STATIONARY NON EQUILIBRIUM STATES FROM A MICROSCOPIC AND A MACROSCOPIC POINT OF VIEW

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References

- L. Bertini; A. De Sole; D. G.; G. Jona-Lasinio; C. Landim

- D. G. From combinatorics to large deviations for the invariant measures of some multiclass particle systems

- L. Bertini; D. G.; G. Jona-Lasinio; C. Landim
  Thermodynamic transformations of nonequilibrium states
SNS: Microscopic description

Lattice: $\Lambda_N$

Configuration of particles: $\eta \in \{0, 1\}^{\Lambda_N}$ or $\eta \in \mathbb{N}^{\Lambda_N}$

$\eta_t(x) =$ number of particles at $x \in \Lambda_N$ at time $t$
SNS: Microscopic description

- Stochastic Markovian dynamics
- \( r(\eta, \eta') = \text{rate of jump from configuration } \eta \text{ to configuration } \eta' \)
- \( \eta' = \text{local perturbation of } \eta \)
- \( \mu_N(\eta) = \text{invariant measure of the process, probability measure on the state space} \)

\[
\mu_N(\eta) \sum_{\eta'} r(\eta, \eta') = \sum_{\eta'} \mu_N(\eta') r(\eta', \eta)
\]

\( \mu_N \Rightarrow \text{MICROSCOPIC description of the SNS} \)
SNS: Macroscopic description

• $\eta \mapsto \pi_N(\eta)$ Empirical measure (positive measure on $[0, 1]$)

$$\pi_N(\eta) = \frac{1}{N} \sum_{x \in \Lambda_N} \eta(x) \delta_x$$

$\delta_x = \text{delta measure (Dirac) at } x \in [0, 1]$; since $x \in \Lambda_N$ we have $x = \frac{i}{N}, i \in \mathbb{N}$. Given $f : [0, 1] \to \mathbb{R}$

$$\int_{[0,1]} f \, d\pi_N = \frac{1}{N} \sum_{x \in \Lambda_N} \eta(x) f(x)$$

$\times = \text{delta measure}$

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When $\eta$ is distributed according to $\mu_N$ and $N$ is large

\[ \text{LAW OF LARGE NUMBERS} \]

\[ \pi_N \rightarrow \bar{\rho}(x) \, dx \]

This means

\[ \int_{[0,1]} f \, d\pi_N \rightarrow \int_{[0,1]} f(x) \bar{\rho}(x) \, dx \]

$\bar{\rho}(x) =$ typical density profile of the SNS
When $\eta$ is distributed according to $\mu_N$ and $N$ is large, a refinement of the law of large numbers

\[ \mathbb{P}\left( \pi_N(\eta) \sim \rho(x)dx \right) \sim e^{-NV(\rho)} \]

$V = \text{Large deviations rate function}$

$V \implies \text{MACROSCOPIC DESCRIPTION OF THE SNS}$

$V$ contains less information than $\mu_N$ but is easier to compute and is independent from microscopic details of the dynamics
Example: Equilibrium SEP

- Equilibrium: \( C_L = C_R = C; \ A_L = A_R = A \)
- Microscopic state: product of Bernoulli measures of parameter \( p = \frac{C}{A+C} \)

\[
\mu_N(\eta) = \prod_{x \in \Lambda_N} p^{\eta(x)} (1 - p)^{1-\eta(x)}
\]
Example: Equilibrium SEP

MACROSCOPIC DESCRIPTION

\[
\mathbb{P}\left( \pi_N(\eta) \sim \rho(x) dx \right) = \sum_{\{\eta, \pi_N(\eta) \sim \rho(x) dx\}} \mu_N(\eta)
\]

\[
= \sum_{\{\eta, \pi_N(\eta) \sim \rho(x) dx\}} e^{-N\left( \int_{[0,1]} \pi_N(\eta) \log \frac{1-p}{p} - \log(1-p) \right)}
\]

Using the combinatorial estimate

\[
\left| \{\eta, \pi_N(\eta) \sim \rho(x) dx\} \right| \simeq e^{-N \int_0^1 \rho(x) \log \rho(x) + (1-\rho(x)) \log(1-\rho(x)) \ dx}
\]

\[
V(\rho) = \int_0^1 \rho(x) \log \frac{\rho(x)}{p} + (1 - \rho(x)) \log \frac{(1-\rho(x))}{(1-p)} \ dx
\]
Contraction

Average number of particles

\[
\frac{1}{N} \sum_i \eta(i) = \int_{[0,1]} d\pi_N(\eta)
\]

satisfies LDP

\[
P \left( \frac{1}{N} \sum_i \eta(i) \sim y \right) \simeq e^{-NJ(y)}
\]

BY CONTRACTION

\[
J(y) = \inf \left\{ \rho : \int_0^1 \rho(x) \, dx = y \right\} V(\rho)
\]
Relative entropy

Relative entropy of the probability measure $\mu^2_N$ with respect to $\mu^1_N$

$$H\left(\mu^2_N \middle| \mu^1_N\right) = \sum_{\eta} \mu^2_N(\eta) \log \frac{\mu^2_N(\eta)}{\mu^1_N(\eta)}$$

$H \geq 0$, not symmetric!!

