Superdiffusive modes in driven diffusive systems

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joint work with R. Grisi (UFABC) and V. Popkov, J. Schmidt (Cologne)

- Introduction
- Onsager-type current symmetries for DDS [Grisi, GMS]
- Superdiffusive dynamical structure function in DDS [Popkov, Schmidt, GMS]
- Conclusions

1. Introduction

Bulk-driven Particle systems with several conservation laws:

- Interacting stochastic particle systems on lattice with biased hopping
  ==> Non-reversible Markovian dynamics
  ==> Invariant measures not known *a priori*

- A few (!) examples:
  (1) Exclusion processes with several species of particles
  (second-class, slow particles, tagged particles, AHR model, ...)
  (2) Multispecies zero-range models
  (3) Bricklayer model
  (4) Multilane exclusion processes

- Rich behaviour, e.g.,
  - Phase transitions (phase separation, spontaneous symmetry breaking)
  - (Conjectured) hydrodynamic equations sensitive to regularization
  - Intricate interplay of shocks and rarefaction waves
  - Universal fluctuations (Diffusive, KPZ)

Generic lattice gas models

- 1-d Torus with N sites (labeled by k,l,m,...)
- finite local state space S (local occupation variable $\omega_k \in S$)
- Particle jumps between two different sites at most M sites apart
- Jump rate depends on configuration up to a distance $M < N/2$

Further definitions:

- Shift operator σ: $(\sigma\omega)_k = \omega_{k+1}$

- Switch operator Θ:

$$\left(\Theta_{k,m}^{\omega',\omega''}\underline{\omega}\right)_n = \begin{cases} \omega_n; & \text{if } n \neq k, k+m \\ \omega'; & \text{if } n = k \\ \omega''; & \text{if } n = k+m. \end{cases}$$

- Jump rate from $\underline{\omega}$ to $\Theta^{\omega',\omega''}_{j,m}(\underline{\omega})$: $\quad r_m(\sigma^j\underline{\omega};\omega',\omega'')$

Generator and conserved quantities:

Infinitesimal generator:

$$\mathscr{L} f(\underline{\omega}) = \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{\omega', \omega'' \in S} r_m(\sigma^k \underline{\omega}; \omega', \omega'')[f(\Theta_{k,m}^{\omega', \omega''} \underline{\omega}) - f(\underline{\omega})]$$

Conserved quantities $\xi^\alpha := \sum_k \xi_k^\alpha(\underline{\omega})$

- Assume at least two conservation laws (particle numbers) ξ^α

- Irreducibility condition on rates to exclude non-ergodicity for fixed ξ^α
  (e.g. no “hidden” conservation laws)

- Microscopic current across bond (k,k+1):

$$j_k^\alpha(\underline{\omega}) := -\sum_{n=0}^{M-1}\sum_{m=n+1}^{M}\sum_{\omega',\omega''\in S} r_m(\sigma^{k-n}\underline{\omega};\omega',\omega'')[\xi_{k-n}^\alpha(\Theta_{k-n,m}^{\omega',\omega''}\underline{\omega}) - \xi_{k-n}^\alpha(\underline{\omega})]$$

(lengthy but straightforward computation)

- Conservation law ==> (Discrete) Noether theorem

$$\mathscr{L}\xi_k^\alpha(\underline{\omega}) = -j_k^\alpha(\underline{\omega}) + j_{k-1}^\alpha(\underline{\omega})$$

(Lattice continuity equation)

Invariant measures:

Translation invariance and ergodicity for fixed values of conserved particle numbers ξ^α

==> “canonical” invariant measure μ is unique and translation invariant

Define “grandcanonical” invariant measures with chemical potentials ϕ^α

$$\mu_\phi(\underline{\omega}) = \mu(\underline{\omega})\, e^{\sum_{k\in\mathbb{T}_N} \phi\cdot\xi_k(\underline{\omega}) - G(\phi)},$$

where

$$G(\phi) = \log \sum_{\underline{\omega}\in\Omega} e^{\sum_{k\in\mathbb{T}_N} \phi\cdot\xi_k(\underline{\omega})} \mu(\underline{\omega})$$

