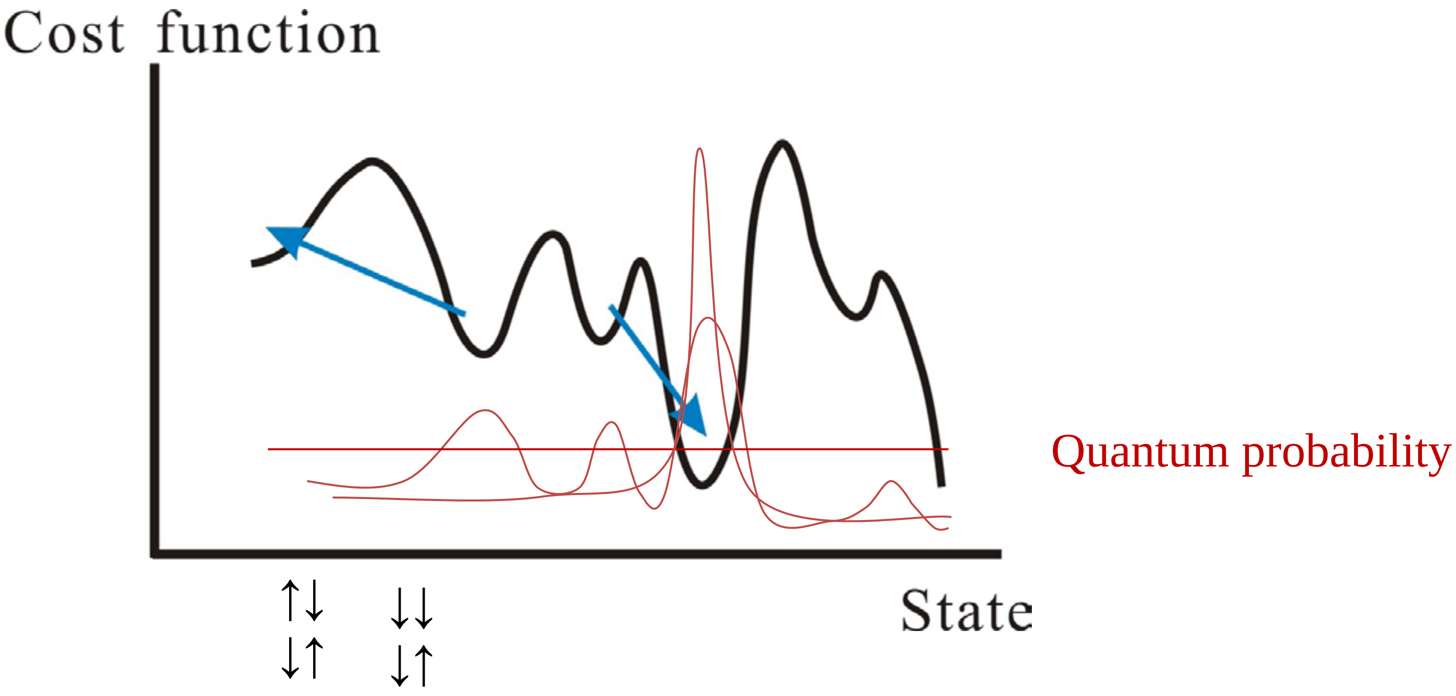
Simulated Annealing (SA)

Search by thermal fluctuations



Quantum Annealing (QA)

- Search by quantum fluctuations



Questions

- *Does quantum annealing work?*

  Yes.

