Superdiffusive modes in driven diffusive systems

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joint work with R. Grisi (UFABC) and V. Popkov, J. Schmidt (Cologne)

• Introduction
• Onsager-type current symmetries for DDS [Grisi, GMS]
• Superdiffusive dynamical structure function in DDS [Popkov, Schmidt, GMS]
• Conclusions
1. Introduction

**Bulk-driven Particle systems with several conservation laws:**

- Interacting stochastic particle systems on lattice with biased hopping
  \[\Rightarrow\] Non-reversible Markovian dynamics
  \[\Rightarrow\] Invariant measures not known \textit{a priori}

- A few (!) examples:
  1. Exclusion processes with several species of particles
     (second-class, slow particles, tagged particles, AHR model, ...)
  2. Multispecies zero-range models
  3. Bricklayer model
  4. Multilane exclusion processes

- Rich behaviour, e.g.,
  - Phase transitions (phase separation, spontaneous symmetry breaking)
  - (Conjectured) hydrodynamic equations sensitive to regularization
  - Intricate interplay of shocks and rarefaction waves
  - Universal fluctuations (Diffusive, KPZ)
**Generic lattice gas models**

- 1-d Torus with N sites (labeled by k,l,m,...)
- Finite local state space $S$ (local occupation variable $\omega_k \in S$)
- Particle jumps between two different sites at most M sites apart
- Jump rate depends on configuration up to a distance $M < N/2$

Further definitions:

- Shift operator $\sigma$: $(\sigma \omega)_k = \omega_{k+1}$

- Switch operator $\Theta$:

\[
(\Theta_{k,m}^{\omega',\omega''} \omega)_n = \begin{cases} 
\omega_n; & \text{if } n \neq k, k+m \\
\omega'; & \text{if } n = k \\
\omega''; & \text{if } n = k + m.
\end{cases}
\]

- Jump rate from $\omega$ to $\Theta^{\omega',\omega'',j,m} (\omega)$:

\[
r_m (\sigma^j \omega; \omega', \omega'')
\]
Generator and conserved quantities:

Infinitesimal generator:

\[
\mathcal{L} f(\omega) = \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{\omega', \omega'' \in S} r_m(\sigma^k \omega; \omega', \omega'') \left[ f(\Theta_{k,m}^{\omega',\omega''} \omega) - f(\omega) \right]
\]

Conserved quantities \(\xi^\alpha := \sum_k \xi_k^\alpha(\omega)\)

- Assume at least two conservation laws (particle numbers) \(\xi^\alpha\)

- Irreducibility condition on rates to exclude non-ergodicity for fixed \(\xi^\alpha\) (e.g. no “hidden” conservation laws)
• Microscopic current across bond \((k,k+1)\):

\[
j_k^\alpha(\omega) := - \sum_{n=0}^{M-1} \sum_{m=n+1}^{M} \sum_{\omega', \omega'' \in S} r_m(\sigma^{k-n}_m \omega; \omega', \omega'') [\xi_{k-n}^\alpha (\Theta^{\omega'}_{k-n,m, \omega} - \xi_{k-n}^\alpha(\omega)]
\]

(lengthy but straightforward computation)

• Conservation law \(\implies\) (Discrete) Noether theorem

\[
\mathcal{L} \xi_k^\alpha(\omega) = -j_k^\alpha(\omega) + j_{k-1}^\alpha(\omega)
\]

(Lattice continuity equation)
**Invariant measures:**

Translation invariance and ergodicity for fixed values of conserved particle numbers $\xi^\alpha$

$\Rightarrow$ “canonical” invariant measure $\mu$ is unique and translation invariant

Define “grandcanonical” invariant measures with chemical potentials $\phi^\alpha$

$$
\mu_\phi(\omega) = \mu(\omega) e^{\sum_{k \in \mathbb{Z}_N} \phi \cdot \xi_k(\omega) - G(\phi)},
$$

where

$$
G(\phi) = \log \sum_{\omega \in \Omega} e^{\sum_{k \in \mathbb{Z}_N} \phi \cdot \xi_k(\omega) \mu(\omega)}
$$

Stationary density of particles of type $\alpha$: $\rho^\alpha(\{\phi\}) = <\xi_k^\alpha >_{\phi}$