Stationary density of particles of type α: $\rho^\alpha(\{\phi\}) = \langle\xi^\alpha_k\rangle_\phi$

Stationary current of particle type α: $j^\alpha(\{\phi\}) = \langle j^\alpha_k\rangle_\phi$

Time-reversed dynamics and currents:

Time-reversed hopping rates:

$$r_m^*(\sigma^k \underline{\omega}; \omega', \omega'') := \mu(\underline{\omega})^{-1} \mu(\Theta_{k,m}^{\omega',\omega''} \underline{\omega}) r_m(\sigma^k \Theta_{k,m}^{\omega',\omega''} \underline{\omega}; \omega_k, \omega_{k+m})$$

==> adjoint process

$$\mathscr{L}^* f(\underline{\omega}) = \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{\omega',\omega'' \in S} r_m^*(\sigma^k \underline{\omega}; \omega', \omega'')[f(\Theta_{k,m}^{\omega',\omega''} \underline{\omega}) - f(\underline{\omega})]$$

Adjoint conservation law $\quad \mathscr{L}^* \xi_k^\alpha(\underline{\omega}) = -j_k^{\alpha *}(\underline{\omega}) + j_{k-1}^{\alpha *}(\underline{\omega})$

$$j_i^{\alpha *}(\underline{\omega}) = -\sum_{n=0}^{M-1} \sum_{m=n+1}^{M} \sum_{\omega',\omega'' \in S} r_m^*(\sigma^{k-n} \underline{\omega}; \omega', \omega'')[\xi_{k-n}^\alpha(\Theta_{k-n,m}^{\omega',\omega''} \underline{\omega}) - \xi_{k-n}^\alpha(\underline{\omega})]$$

Notice: $j^{\alpha *} = - j^\alpha$

Hydrodynamics under Eulerian Scaling

Study large-scale dynamics under coarse-graining x = ka, t = τa:

==> Law large numbers:

Occupation numbers on lattice $n^{\alpha}_k(t) \to \rho^{\alpha}(x,t)$ (Conserved particle densities)

==> Local stationarity:

Microscopic current $j^{\alpha}_k(t) \to j^{\alpha}(\{\rho\})$: Associated locally stationary currents

==> Lattice continuity equation $\to$ System of hyperbolic conservation laws

$$\partial_t \rho^{\alpha} + \partial_x j^{\alpha} = 0$$

Origin of hyperbolicity: Onsager-type symmetry

$$\frac{\partial j^{\alpha}}{\partial \phi^{\beta}} = \frac{\partial j^{\beta}}{\partial \phi^{\alpha}}$$

$\phi^{\alpha}(\{\rho\})$: Fugacities associated with the densities (Legrende transformation)

Proof (Toth, Valko, 2003) for family of models with invariant product measure

2. Onsager-type symmetries

Full proof for generic lattice gas models (Grisi, GMS (2011)):

Lemma 1: For N > 2M we have

$$\langle (\xi^{\alpha}_{\frac{N}{2}} - \rho^{\alpha}) j_0^{\beta *} \rangle_{\phi} = - \langle (\xi^{\alpha}_{\frac{N}{2}} - \rho^{\alpha}) j_0^{\beta} \rangle_{\phi}$$

Proof: Straightforward computation, but requires

(1) no overlap of expression of current (adjoint current) at site N/2
with conserved quantity at site 0

(2) for $0 \leq n < m \leq M$ $\quad \xi^{\alpha}_{\frac{N}{2}} (\Theta^{\omega', \omega''}_{-n,m} \underline{\omega}) = \xi^{\alpha}_{\frac{N}{2}} (\underline{\omega})$

(guaranteed by condition on interaction range M < N/2)

Main result:

Theorem: For finite system with N sites we have

$$\frac{\partial j^\alpha}{\partial \phi^\beta} - \frac{\partial j^\beta}{\partial \phi^\alpha} = N[\langle j^\alpha_{\frac{N}{2}} (\xi^\beta_1 - \rho^\beta)\rangle_\phi - \langle j^\beta_{\frac{N}{2}} (\xi^\alpha_0 - \rho^\alpha)\rangle_\phi]$$

Proof: (a) By construction of grandcanonical measure

$$\frac{\partial \mu_\phi(\underline{\omega})}{\partial \phi^\alpha} = \mu_\phi(\underline{\omega})\Big[\sum_{k\in\mathbb{T}_N} (\xi^\alpha_k(\underline{\omega}) - \rho^\alpha)\Big]$$

Therefore with translation invariance:

$$\frac{\partial j^\beta}{\partial \phi^\alpha} = \sum_{k\in\mathbb{T}_N} \langle (\xi^\alpha_k - \rho^\alpha) j^\beta_0 \rangle_\phi$$

(b) Conservation law, translation invariance, time-reversal:

$$
\begin{aligned}
\langle(\xi_k^\beta-\rho^\beta)(j_{-1}^\alpha-j_0^\alpha)\rangle_\phi &= \langle(\xi_k^\beta-\rho^\beta)\mathscr{L}(\xi_0^\alpha-\rho^\alpha)\rangle_\phi \\
&= \langle(\xi_0^\alpha-\rho^\alpha)\mathscr{L}^*(\xi_k^\beta-\rho^\beta)\rangle_\phi \\
&= \langle(\xi_0^\alpha-\rho^\alpha)(j_{k-1}^{\beta *}-j_k^{\beta *})\rangle_\phi .
\end{aligned}
$$

(c) Partial summation and translation invariance:

$$
\begin{aligned}
&\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k\langle(\xi_k^\beta-\rho^\beta)(j_{-1}^\alpha-j_0^\alpha)\rangle_\phi \\
&\qquad\qquad = -\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \langle(\xi_0^\beta-\rho^\beta)j_{-k}^\alpha\rangle_\phi + N\langle(\xi_0^\beta-\rho^\beta)j_{\frac{N}{2}-1}^\alpha\rangle_\phi \\
&\qquad\qquad = -\frac{\partial j^\alpha}{\partial\phi^\beta} + N\langle(\xi_1^\beta-\rho^\beta)j_{\frac{N}{2}}^\alpha\rangle_\phi
\end{aligned}
$$

Similarly for time-reversed process

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k\langle(\xi_0^\alpha - \rho^\alpha)(j_{k-1}^{\beta *} - j_k^{\beta *})\rangle_\phi = -\frac{\partial j^\beta}{\partial \phi^\alpha} + N\langle(\xi_0^\alpha - \rho^\alpha)j_{\frac{N}{2}}^\beta\rangle_\phi$$

Lemma 1 completes the proof. □

Remark 1: Valid for finite size, no assumption of product measure

Remark 2: Product measure yields Toth/Valko

Corollary 1: For sufficiently fast decaying stationary current-density correlations (o(1/N)) one has in the thermodynamic limit $N \to \infty$ the current symmetry

$$\frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha}$$

Remark: Can be written $S = S^T$ with $S_{\alpha\beta} = \partial j^\alpha / \partial \phi^\beta$

Corollary 2: Define

Current Jacobian A with matrix elements $A_{\alpha\beta} = \partial j^\alpha / \partial \rho^\beta$

Compressibility matrix C with matrix elements

$C_{\alpha\beta} = 1/N \langle(\xi^\alpha - N\rho^\alpha)(\xi^\beta - N \rho^\beta)\rangle = \partial \rho^\alpha / \partial \phi^\beta$

Then $C A^T = A C$

Remark: Cf. Ferrari, Sasamoto, Spohn (2013) for heuristic proof.