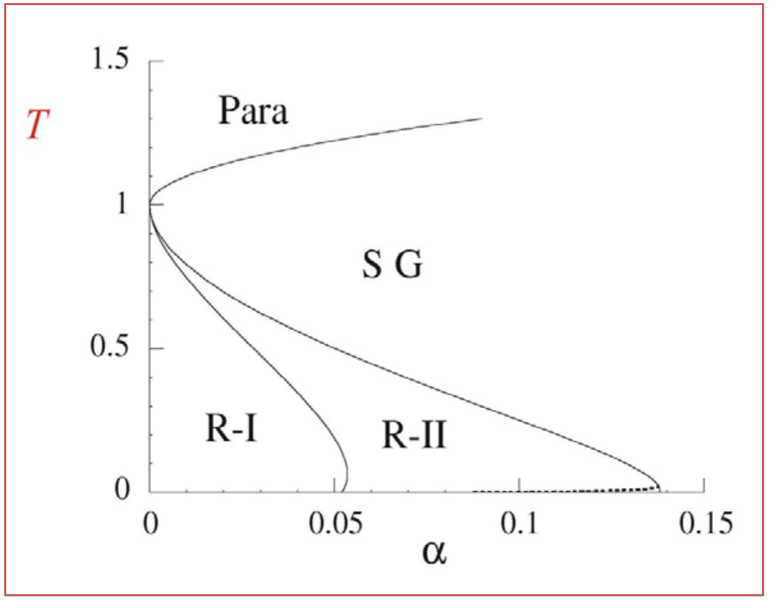
- *Is it better than simulated annealing?*

  Yes, in some sense ....

T vs Γ: Hopfield model

$$H = -\sum J_{ij}\sigma_i\sigma_j \quad (\text{Finite } T)$$

$$J_{ij} = \sum_{\mu=1}^{p} \xi_i^{\mu}\xi_j^{\mu}$$

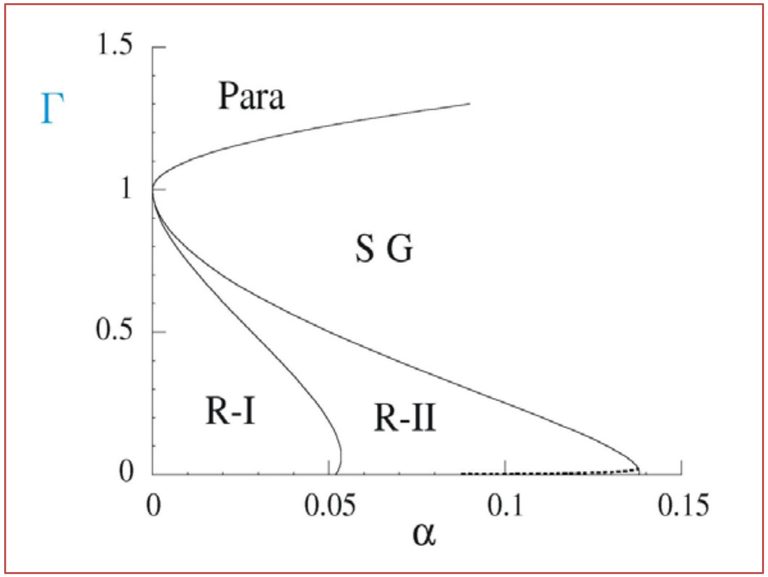


$$\alpha = \frac{p}{N}$$

Amit, Gutfreunt, Sompolinsky (1985)

T vs *Γ*: Hopfield model

$$H = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma\sum \sigma_i^x \quad (T=0)$$

