STATIONARY NON EQULIBRIUM STATES FROM A MICROSCOPIC AND A MACROSCOPIC POINT OF VIEW

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References

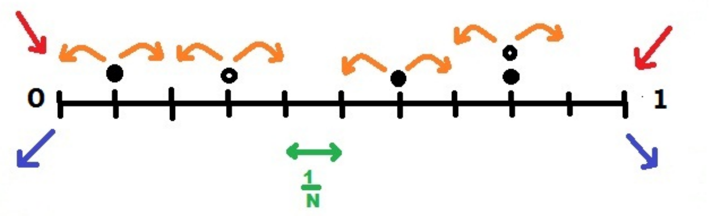
- L. Bertini; A. De Sole; D. G. ; G. Jona-Lasinio; C. Landim Stochastic interacting particle systems out of equilibrium J. Stat. Mech. (2007)
- D. G. From combinatorics to large deviations for the invariant measures of some multiclass particle systems Markov Processes Relat. (2008)
- L. Bertini; D. G.; G. Jona-Lasinio; C. Landim Thermodynamic transformations of nonequilibrium states J. Stat. Phys. (2012)

SNS: Microscopic description

Lattice: Λ_N

Configuration of particles: $\eta \in \{0,1\}^{\Lambda_N}$ or $\eta \in \mathbb{N}^{\Lambda_N}$

$\eta_t(x) =$ number of particles at $x \in \Lambda_N$ at time t



SNS: Microscopic description

- Stochastic Markovian dynamics
- $r(\eta, \eta')$ = rate of jump from configuration η to configuration η'
- η' = local perturbation of η
- $\mu_N(\eta)$ = invariant measure of the process, probability measure on the state space

$$\boxed{\mu_N(\eta) \sum_{\eta'} r(\eta, \eta') = \sum_{\eta'} \mu_N(\eta') r(\eta', \eta)}$$

$\mu_N \Longrightarrow$ MICROSCOPIC description of the SNS

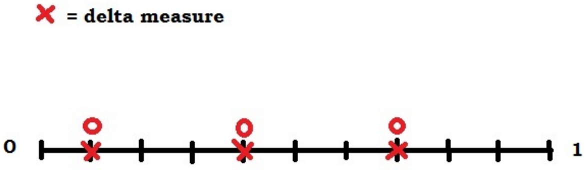
SNS: Macroscopic description

- $\eta \Longrightarrow \pi_N(\eta)$ Empirical measure (positive measure on $[0,1]$)

$$\boxed{\pi_N(\eta) = \frac{1}{N}\sum_{x\in\Lambda_N} \eta(x)\delta_x}$$

$\delta_x =$ delta measure (Dirac) at $x \in [0,1]$; since $x \in \Lambda_N$ we have $x = \frac{i}{N}$, $i \in \mathbb{N}$. Given $f : [0,1] \to \mathbb{R}$

$$\int_{[0,1]} f d\pi_N = \frac{1}{N}\sum_{x\in\Lambda_N} \eta(x) f(x)$$



SNS: Macroscopic description

When η is distributed according to μ_N and N is large

LAW OF LARGE NUMBERS

$$\boxed{\pi_N \to \bar{\rho}(x)dx}$$

This means

$$\int_{[0,1]} f d\pi_N \to \int_{[0,1]} f(x)\bar{\rho}(x)\, dx$$

$\bar{\rho}(x) =$ typical density profile of the SNS

SNS: Macroscopic description

When η is distributed according to μ_N and N is large, a refinement of the law of large numbers