Stationary current of particle type $\alpha$: $j^\alpha(\{\phi\}) = <j_k^\alpha >_{\phi}$
Time-reversed dynamics and currents:

Time-reversed hopping rates:

\[ r_m^* (\sigma^k \omega; \omega', \omega'') := \mu(\omega)^{-1} \mu(\Theta_{k,m}^{\omega'} \omega) r_m (\sigma^k \Theta_{k,m}^{\omega'} \omega; \omega, \omega_{k+m}) \]

\[ \Rightarrow \text{adjoint process} \]

\[ \mathcal{L}^* f(\omega) = \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{\omega', \omega'' \in S} r_m^* (\sigma^k \omega; \omega', \omega'') [f(\Theta_{k,m}^{\omega'} \omega) - f(\omega)] \]

Adjoint conservation law

\[ \mathcal{L}^* \xi_k^\alpha (\omega) = -j_k^{\alpha*} (\omega) + j_{k-1}^{\alpha*} (\omega) \]

\[ j_{i}^{\alpha*} (\omega) = - \sum_{n=0}^{M-1} \sum_{m=n+1}^{M} \sum_{\omega', \omega'' \in S} r_m^* (\sigma^{k-n} \omega; \omega', \omega'') [\xi_{k-n}^\alpha (\Theta_{k-n,m}^{\omega'} \omega) - \xi_{k-n}^\alpha (\omega)] \]

Notice: \[ j^{\alpha*} = - j^{\alpha} \]
Hydrodynamics under Eulerian Scaling

Study large-scale dynamics under coarse-graining $x = ka$, $t = \tau a$:

==> Law large numbers:
Occupation numbers on lattice $n^\alpha_k(t) \to \rho^\alpha(x,t)$ (Conserved particle densities)

==> Local stationarity:
Microscopic current $j^\alpha_k(t) \to j^\alpha(\{\rho\})$: Associated locally stationary currents

==> Lattice continuity equation $\to$ System of hyperbolic conservation laws

$$\partial_t \rho^\alpha + \partial_x j^\alpha = 0$$

Origin of hyperbolicity: Onsager-type symmetry

$$\frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha}$$

$\phi^\alpha(\{\rho\})$: Fugacities associated with the densities (Legrende transformation)

Proof (Toth, Valko, 2003) for family of models with invariant product measure
2. Onsager-type symmetries

Full proof for generic lattice gas models (Grisi, GMS (2011)):

Lemma 1: For $N > 2M$ we have

$$\langle (\xi_N^\alpha - \rho^\alpha) j_0^\beta \rangle \phi = -\langle (\xi_N^\alpha - \rho^\alpha) j_0^\beta \rangle \phi$$

Proof: Straightforward computation, but requires

(1) no overlap of expression of current (adjoint current) at site $N/2$ with conserved quantity at site 0

(2) for $0 \leq n < m \leq M$ \[ \xi_N^\alpha \left( \Theta_{-n, m} \omega' \omega'' \right) = \xi_N^\alpha (\omega) \]

(guaranteed by condition on interaction range $M < N/2$)
Main result:

Theorem: For finite system with $N$ sites we have

$$
\frac{\partial j^\alpha}{\partial \phi^\beta} - \frac{\partial j^\beta}{\partial \phi^\alpha} = N[\langle j^\alpha_N (\xi^\beta_1 - \rho^\beta) \rangle_\phi - \langle j^\beta_N (\xi^\alpha_0 - \rho^\alpha) \rangle_\phi]
$$

Proof: (a) By construction of grandcanonical measure

$$
\frac{\partial \mu_\phi(\omega)}{\partial \phi^\alpha} = \mu_\phi(\omega) \left[ \sum_{k \in T_N} (\xi^\alpha_k(\omega) - \rho^\alpha) \right]
$$