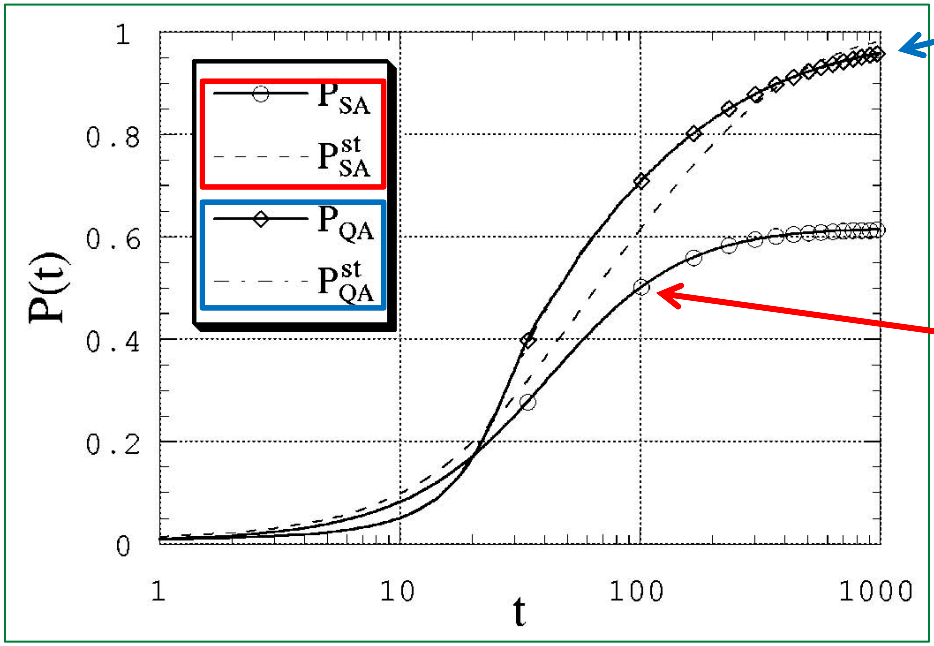


Nishimori & Nonomura (1996)

Numerical evidence

Master eqn vs. Schrödinger eqn

Random J_{ij}, 8 spins



$$\Gamma(t) = \frac{3}{\sqrt{t}}$$

Schrödinger eqn.

$$T(t) = \frac{3}{\sqrt{t}}$$

Master eqn.

Kadowaki & Nishimori (1998)

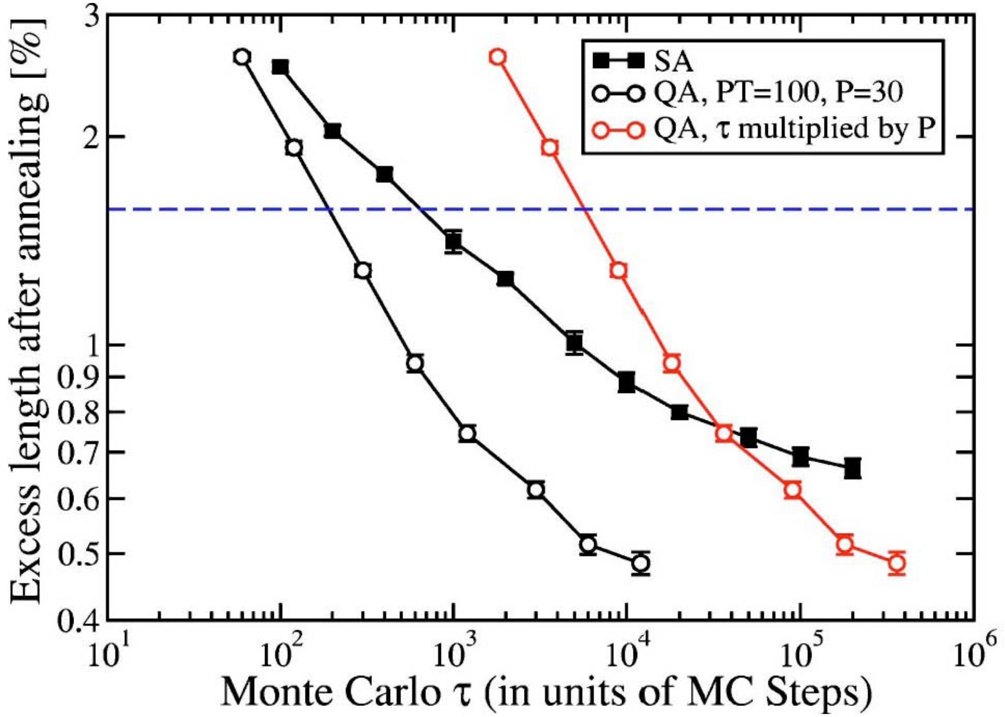
Monte Carlo for TSP (1002 cities)

$$H(t) = \frac{t}{\tau} H_0 + \left(1 - \frac{t}{\tau}\right) H_{\text{quantum}}$$