3. Superdiffusive structure function in DDS

Go beyond LLN and study fluctuations:

• Dynamical structure function

$$S_{\alpha\beta}(p,t) = \sum_k e^{ikp} <(\xi^\alpha_k(t) - \rho^\alpha)(\xi^\beta_0(0) - \rho^\beta)> = <u^\alpha(p,t)\, u^\beta(-p,t)>$$

where $u^\alpha(p,t)$ = Fourier transform of locally conservd quantity $\xi^\alpha_k(t)$

• <u>One conservation law:</u> Scaling form $S(p,t) = F(p^z t)$

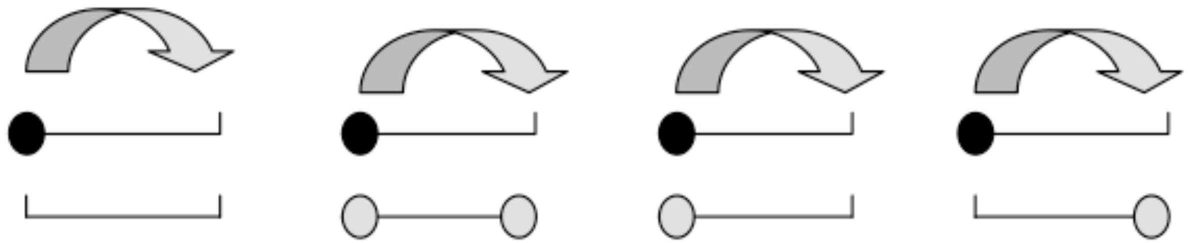
• KPZ universality class z=3/2, universal scaling function F [Praehofer, Spohn (2002)]

• <u>Several conservation laws:</u> Different universality classes in the same DDS

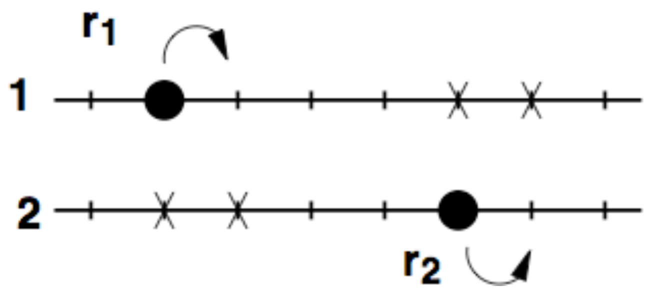
• Known cases for two-component DDS: (a) Both KPZ (generic)
(b) KPZ and Diffusive (z=2)
[Das et al (2001), Rakos, GMS (2005)]

==> Is that all there is?

- Model: Interacting two-lane TASEPs with densities $\rho_{1,2}$



- Product measure: $r_1 = 1 + \gamma\, n^{(2)}/2$, $r_2 = b + \gamma\, n^{(1)}/2$ [Popkov, Salerno (2004)]



- Stationary currents:

$$j_1(\rho_1, \rho_2) = \rho_1(1-\rho_1)(1+\gamma\rho_2)$$
$$j_2(\rho_1, \rho_2) = \rho_2(1-\rho_2)(b+\gamma\rho_1)$$

Nonlinear fluctuating hydrodynamics (non-rigorous):

- Starting point: (Deterministic) hyperbolic system of conservation laws

$$\frac{\partial}{\partial t}\vec{\rho} + A\frac{\partial}{\partial x}\vec{\rho} = 0$$

with density vector and current Jacobian A

- Stationary solutions: ρ_i

- Introduce fluctuation fields $u_i(x,t) = \rho_i(x,t) - \rho_i$ and expand in u_i

A) Linear theory:

- Diagonalize A: $RAR^{-1} = \mathrm{diag}(c_i)$, Normalization $RCR^T = 1$

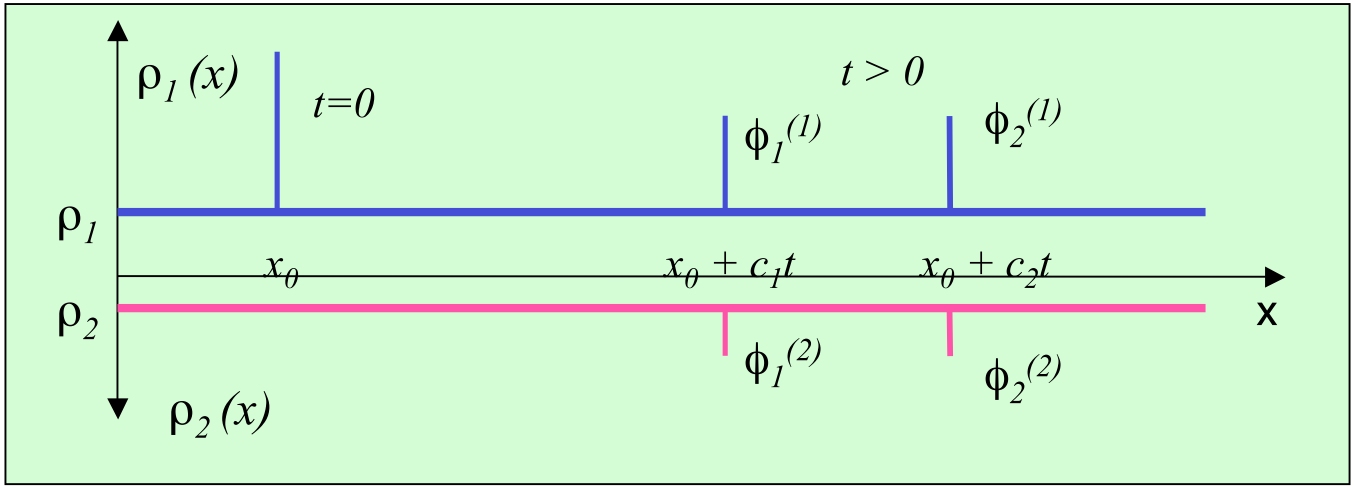
==> Eigenmode equation $\partial_t \phi_i = - c_i \partial_x \phi_i$

- Travelling waves (eigenmodes) $\phi_i(x,t) = \phi_i(x-c_i t)$

- Characteristic speeds $c_{1,2}(\rho_1, \rho_2)$ = eigenvalues of current Jacobian A

- Strict hyperbolicity for two-lane model: $c_1 \neq c_2 \quad \forall (\rho_1, \rho_2) \in (0,1) \times (0,1)$

- Microscopic: Stationary center of mass motion of localized perturbation
  [Popkov, GMS (2003)]



B) Nonlinear fluctuating theory

• Expand to second order, add phenomenological diffusion term and noise [Spohn]

$$\partial_t \phi_i = -\partial_x [c_i \phi_i + \langle \vec{\phi}, G^{(i)} \vec{\phi} \rangle - \partial_x (D\vec{\phi})_i + (B\vec{\xi})_i]$$

diffusion = regularization, noise B and diffusion matrix D related by FDT

- Mode coupling coefficients for eigenmodes

$$G^{(i)} = (1/2) \sum_j R_{ij} (R^{-1})^T H^{(j)} R^{-1}$$

- Hessian $H^{(\gamma)}$ with matrix elements $\partial^2 j_\gamma / (\partial \rho_\alpha \partial \rho_\beta)$

One component: $\partial_t \phi = - \partial_x [c \phi + g \phi^2 - D \partial_x \phi + B \xi]$ (KPZ equation, $g = j''/2$)

Two components ==> Two coupled KPZ equations

Remarks:

1) Higher order terms irrelevant in RG sense (if second order non-zero)

2) Offdiagonal terms neglible for strictly hyperbolic systems (no overlap between modes)