Density of relative entropy

$$h = \lim_{N \to +\infty} \frac{1}{N} H\left(\mu^2_N \middle| \mu^1_N\right)$$
An example

\[ \mu_N^1(\eta) = \prod_{x \in \Lambda_N} p^{\eta(x)} (1 - p)^{1 - \eta(x)}, \text{ product of Bernoulli measures of parameter } p \]

\[ \mu_N^2(\eta) = \prod_{x \in \Lambda_N} \rho(x)^{\eta(x)} (1 - \rho(x))^{1 - \eta(x)}, \text{ slowly varying product of Bernoulli measures associated to the density profile } \rho(x) \]

\[ \frac{1}{N} H \left( \mu_N^2 \mid \mu_N^1 \right) \]

\[ = \sum_{\eta} \mu_N^2(\eta) \left( \frac{1}{N} \sum_{x \in \Lambda_N} \eta(x) \log \frac{\rho(x)}{p} + (1 - \eta(x)) \log \frac{(1 - \rho(x))}{(1 - p)} \right) \]

\[ = \frac{1}{N} \sum_{x \in \Lambda_N} \rho(x) \log \frac{\rho(x)}{p} + (1 - \rho(x)) \log \frac{(1 - \rho(x))}{(1 - p)} \]

Riemann sums, convergence when \( N \to +\infty \) to \( V(\rho) \)
• Driving parameters \((\lambda, E)\)
• \(\lambda \mapsto\) rates of injection and annihilation at the boundary
• \(E \mapsto\) external field driving the particles on the bulk
• \(\mu_{\lambda,E}^{N} \mapsto\) corresponding invariant measure
• \(\bar{\rho}_{\lambda,E} \mapsto\) corresponding typical density profile
• \(V_{\lambda,E}(\rho) \mapsto\) corresponding LD rate function

\[
V_{\lambda_1,E_1}(\bar{\rho}_{\lambda_2,E_2}) = \lim_{N \to +\infty} \frac{1}{N} H\left(\mu_{\lambda_2,E_2}^{N} \mid \mu_{\lambda_1,E_1}^{N}\right)
\]
From microscopic to MACROSCOPIC

- This relation between relative entropy and LD rate function can be easily verified for the boundary driven Zero Range Process.
- It is true also for boundary driven SEP; proof based on matrix representation of $\mu_N$.
- In general the computation of $V$ through relative entropy is difficult.
- An alternative powerful approach to compute $V$ is the dynamic variational one of the Macroscopic Fluctuation Theory.
Boundary driven TASEP: a microscopic view
Boundary driven TASEP: a microscopic view

Boundary driven TASEP: a microscopic view

- $\eta \mapsto$ configuration of particles above
- $\xi \mapsto$ configuration of particles below
- $(\eta, \xi) \mapsto$ full configuration of particles
- Stochastic Markov dynamics for $(\eta, \xi)$
- Observing just $\eta \mapsto$ still Markov and boundary driven TASEP
- $\nu_N(\eta, \xi) \mapsto$ invariant measure for the joint dynamics, it has a combinatorial representation

$$\mu_N(\eta) = \sum_\xi \nu_N(\eta, \xi)$$
Boundary driven TASEP: a microscopic view

Complete configurations

\[ E(x) = \sum_{y \leq x} (\eta(y) + \xi(y)) - Nx - 1 \]

\((\eta, \xi)\) is a complete configuration if

\[
\begin{cases}
E(x) \geq 0 \\
E(1) = 0
\end{cases}
\]

\(\nu_N\) is concentrated on complete configurations

\(\eta, \xi\) complete \(\implies\) \(N_1(\eta, \xi), N_2(\eta, \xi)\)

\[
\nu_N(\eta, \xi) = \frac{1}{Z_N} A^{N_1(\eta, \xi)} C^{N_2(\eta, \xi)}
\]

Special case \(A = C = 1 \implies \nu_N\) uniform measure on complete configurations
Boundary driven TASEP: a macroscopic view