$$H(0) = H_{\text{quantum}} \Rightarrow H(\tau) = H_0$$

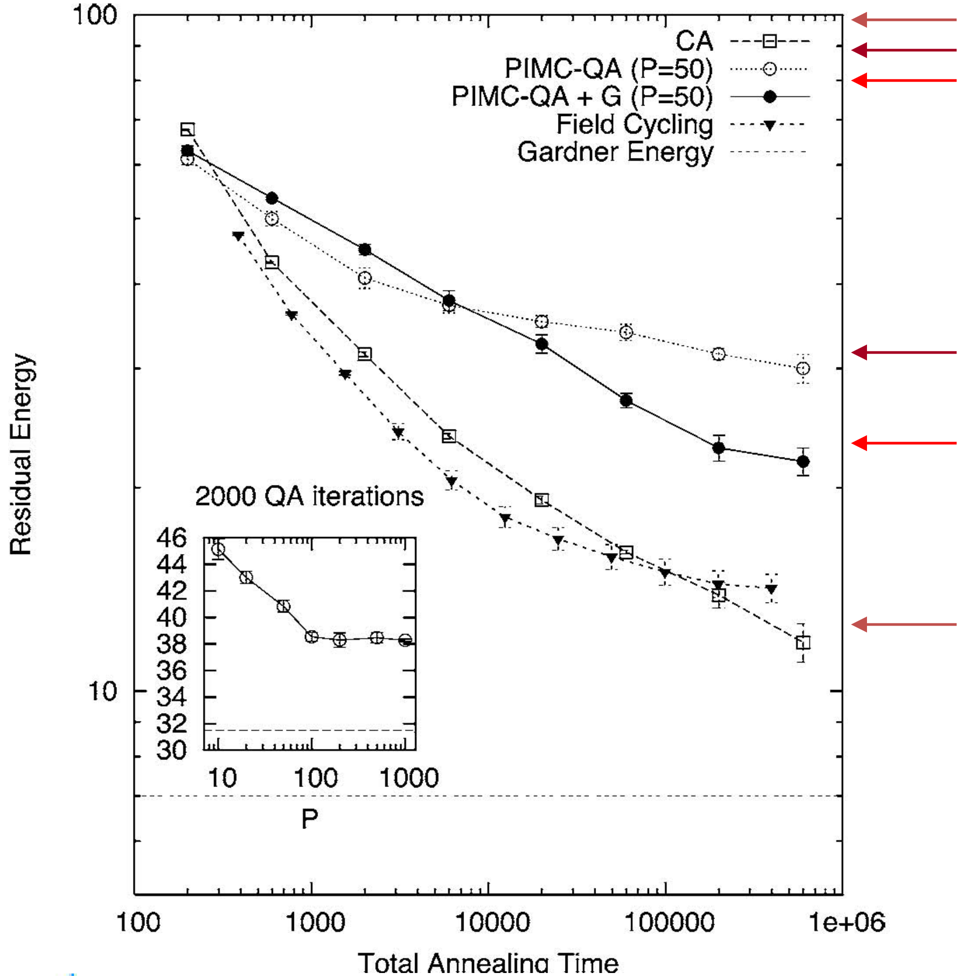
Residual energy

$H(\tau) - E_{\text{true}}$



Martonak, Santoro &Tosatti (2004)

Monte Carlo for 3SAT



QA

QA

SA

Battaglia, Santoro, Tosatti (2005)