LARGE DEVIATIONS

$$\boxed{\mathbb{P}\Big(\pi_N(\eta) \sim \rho(x)dx\Big) \simeq e^{-NV(\rho)}}$$

$V =$ Large deviations rate function

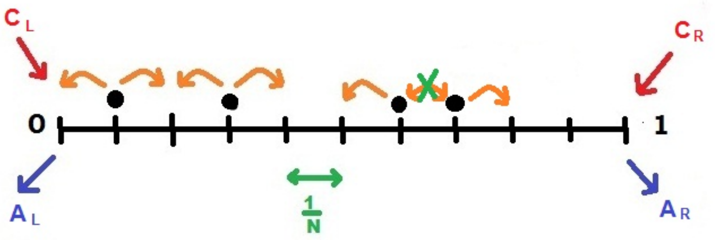
$V \Longrightarrow$ MACROSCOPIC DESCRIPTION OF THE SNS

V contains less information than μ_N but is easier to compute and is independent from microscopic details of the dynamics

Example: Equilibrium SEP

- Equilibrium: $C_L = C_R = C$; $A_L = A_R = A$
- Microscopic state: product of Bernoulli measures of parameter $p = \frac{C}{A+C}$

$$\mu_N(\eta) = \prod_{x \in \Lambda_N} p^{\eta(x)} \left(1-p\right)^{1-\eta(x)}$$



Example: Equilibrium SEP

MACROSCOPIC DESCRIPTION

$$\mathbb{P}\Big(\pi_N(\eta) \sim \rho(x)dx\Big) = \sum_{\{\eta,:\, \pi_N(\eta)\sim\rho(x)dx\}} \mu_N(\eta)$$

$$= \sum_{\{\eta,:\, \pi_N(\eta)\sim\rho(x)dx\}} e^{-N\left(\int_{[0,1]} d\pi_N(\eta)\log\frac{1-p}{p} - \log(1-p)\right)}$$

Using the combinatorial estimate

$$\Big|\{\eta,:\ \pi_N(\eta) \sim \rho(x)dx\}\Big| \simeq e^{-N\int_0^1 \rho(x)\log\rho(x) + (1-\rho(x))\log(1-\rho(x))\,dx}$$

$$\boxed{V(\rho) = \int_0^1 \rho(x)\log\frac{\rho(x)}{p} + (1-\rho(x))\log\frac{(1-\rho(x))}{(1-p)}\,dx}$$

Contraction

Average number of particles

$$\frac{1}{N}\sum_i \eta(i) = \int_{[0.1]} d\pi_N(\eta)$$

satisfies LDP

$$\mathbb{P}\left(\frac{1}{N}\sum_i \eta(i) \sim y\right) \simeq e^{-NJ(y)}$$

BY CONTRACTION

$$\boxed{J(y) = \inf_{\left\{\rho:\int_0^1 \rho(x)\,dx=y\right\}} V(\rho)}$$

Relative entropy

Relative entropy of the probability measure μ_N^2 with respect to μ_N^1

$$H\left(\mu_N^2\middle|\mu_N^1\right) = \sum_\eta \mu_N^2(\eta) \log \frac{\mu_N^2(\eta)}{\mu_N^1(\eta)}$$

$H \geq 0$, not symmetric!!

Density of relative entropy

$$h = \lim_{N\to+\infty} \frac{1}{N} H\left(\mu_N^2\middle|\mu_N^1\right)$$

An example

$\mu_N^1(\eta) = \prod_{x\in\Lambda_N} p^{\eta(x)}\,(1-p)^{1-\eta(x)}$, product of Bernoulli measures of parameter p

$\mu_N^2(\eta) = \prod_{x\in\Lambda_N} \rho(x)^{\eta(x)}\,(1-\rho(x))^{1-\eta(x)}$, slowly varying product of Bernoulli measures associated to the density profile $\rho(x)$

$$\frac{1}{N} H\left(\mu_N^2 \middle| \mu_N^1\right)$$

$$= \sum_\eta \mu_N^2(\eta) \left(\frac{1}{N}\sum_{x\in\Lambda_N} \eta(x)\log\frac{\rho(x)}{p} + (1-\eta(x))\log\frac{(1-\rho(x))}{(1-p)}\right)$$

$$= \frac{1}{N}\sum_{x\in\Lambda_N} \rho(x)\log\frac{\rho(x)}{p} + (1-\rho(x))\log\frac{(1-\rho(x))}{(1-p)}$$