Therefore with translation invariance:

$$
\frac{\partial j^\beta}{\partial \phi^\alpha} = \sum_{k \in T_N} \langle (\xi^\alpha_k - \rho^\alpha) j^\beta_0 \rangle_\phi
$$
(b) Conservation law, translation invariance, time-reversal:

\[
\langle (\xi_0^\beta - \rho^\beta)(j_1^\alpha - j_0^\alpha) \rangle_\phi = \langle (\xi_k^\beta - \rho^\beta) \mathcal{L}(\xi_0^\alpha - \rho^\alpha) \rangle_\phi \\
= \langle (\xi_0^\alpha - \rho^\alpha) \mathcal{L}^*(\xi_k^\beta - \rho^\beta) \rangle_\phi \\
= \langle (\xi_0^\alpha - \rho^\alpha)(j_{k-1}^{\beta*} - j_k^{\beta*}) \rangle_\phi.
\]

(c) Partial summation and translation invariance:

\[
\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k \langle (\xi_k^\beta - \rho^\beta)(j_1^\alpha - j_0^\alpha) \rangle_\phi \\
= - \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \langle (\xi_0^\beta - \rho^\beta) j_{-k}^\alpha \rangle_\phi + N \langle (\xi_0^\beta - \rho^\beta) j_{\frac{N}{2} - 1}^\alpha \rangle_\phi \\
= - \frac{\partial j^\alpha}{\partial \phi^\beta} + N \langle (\xi_1^\beta - \rho^\beta) j_{\frac{N}{2}}^\alpha \rangle_\phi.
\]
Similarly for time-reversed process

\[
\sum_{k=-N/2+1}^{N/2} k \langle (\xi_0^\alpha - \rho^\alpha) (j_{k-1}^\beta - j_k^\beta) \rangle_\phi = -\frac{\partial j_\beta}{\partial \phi_\alpha} + N \langle (\xi_0^\alpha - \rho^\alpha) j_{N/2}^\beta \rangle_\phi
\]

Lemma 1 completes the proof. \( \square \)

Remark 1: Valid for finite size, no assumption of product measure

Remark 2: Product measure yields Toth/Valko
Corollary 1: For sufficiently fast decaying stationary current-density correlations \(o(1/N)\) one has in the thermodynamic limit \(N \to \infty\) the current symmetry

\[
\frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha}
\]

Remark: Can be written \(S = S^T\) with \(S_{\alpha\beta} = \partial j^\alpha / \partial \phi^\beta\)

Corollary 2: Define

Current Jacobian \(A\) with matrix elements \(A_{\alpha\beta} = \partial j^\alpha / \partial \rho^\beta\)

Compressibility matrix \(C\) with matrix elements

\(C_{\alpha\beta} = 1/N \langle (\bar{\xi}^\alpha - N \rho^\alpha)(\bar{\xi}^\beta - N \rho^\beta) \rangle = \partial \rho^\alpha / \partial \phi^\beta\)

Then \(C A^T = A C\)

3. Superdiffusive structure function in DDS

Go beyond LLN and study fluctuations:

• Dynamical structure function

\[ S_{\alpha\beta}(p,t) = \sum_k e^{ikp} \langle (\xi_k^\alpha(t) - \rho^\alpha)(\xi_0^\beta(0) - \rho^\beta) \rangle = \langle u^\alpha(p,t) u^\beta(-p,t) \rangle \]

where \( u^\alpha(p,t) \) = Fourier transform of locally conserved quantity \( \xi_k^\alpha(t) \)

• One conservation law: Scaling form \( S(p,t) = F(p^z t) \)

• KPZ universality class \( z=3/2 \), universal scaling function \( F \) [Praehofer, Spohn (2002)]

• Several conservation laws: Different universality classes in the same DDS

• Known cases for two-component DDS: (a) Both KPZ (generic)
  (b) KPZ and Diffusive (\( z=2 \))
  [Das et al (2001), Rakos, GMS (2005)]

===> Is that all there is?
- Model: Interacting two-lane TASEPs with densities $\rho_{1,2}$

- Product measure: $r_1 = 1 + \gamma n^{(2)}/2$, $r_2 = b + \gamma n^{(1)}/2$ [Popkov, Salerno (2004)]

- Stationary currents:
  
  \[
  j_1(\rho_1, \rho_2) = \rho_1(1 - \rho_1)(1 + \gamma \rho_2)
  
  j_2(\rho_1, \rho_2) = \rho_2(1 - \rho_2)(b + \gamma \rho_1)
  \]
Nonlinear fluctuating hydrodynamics (non-rigorous):