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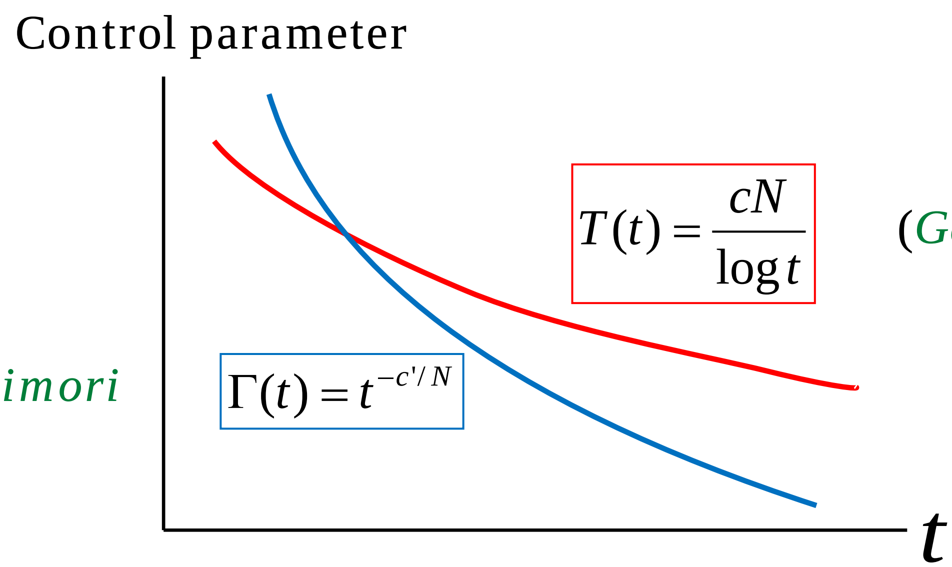
Theoretical background

Convergence theorem



(*Geman-Geman* for SA)

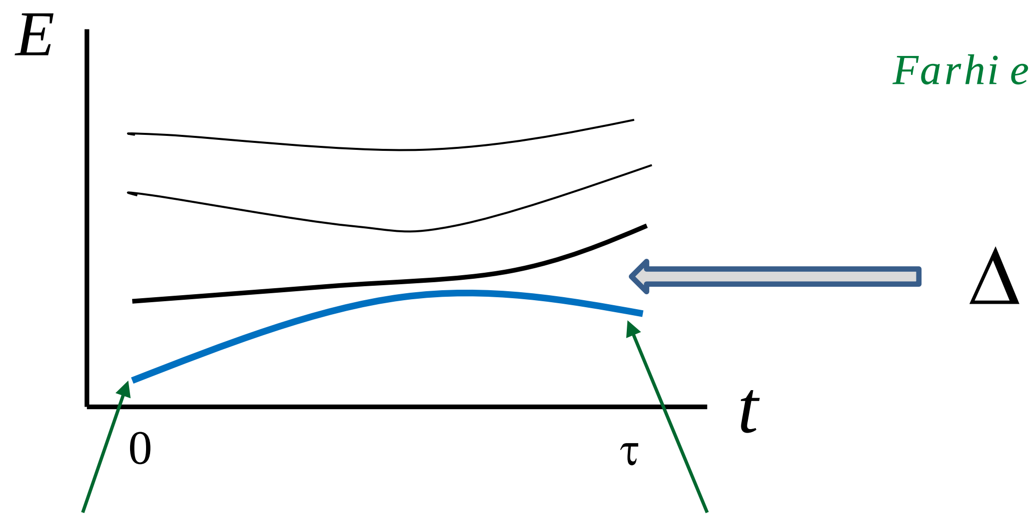
Morita & Nishimori

$$H = H_0 + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma(t)\sum \sigma_i^x$$

Adiabatic computation

Quantum adiabatic computation

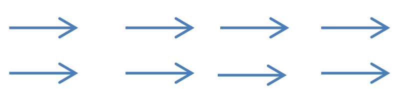
Farhi et al (2001)

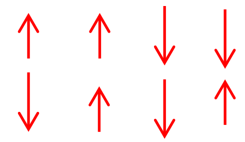


Trivial initial state

Non-trivial final state

$$H(t) = -\left(1-\frac{t}{\tau}\right)\sum \sigma_i^x - \frac{t}{\tau}\sum J_{ij}\sigma_i^z\sigma_j^z$$