Riemann sums, convergence when $N \to +\infty$ to $V(\rho)$

From microscopic to MACROSCOPIC

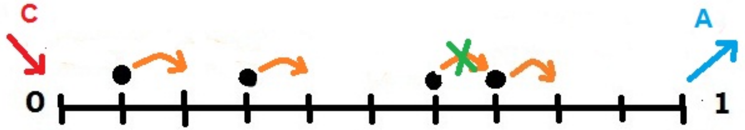
- Driving parameters (λ, E)
- $\lambda \Longrightarrow$ rates of injection and annihilation at the boundary
- $E \Longrightarrow$ external field driving the particles on the bulk
- $\mu_N^{\lambda,E} \Longrightarrow$ corresponding invariant measure
- $\bar{\rho}_{\lambda,E} \Longrightarrow$ corresponding typical density profile
- $V_{\lambda,E}(\rho) \Longrightarrow$ corresponding LD rate function

$$\boxed{V_{\lambda_1,E_1}(\bar{\rho}_{\lambda_2,E_2}) = \lim_{N\to+\infty} \frac{1}{N} H\left(\mu_N^{\lambda_2,E_2} \middle| \mu_N^{\lambda_1,E_1}\right)}$$

From microscopic to MACROSCOPIC

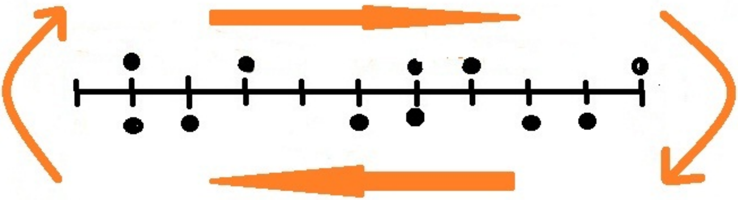
- This relation between relative entropy and LD rate function can be easily verified for the boundary driven Zero Range Process
- It is true also for boundary driven SEP; proof based on matrix representation of μ_N
- In general the computation of V through relative entropy is difficult
- An alternative powerful approach to compute V is the dynamic variational one of the Macroscopic Fluctuation Theory

Boundary driven TASEP: a microscopic view



Boundary driven TASEP: a microscopic view

- Duchi E., Schaeffer G A combinatorial approach to jumping particles, J. Comb. Theory A (2005)



Boundary driven TASEP: a microscopic view

- $\eta \Longrightarrow$ configuration of particles above
- $\xi \Longrightarrow$ configuration of particles below
- $(\eta, \xi) \Longrightarrow$ full configuration of particles
- Stochastic Markov dynamics for (η, ξ)
- Observing just $\eta \Longrightarrow$ still Markov and boundary driven TASEP
- $\nu_N(\eta, \xi) \Longrightarrow$ invariant measure for the joint dynamics, it has a combinatorial representation

$$\boxed{\mu_N(\eta) = \sum_\xi \nu_N(\eta, \xi)}$$

Boundary driven TASEP: a microscopic view

Complete configurations

$$E(x) = \sum_{y \le x} (\eta(y) + \xi(y)) - Nx - 1$$

(η, ξ) is a complete configuration if

$$\begin{cases} E(x) \ge 0 \\ E(1) = 0 \end{cases}$$

ν_N is concentrated on complete configurations
(η, ξ) complete $\Longrightarrow N_1(\eta, \xi), N_2(\eta, \xi)$

$$\boxed{\nu_N(\eta, \xi) = \frac{1}{Z_N} A^{N_1(\eta,\xi)} C^{N_2(\eta,\xi)}}$$

Special case $A = C = 1 \Longrightarrow \nu_N$ uniform measure on complete configurations