Joint Large deviations

\[ \mathbb{P}\left( (\pi_N(\eta), \pi_N(\xi)) \sim (\rho(x), f(x)) \right) \simeq e^{-NG(\rho,f)} \]

Contraction principle

\[ \mathbb{P}\left( \pi_N(\eta) \sim \rho(x) \right) \simeq e^{-NV(\rho)} \]

\[ V(\rho) = \inf_f G(\rho, f) \]
Boundary driven TASEP: a macroscopic view

Complete density profiles

$$\mathcal{E}(x) = \int_0^x (\rho(y) + f(y)) \, dy - x$$

The pair $$(\rho, f)$$ is a complete density profile if

$$\begin{cases} \mathcal{E}(x) \geq 0 \\ \mathcal{E}(1) = 0 \end{cases}$$

When $C = A = 1$ since $\nu_N$ is uniform on complete configurations a classic simple computation gives

$$\mathcal{G}(\rho, f) = \int_0^1 \left[ h_{\frac{1}{2}}(\rho(x)) + h_{\frac{1}{2}}(f(x)) \right] \, dx$$

if $$(\rho, f)$$ is complete; here

$$h_p(\alpha) = \alpha \log \frac{\alpha}{p} + (1 - \alpha) \log \frac{1 - \alpha}{1 - p}$$
Boundary driven TASEP: a macroscopic view

\[ V(\rho) = \inf_{f : (\rho, f) \in \mathcal{C}} \int_0^1 \left[ h_{\frac{1}{2}}(\rho(x)) + h_{\frac{1}{2}}(f(x)) \right] \, dx \]

To be compared with B. Derrida, J.L. Lebowitz, E.R. Speer


\[ V(\rho) = \sup_f \int_0^1 \left\{ \rho(x) \log [\rho(x)(1 - f(x))] \\
+ (1 - \rho(x)) \log [(1 - \rho(x))f(x)] \right\} \, dx + \log 4 \]

where \( f(0) = 1, \, f(1) = 0 \) and \( f \) is monotone

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Both variational problems have the same minimizer

\[ f_\rho(x) = CE\left( \int_0^x (1 - \rho(y)) \, dy \right) \]

\[ V(\rho) = G(\rho, f_\rho) \]

See Bahadoran C. A quasi-potential for conservation laws with boundary conditions arXiv:1010.3624 for a dynamic variational approach, using MFT
2-class TASEP

\[ \text{= first class} \]
\[ \text{= second class} \]
The invariant measure

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Collapsing particles

\[ (\tilde{\eta}_1, \tilde{\eta}_T) : \sum_x \tilde{\eta}_1(x) \leq \sum_x \tilde{\eta}_T(x) \implies (\eta_1, \eta_T) = C[ (\tilde{\eta}_1, \tilde{\eta}_T) ] \]

Flux across bond \((x, x + 1)\)

\[ J(x) = \sup_y \left[ \sum_{z \in [y, x]} \tilde{\eta}_1(z) - \tilde{\eta}_T(z) \right]_+ \]
Collapsing measures

\[(\tilde{\rho}_1, \tilde{\rho}_T) : \int_{S^1} d\tilde{\rho}_1 \leq \int_{S^1} d\tilde{\rho}_T \implies (\rho_1, \rho_T) = C[(\tilde{\rho}_1, \tilde{\rho}_T))] \]

Definition

\[
\int_{(a,b]} d\rho_1 = \int_{(a,b]} d\tilde{\rho}_1 + J(a) - J(b)
\]

where

\[
J(x) := \sup_y \left[ \int_{(y,x]} d\tilde{\rho}_1 - \int_{(y,x]} d\tilde{\rho}_2 \right]
\]
Collapsing measures
Large deviations

LD for the \((\tilde{\eta}_1, \tilde{\eta}_T)\) variables

\[
\tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T) = \int_{S^1} [h_{m_1} (\tilde{\rho}_1) + h_{m_2} (\tilde{\rho}_T))] \, dx
\]

LD for the SNS (not convex!)

\[
V(\rho_1, \rho_T) = \inf_{\{(\tilde{\rho}_1, \tilde{\rho}_T) : C[(\tilde{\rho}_1, \tilde{\rho}_T)] = (\rho_1, \rho_T)\}} \tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T)
\]

\[
= \int_{S^1} [h_{m_1} (\hat{\rho}_1) + h_{m_2} (\rho_T))] \, dx
\]

On any \((a, b)\) where \(\rho_1 = \rho_T\)

\[
\int_a^x \hat{\rho}_1(y) \, dy = CE \left[ \int_a^x \rho_1(y) \, dy \right]
\]