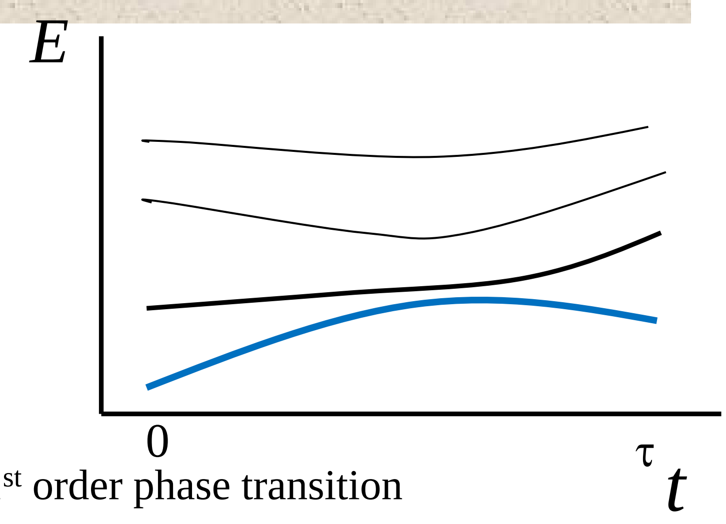




Computational complexity

Finite-size analysis

Adiabatic theorem $\tau \propto \Delta^{-2}$



1st order phase transition

2nd order phase transition

Gap scaling $\Delta \propto \begin{cases} e^{-aN} \\ N^{-b} \end{cases}$

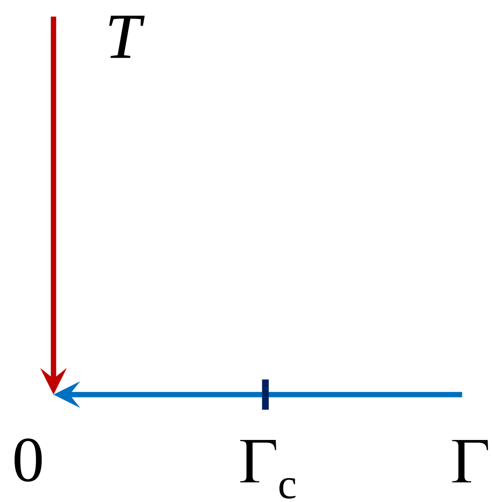
Complexity $\tau \propto \begin{cases} e^{2aN} & \text{(hard)} \\ N^{2b} & \text{(easy)} \end{cases}$

Summary so far

✓QA works and is better than SA.

✓1st order quantum transitions is problematic.

✓Question: What happens when there exists no classical phase transition but there is a quantum transition?



Classical dynamics and quantum Hamiltonian

Hidetoshi Nishimori

Tokyo Institute of Technology

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Classical dynamics to quantum Hamiltonian

(Classical) Ising model $H_0(\sigma)$, $(\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_N\})$

Master equation (fixed T, single-spin flip)

$$\frac{dP_\sigma(t)}{dt} = \sum_{\sigma'} W_{\sigma\sigma'} P_{\sigma'}(t)$$

Transverse-field Ising model

$$H_{\sigma\sigma'} := -e^{\frac{1}{2}\beta H_0(\sigma)} W_{\sigma\sigma'} e^{-\frac{1}{2}\beta H_0(\sigma')}$$

- Eigenvalue spectrum
  $W, -H : \lambda_0 = 0 > -\lambda_1 > -\lambda_2 > \cdots$
- $W : \lambda_1 = \tau^{-1}$ (inverse relaxation time; $P(t) \sim P_{\rm eq} + a\, e^{-\lambda_1 t}$)
  H: $\lambda_1 = \Delta$

cf. Castelnovo, Chamon, Mudry, and Pujol, Ann. Phys. (2005)

Example

- 1d ferromagnetic Ising model

$$H_0(\sigma) = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1}$$

- W of heat-bath dynamics (at fixed T) is equivalent to:

$$\hat{H} = -\frac{1}{2} \tanh 2K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z$$

$$- \frac{1}{2\cosh 2K} \sum_{j=1}^{N} \left(\cosh^2 K - \sinh^2 K \, \sigma_{j-1}^z \sigma_{j+1}^z \right) \sigma_j^x$$

Quantize it!