Boundary driven TASEP: a macroscopic view

Joint Large deviations

$$\mathbb{P}\Big((\pi_N(\eta), \pi_N(\xi)) \sim (\rho(x), f(x)) \Big) \simeq e^{-N\mathcal{G}(\rho,f)}$$

Contraction principle

$$\mathbb{P}\Big(\pi_N(\eta) \sim \rho(x)\Big) \simeq e^{-NV(\rho)}$$

$$\boxed{V(\rho) = \inf_f \mathcal{G}(\rho, f)}$$

Boundary driven TASEP: a macroscopic view

Complete density profiles

$$\mathcal{E}(x) = \int_0^x (\rho(y) + f(y))\, dy - x$$

The pair (ρ, f) is a complete density profile if

$$\begin{cases} \mathcal{E}(x) \geq 0 \\ \mathcal{E}(1) = 0 \end{cases}$$

When $C = A = 1$ since ν_N is uniform on complete configurations a classic simple computation gives

$$\mathcal{G}(\rho, f) = \int_0^1 \left[h_{\frac{1}{2}}(\rho(x)) + h_{\frac{1}{2}}(f(x)) \right] dx$$

if (ρ, f) is complete; here

$$h_p(\alpha) = \alpha \log \frac{\alpha}{p} + (1-\alpha) \log \frac{(1-\alpha)}{1-p}$$

Boundary driven TASEP: a macroscopic view

$$V(\rho) = \inf_{f\,:\,(\rho,f)\in\mathcal{C}} \int_0^1 \left[h_{\frac{1}{2}}(\rho(x)) + h_{\frac{1}{2}}(f(x)) \right] dx$$

To be compared with B. Derrida, J.L. Lebowitz, E.R. Speer Exact large deviation functional of a stationary open driven diffusive system: the asymmetric exclusion process J. Stat. Phys. (2003)

$$\begin{aligned} V(\rho) &= \sup_f \int_0^1 \Big\{ \rho(x) \log\left[\rho(x)(1-f(x))\right] \\ &+ (1-\rho(x)) \log\left[(1-\rho(x))f(x)\right] \Big\} dx + \log 4 \end{aligned}$$

where $f(0)=1$, $f(1)=0$ and f is monotone

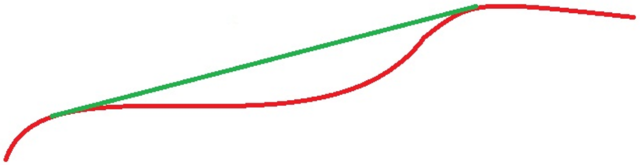
Boundary driven TASEP: a macroscopic view

Both variational problems have the same minimizer

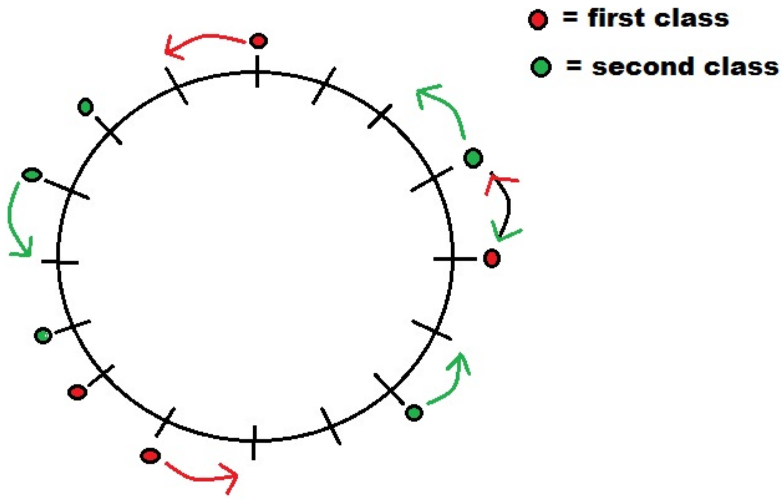
$$f_\rho(x) = CE\Big(\int_0^x (1-\rho(y))\,dy\Big)$$