3) Self-coupling terms $G^{(\alpha)}_{\alpha\alpha}$ leading, other diagonal terms $G^{(\alpha)}_{\beta\beta}$ subleading

$$
\Longrightarrow \begin{cases} \partial_t \phi_1 = - \partial_x [c_1 \phi_1 + G^{(1)}_{11} (\phi_1)^2 + G^{(1)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}] \\ \partial_t \phi_2 = - \partial_x [c_2 \phi_2 + G^{(2)}_{11} (\phi_1)^2 + G^{(2)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}] \end{cases}
$$

Mode coupling theory

[van Beijeren (2012), Spohn (2013)]

Some scenarios:

A) Both self-coupling coefficients nonzero: $G^{(1)}_{11} \neq 0$, $G^{(2)}_{22} \neq 0$

==> two KPZ modes ($z_1 = 3/2$, $z_2=3/2$)

B) One self-coupling coefficient nonzero, all diagonal terms of other mode-coupling matrix 0, e.g., $G^{(1)}_{11} \neq 0$, $G^{(2)}_{22} = G^{(2)}_{11} = 0$

==> one KPZ mode, one diffusive mode ($z_1 = 3/2$, $z_2=2$)

C) One self-coupling coefficient nonzero, subleading diagonal of other mode-coupling matrix 0, e.g., $G^{(1)}_{11} \neq 0$, $G^{(2)}_{11} \neq 0$, $G^{(2)}_{22} = 0$

==> one KPZ mode, second non-KPZ superdiffusive mode ($z_1 = 3/2$, $z_2=5/3$)

Remark: Heat mode with $z=5/3$, two KPZ sound modes in Hamiltonian dynamics with three conservation laws [van Beijeren (2012)]

Monte-Carlo simulations

[Popkov, Schmidt, GMS (PRL, 2014)]

Measure dynamical exponents z_i

1) Average over 10^7 - 10^8 runs with uniform random initial conditions with densities ρ_i

2) Excite each mode independently at site k=N/2 at t=0 and measure dynamical structure function of each mode

3) Compute center of mass motion $\langle X_i(t) \rangle$ of excitation ==> $c_i t$

4) Compute CM variances $V_i(t)$:
   Scaling hypothesis $V_i(t) \sim t^{2/z_i}$ ==> z_i
   (no assumption on existence for asymptotic scaling function)

5) Measure amplitudes $A_i(t)$ at maximum:
   Mass conservation $A_i(t) \sim 1/t^{1/z_i}$ ==> z_i

Monte-Carlo simulations

[Popkov, Schmidt, GMS (PRL, 2014)]

Choose equal densities $\rho_1 = \rho_2 = \rho$, and interaction strength $\gamma = 1$

Set b = 2 (inequivalent lanes)

$$G^{(1)}_{11} = - 2\, g\, (6\, \rho^4 - 8\, \rho^3 + 5\, \rho^2 + \rho - 1)$$
$$G^{(1)}_{12} = G^{(1)}_{21} = g\, (4\, \rho^3 - 10\, \rho^2 + 8\, \rho - 1)$$
$$G^{(1)}_{22} = - 2\, g\, \rho\, (1-\rho)\, (2\, \rho^2 - 6\, \rho + 3)$$

$$g = -1/2\, \{\rho\, (1-\rho) / [2\, \rho^2 - 2\, \rho + 1]^3\}^{1/2}$$

$$G^{(2)}_{11} = 4\, g\, \rho\, (1-\rho)$$
$$G^{(2)}_{12} = G^{(1)}_{21} = - g\, (1 - 2\, \rho^2)^2$$
$$G^{(2)}_{22} = 4\, g\, (3\, \rho^2 - 3\, \rho + 1)$$