- Quantum Hamiltonian: Real symmetric (Hermitian)

$$H_{\sigma\sigma'} = \left(\hat{H}\right)_{\sigma\sigma'} = -e^{\frac{1}{2}\beta H_0(\sigma)} W_{\sigma\sigma'} e^{-\frac{1}{2}\beta H_0(\sigma')}$$
$$\Rightarrow H_{\sigma\sigma'} = H_{\sigma'\sigma} \quad (\leftarrow \text{detailed balance})$$

- Eigenvector and eigenvalue

$$\hat{W}\hat{\psi}^{(R,n)} = -\lambda_n \hat{\psi}^{(R,n)}, \quad \hat{H} = -e^{\frac{1}{2}\beta\hat{H}_0}\hat{W}e^{-\frac{1}{2}\beta\hat{H}_0}$$

$$\hat{\phi}^{(n)} := e^{\frac{1}{2}\beta\hat{H}_0}\hat{\psi}^{(R,n)}$$

$$\Longrightarrow \hat{H}\hat{\phi}^{(n)} = \lambda_n \hat{\phi}^{(n)}$$

Matrix elements of $\hat{H}$

Off-diagonal

$$H_{\sigma\sigma'} = -e^{\frac{1}{2}\beta H_0(\sigma)} W_{\sigma\sigma'} e^{-\frac{1}{2}\beta H_0(\sigma')} = -w_{\sigma\sigma'} \ (< 0)$$

$$\left(W_{\sigma\sigma'} = w_{\sigma\sigma'} e^{-\frac{1}{2}\beta(H_0(\sigma) - H_0(\sigma'))}\right)$$

Diagonal

$$H_{\sigma\sigma} = -W_{\sigma\sigma} = \sum_{\sigma' \in \mathcal{N}(\sigma)} w_{\sigma\sigma'} e^{-\frac{1}{2}\beta(H_0(\sigma') - H_0(\sigma))}$$

Combined: operator representation

$$\hat{H} = \sum_{\sigma} \sum_{\sigma' \in \mathcal{N}(\sigma)} \left(w_{\sigma\sigma'} e^{-\frac{1}{2}\beta(H_0(\sigma') - H_0(\sigma))} |\sigma\rangle\langle\sigma| - w_{\sigma\sigma'} |\sigma'\rangle\langle\sigma| \right)$$

Locality + single-spin flip

- Assume $H_0(\sigma)$ is local.

$$H_0(\sigma) = \sum_j H_j, \quad \Big(H_j = -h_j\sigma_j - \sigma_j \sum_{k\in\mathcal{N}(j)} J_{jk}\sigma_k - \cdots\Big)$$

- Assume $\sigma \to \sigma'$: $\sigma_j \to -\sigma_j$ (single-spin flip)

$$H_0(\sigma) - H_0(\sigma') = H_j - (-H_j) = 2H_j \quad \text{(local)}$$

- Operator representation

$$\hat{H} = \sum_\sigma \sum_{\sigma'\in\mathcal{N}(\sigma)} \Big(w_{\sigma\sigma'} e^{\frac{1}{2}\beta(H_0(\sigma)-H_0(\sigma'))}|\sigma\rangle\langle\sigma| - w_{\sigma\sigma'}|\sigma'\rangle\langle\sigma|\Big)$$

$$= \sum_j w(\sigma_j^z \to -\sigma_j^z)\big(e^{\beta H_j}\mathbb{I} - \sigma_j^x\big)$$

Local Hamiltonian!