$$V(\rho) = \mathcal{G}(\rho, f_\rho)$$

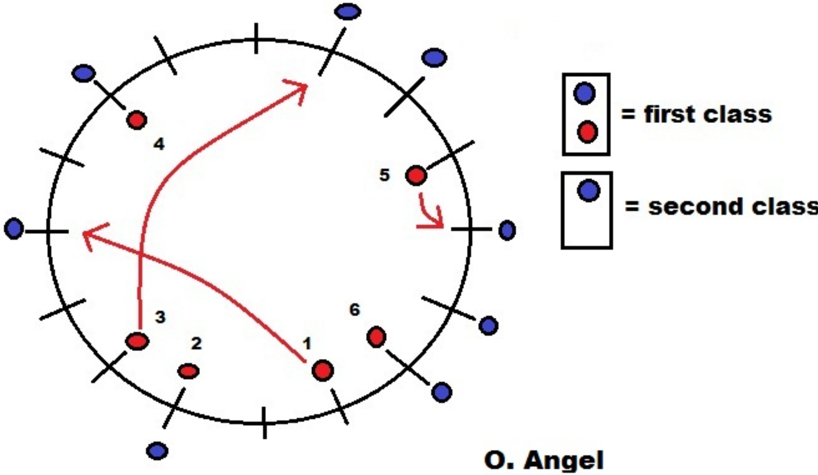
See Bahadoran C. A quasi-potential for conservation laws with boundary conditions arXiv:1010.3624 for a dynamic variational approach, using MFT



2-class TASEP



The invariant measure



Collapsing particles

$$(\tilde{\eta}_1, \tilde{\eta}_T) \, : \, \sum_x \tilde{\eta}_1(x) \leq \sum_x \tilde{\eta}_T(x) \quad \Longrightarrow \quad (\eta_1, \eta_T) = \mathcal{C}\big[\, (\tilde{\eta}_1, \tilde{\eta}_T) \big)\, \big]$$

Flux across bond $(x, x+1)$

$$J(x) = \sup_y \Big[\sum_{z \in [y,x]} \tilde{\eta}_1(z) - \tilde{\eta}_T(z) \Big]_+$$

Collapsing measures

$$(\tilde{\rho}_1, \tilde{\rho}_T)) \, : \, \int_{\mathbb{S}^1} d\tilde{\rho}_1 \le \int_{\mathbb{S}^1} d\tilde{\rho}_T \implies (\rho_1, \rho_T) = \mathcal{C}\big[\, (\tilde{\rho}_1, \tilde{\rho}_T))\,\big]$$

Definition

$$\boxed{\int_{(a,b]} d\rho_1 = \int_{(a,b]} d\tilde{\rho}_1 + J(a) - J(b)}$$

where

$$J(x) := \sup_y \left[\int_{(y,x]} d\tilde{\rho}_1 - \int_{(y,x]} d\tilde{\rho}_2 \right]_+$$

Collapsing measures

Large deviations

LD for the $(\tilde{\eta}_1, \tilde{\eta}_T)$ variables

$$\tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T) = \int_{\mathbb{S}^1} \left[h_{m_1}(\tilde{\rho}_1) + h_{m_2}(\tilde{\rho}_T)\right] dx$$

LD for the SNS (not convex!)

$$\begin{aligned} V(\rho_1, \rho_T) &= \inf_{\{(\tilde{\rho}_1, \tilde{\rho}_T) : \mathcal{C}[(\tilde{\rho}_1, \tilde{\rho}_T)] = (\rho_1, \rho_T)\}} \tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T) \\ &= \int_{\mathbb{S}^1} \left[h_{m_1}(\hat{\rho}_1) + h_{m_2}(\rho_T)\right] dx \end{aligned}$$

On any (a, b) where $\rho_1 = \rho_T$

$$\int_a^x \hat{\rho}_1(y)dy = CE\left[\int_a^x \rho_1(y)dy\right]$$