==> Generically case A (two KPZ modes) with $c_1 \neq c_2$ and $z_{1,2} = 3/2$

Good agreement with Monte-Carlo data for $c_{1,2}$ and $z_{1,2}$

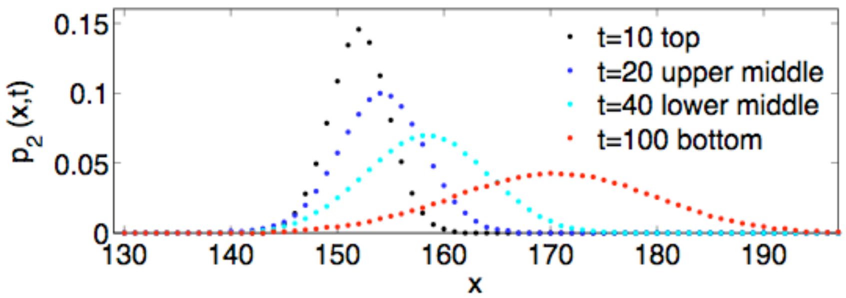
Monte-Carlo simulations (cont')

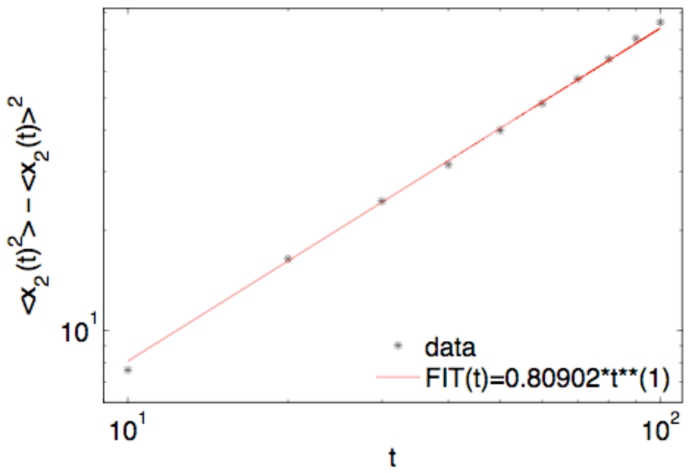
$\rho_1 = \rho_2 = 0.5$, γ = - 0.8 , b = 1 (symmetry between lanes) ==> $c_1 = 0.2$, $c_2 = 0.2$

Mode coupling matrices: $$G^1 = 0.2121 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G^2 = 0.2121 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

==> KPZ with $z_1 = 3/2$ (mode 1) and diffusive with $z_2 = 2$ (mode 2)

Monte-Carlo data for diffusive mode:





Error in velocities < 1%

Variance: $2/z_1 = 1.343 \approx 4/3$

$2/z_2 = 1.030 \approx 1$

Monte-Carlo simulations (cont')

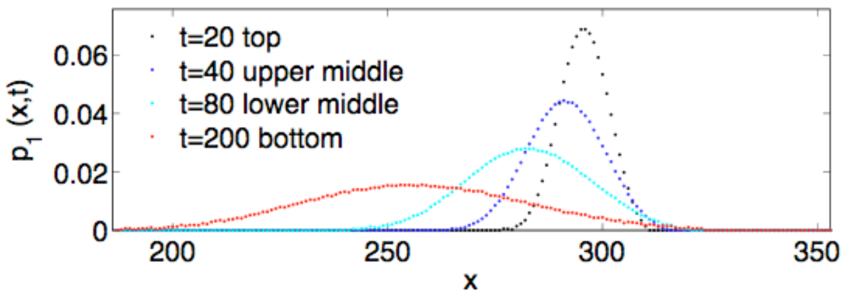
$\rho_1 = \rho_2 = \rho$, γ =-052588 , b = 1.3

$G^{(2)}_{22} = 0$ at $\rho^* = 0.55000...$, $G^{(2)}_{11}(\rho^*) \neq 0$, $G^{(1)}_{11}(\rho^*) \neq 0$