Example: 1d ferromagnetic Ising model

- Heat-bath dynamics

$$\hat{H} = (\text{const}) - \frac{1}{2}\tanh 2K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z$$

$$- \frac{1}{2\cosh 2K} \sum_{j=1}^{N} \left(\cosh^2 K - \sinh^2 K\, \sigma_{j-1}^z \sigma_{j+1}^z \right) \sigma_j^x$$

Adaptive change of local transverse fields

- Transverse-field term

$$-\frac{1}{2\cosh 2K}\sum_{j=1}^{N}\left(\cosh^2 K - \sinh^2 K\,\sigma_{j-1}^z\sigma_{j+1}^z\right)\sigma_j^x$$

- $\sigma_{j-1}^z\sigma_{j+1}^z = 1: \quad \cosh^2 K - \sinh^2 K$ Weak field
- $\sigma_{j-1}^z\sigma_{j+1}^z = -1: \cosh^2 K + \sinh^2 K$ Strong field
- Weak/strong field for desirable/undesirable configuration
- Adaptive transverse field→ no phase transition

  (no transition in dynamics in 1d)
- cf. Uniform transverse field → quantum phase transition

Simulated annealing with β as a function of t

Master equation with $\hat{W}(t)$

$$\frac{d\hat{P}(t)}{dt} = \hat{W}(t)\hat{P}(t)$$

$$\hat{\phi}(t) := e^{\frac{1}{2}\beta(t)\hat{H}_0}\hat{P}(t), \quad \hat{H}(t) = -e^{\frac{1}{2}\beta(t)\hat{H}_0}\hat{W}(t)e^{-\frac{1}{2}\beta(t)\hat{H}_0}$$

Rewrite the master equation in terms of $\hat{\phi}(t)$ and $\hat{H}(t)$

$$\frac{d\hat{\phi}(t)}{dt} = -\hat{H}(t)\hat{\phi}(t) + \frac{1}{2}\dot{\beta}(t)\hat{H}_0\hat{\phi}(t)$$

$t \longrightarrow it$: Schrödinger equation

$$i\frac{d\hat{\phi}(t)}{dt} = \left(\hat{H}(t) - \frac{1}{2}\dot{\beta}(t)\hat{H}_0\right)\hat{\phi}(t)$$

Quantum to classical: Construction of transition matrix

- Given $\hat{H}$: $(\hat{H})_{\sigma\sigma'} \le 0 \ (\sigma \ne \sigma')$

$$\hat{H} = -\sum_{ij} J_{ij}\sigma_i^z\sigma_j^z - \Gamma_1 \sum_i \sigma_i^x + \Gamma_2\Big(\sum_i \sigma_i^x\Big)^2 \quad \text{excluded}$$

- Shift the energy: $\hat{H}\hat{\phi}^{(0)} = 0$ (ground state)
- Perron-Frobenius: $\phi_\sigma^{(0)} > 0 \quad (\forall\sigma)$
- Define the Ising model: $H_0(\sigma) := -2\ln\phi_\sigma^{(0)}$

  cf. Classical to quantum: $\hat{\phi}^{(0)} = e^{-\frac{1}{2}\beta\hat{H}_0}/\sqrt{Z}$

Quantum to classical (2)

- Define the Ising model: $H_0(\sigma) := -2 \ln \phi_\sigma^{(0)}$

  cf. Classical to quantum: $\hat{\phi}^{(0)} = e^{-\frac{1}{2}\beta \hat{H}_0}/\sqrt{Z}$

- Non-local

$$H_0(\sigma) = \sum_j h_j \sigma_j + \sum_{ij} J_{ij} \sigma_i \sigma_j + \cdots + J \sigma_1 \sigma_2 \cdots \sigma_N$$

- Transition matrix: $\hat{W} := -e^{-\frac{1}{2}\hat{H}_0} \hat{H} e^{\frac{1}{2}\hat{H}_0}$

Summary

- Equivalence: Eigenvalue spectrum (fixed T), Time-dependent $T(t)$

$$\hat{W}, -\hat{H} : \lambda_0 = 0 > -\lambda_1 > -\lambda_2 > \cdots$$

$$(\text{Relaxation time})^{-1} = \text{Energy gap}$$

- Inquivalence: Interaction range, $H_{\sigma\sigma'} \leq 0$

Original system	short
classical $\rightarrow$ quantum	short
quantum $\rightarrow$ classical	long

- Thanks to Junichi Tsuda and Sergey Knysh