• Starting point: (Deterministic) hyperbolic system of conservation laws

\[
\frac{\partial}{\partial t} \mathbf{\rho} + A \frac{\partial}{\partial x} \mathbf{\rho} = 0
\]

with density vector and current Jacobian \(A\)

• Stationary solutions: \(\rho_i\)

• Introduce fluctuation fields \(u_i(x,t) = \rho_i(x,t) - \rho_i\) and expand in \(u_i\)
A) **Linear theory:**

- Diagonalize $A$: $RAR^{-1} = \text{diag}(c_i)$, Normalization $RCR^T = 1$

$$\Rightarrow \text{Eigenmode equation } \partial_t \phi_i = -c_i \partial_x \phi_i$$

- Travelling waves (eigenmodes) $\phi_i(x,t) = \phi_i(x-c_it)$

- Characteristic speeds $c_{1,2}(\rho_1, \rho_2) = \text{eigenvalues of current Jacobian } A$

- Strict hyperbolicity for two-lane model: $c_1 \neq c_2 \ \forall \ (\rho_1, \rho_2) \in (0,1) \times (0,1)$

- Microscopic: Stationary center of mass motion of localized perturbation

[Popkov, GMS (2003)]
B) Nonlinear fluctuating theory

- Expand to second order, add phenomenological diffusion term and noise [Spohn]

\[
\partial_t \phi_i = -\partial_x [c_i \phi_i + \langle \vec{\phi}, G^{(i)} \vec{\phi} \rangle - \partial_x (D \vec{\phi})_i + (B \vec{\xi})_i]
\]

diffusion = regularization, noise B and diffusion matrix D related by FDT

- Mode coupling coefficients for eigenmodes

\[
G^{(i)} = \frac{1}{2} \sum_j R_{ij} (R^{-1})^T H^{(j)} R^{-1}
\]

- Hessian \( H^{(\gamma)} \) with matrix elements \( \partial^2 j_\gamma / (\partial \rho_\alpha \partial \rho_\beta) \)
One component: $\partial_t \phi = -\partial_x [c \phi + g \phi^2 - D \partial_x \phi + B \xi]$ (KPZ equation, $g = j''/2$)

Two components $\Rightarrow$ Two coupled KPZ equations

Remarks:

1) Higher order terms irrelevant in RG sense (if second order non-zero)

2) Offdiagonal terms negligible for strictly hyperbolic systems (no overlap between modes)

3) Self-coupling terms $G^{(\alpha)}_{\alpha \alpha}$ leading, other diagonal terms $G^{(\alpha)}_{\beta \beta}$ subleading

$$\partial_t \phi_1 = -\partial_x [c_1 \phi_1 + G^{(1)}_{11} (\phi_1)^2 + G^{(1)}_{22} (\phi_2)^2 + \text{diff. + noise}]$$

$$\partial_t \phi_2 = -\partial_x [c_2 \phi_2 + G^{(2)}_{11} (\phi_1)^2 + G^{(2)}_{22} (\phi_2)^2 + \text{diff. + noise}]$$
Some scenarios:

A) Both self-coupling coefficients nonzero: \( G^{(1)}_{11} \neq 0, G^{(2)}_{22} \neq 0 \)

\[ \Rightarrow \text{two KPZ modes (} z_1 = 3/2, z_2 = 3/2) \]

B) One self-coupling coefficient nonzero, all diagonal terms of other mode-coupling matrix 0, e.g., \( G^{(1)}_{11} \neq 0, G^{(2)}_{22} = G^{(2)}_{11} = 0 \)

\[ \Rightarrow \text{one KPZ mode, one diffusive mode (} z_1 = 3/2, z_2 = 2) \]

C) One self-coupling coefficient nonzero, subleading diagonal of other mode-coupling matrix 0, e.g., \( G^{(1)}_{11} \neq 0, G^{(2)}_{11} \neq 0, G^{(2)}_{22} = 0 \)

\[ \Rightarrow \text{one KPZ mode, second non-KPZ superdiffusive mode (} z_1 = 3/2, z_2 = 5/3) \]

Remark: Heat mode with \( z = 5/3 \), two KPZ sound modes in Hamiltonian dynamics with three conservation laws [van Beijeren (2012)]
Monte-Carlo simulations