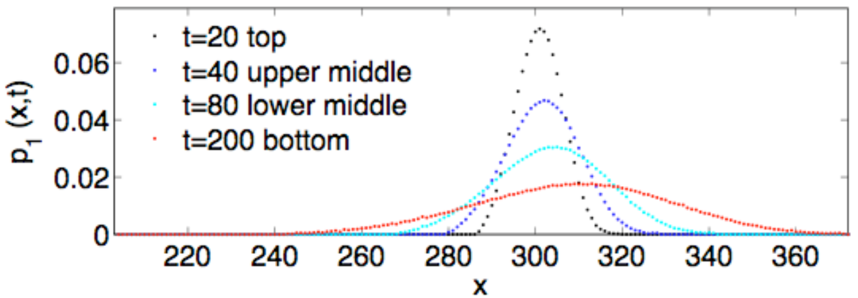
==> At ρ^*: KPZ with $z_1 = 3/2$ (mode 1) and superdiffusive with $z_2 = 5/3$ (mode 2)

Monte-Carlo data:

KPZ mode:



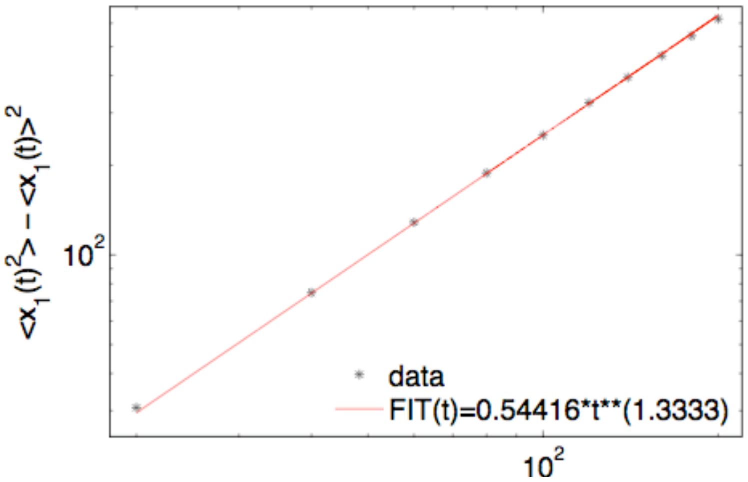
Superdiffusive non-KPZ mode:



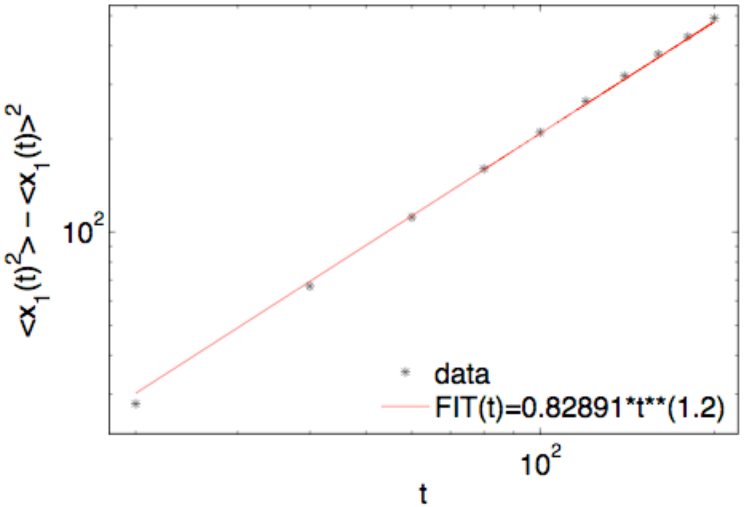
Monte-Carlo simulations (cont')

Measurement of center of mass: $c_1 \approx -0.22$, $c_2 \approx 0.045$ (error < 1%)

Variances:



KPZ mode $2/z_1 = 1.302 \approx 4/3$



Non-KPZ: $2/z_2 = 1.19 \approx 6/5$

Amplitudes: $1/z_2 = 0.58 \approx 3/5$

4. Conclusions

- Onsager type current symmetry without assumption of invariant product measure (rigorous)

  ==> Hyperbolicity of associated system of conservation laws

- Numerical observation of new universality class for fluctuations in strictly hyperbolic two-component systems (z=5/3)

Open question: Universality classes at umbilical points $c_1 = c_2$?