Measure dynamical exponents $z_i$

1) Average over $10^7 - 10^8$ runs with uniform random initial conditions with densities $\rho_i$

2) Excite each mode independently at site $k=N/2$ at $t=0$ and measure dynamical structure function of each mode

3) Compute center of mass motion $<X_i(t)>$ of excitation $\Rightarrow c_i t$

4) Compute CM variances $V_i(t)$:
   Scaling hypothesis $V_i(t) \sim t^{2/z_i} \Rightarrow z_i$
   (no assumption on existence for asymptotic scaling function)

5) Measure amplitudes $A_i(t)$ at maximum:
   Mass conservation $A_i(t) \sim 1/t^{1/z_i} \Rightarrow z_i$
Choose equal densities $\rho_1 = \rho_2 = \rho$, and interaction strength $\gamma = 1$

Set $b = 2$ (inequivalent lanes)

$$
\begin{align*}
G^{(1)}_{11} &= -2g(6\rho^4 - 8\rho^3 + 5\rho^2 + \rho -1) \\
G^{(1)}_{12} &= G^{(1)}_{21} = g(4\rho^3 - 10\rho^2 + 8\rho -1) \\
G^{(1)}_{22} &= -2g\rho(1-\rho)(2\rho^2 - 6\rho +3)
\end{align*}
$$

$$
g = -1/2 \{(\rho(1-\rho)) / [2\rho^2 - 2\rho +1]^3\}^{1/2}$$

$$
\begin{align*}
G^{(2)}_{11} &= 4g\rho(1-\rho) \\
G^{(2)}_{12} &= G^{(1)}_{21} = -g(1 - 2\rho^2)^2 \\
G^{(2)}_{22} &= 4g(3\rho^2 - 3\rho +1)
\end{align*}
$$

===> Generically case A (two KPZ modes) with $c_1 \neq c_2$ and $z_{1,2} = 3/2$

Good agreement with Monte-Carlo data for $c_{1,2}$ and $z_{1,2}$
Monte-Carlo simulations (cont')

\[ \rho_1 = \rho_2 = 0.5, \quad \gamma = -0.8, \quad b = 1 \text{ (symmetry between lanes)} \Rightarrow c_1 = 0.2, \quad c_2 = 0.2 \]

Mode coupling matrices:

\[ G^1 = 0.2121 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G^2 = 0.2121 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \Rightarrow \text{KPZ with } z_1 = 3/2 \text{ (mode 1) and diffusive with } z_2 = 2 \text{ (mode 2)} \]

Monte-Carlo data for diffusive mode:

Error in velocities < 1%

Variance: \[ 2/z_1 = 1.343 \approx 4/3 \]

\[ 2/z_2 = 1.030 \approx 1 \]
\[ \rho_1 = \rho_2 = \rho, \gamma = -0.52588, b = 1.3 \]

\[ G^{(2)}_{22} = 0 \text{ at } \rho^* = 0.55000..., G^{(2)}_{11}(\rho^*) \neq 0, G^{(1)}_{11}(\rho^*) \neq 0 \]

\[ \Rightarrow \text{At } \rho^* \text{: KPZ with } z_1 = 3/2 \text{ (mode 1) and superdiffusive with } z_2 = 5/3 \text{ (mode 2)} \]

Monte-Carlo data:

**KPZ mode:**

**Superdiffusive non-KPZ mode:**
Monte-Carlo simulations (cont’)

Measurement of center of mass: \( c_1 \approx -0.22, c_2 \approx 0.045 \) (error < 1%)

Variances:

KPZ mode \( 2/z_1 = 1.302 \approx 4/3 \)

Non-KPZ: \( 2/z_2 = 1.19 \approx 6/5 \)

Amplitudes: \( 1/z_2 = 0.58 \approx 3/5 \)
4. Conclusions

- Onsager type current symmetry without assumption of invariant product measure (rigorous)

  ==> Hyperbolicity of associated system of conservation laws

- Numerical observation of new universality class for fluctuations in strictly hyperbolic two-component systems ($z=5/3$)

Open question: Universality classes at umbilical points $c_1 = c_2$?