# Deterministic walks on a square lattice

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# Contents

## Deterministic walks

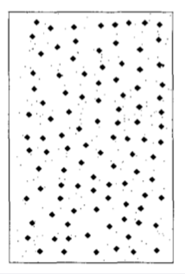
- Lorentz gas.
- A walker on a landscape.
- The walker interacts with the landscape during the walk.
- Landscape = $2d$ square lattice with obstacles.
- Complex system.
- Simple model of anomalous transport.

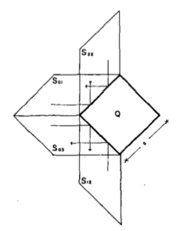
H. A. Lorentz, Proc. Amst. Acad. **7** 438 (1905).

E. G. D. Cohen, L. Bunimovich, J. P. Boon, X. P. Kong, P. M. Binder, H-F. Meng.

# Introduction

## The Ehrenfest's wind-tree model (1911)



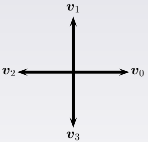


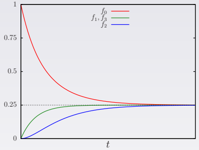

$$\frac{df_i}{dt} = k(f_{i+1} + f_{i-1} - 2f_i), \qquad i = 0, \ldots 3$$

$$f_0(0) = 1, \; f_1(0) = f_2(0) = f_3(0) = 0, \quad \sum_i f_i = 1,$$

$$f_0 = \frac{1}{4}\left[1 + e^{(-2kt)}\right]^2, \quad f_1 = f_3 = \frac{1}{4}\left[1 + e^{(-2kt)}\right]\left[1 - e^{(-2kt)}\right],$$

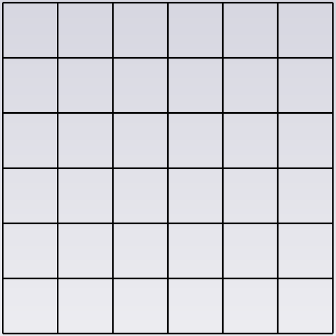
$$f_2 = \frac{1}{4}\left[1 - e^{(-2kt)}\right]^2.$$



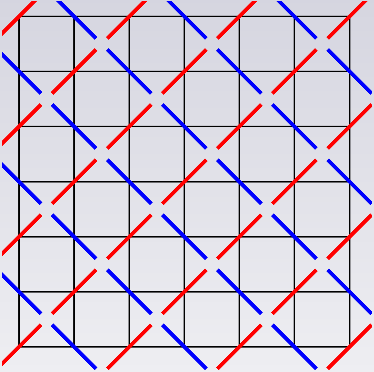

P. Ehenfest, T. Ehrenfest, Begriffliche Grundlagen der Statistische Auffassung in der Mechanik, *Encyklopädie der Mathematische Wissenschaften vol. 4 pt 32 (Leipzig: Teubner)*, 1911. Engl. Trans. M. J. Moravcsik, *The Conceptual Foundations of the Statistical Approach in Mechanics*, Ithaca, Cornell University Press, 1959. R. Rechtman, A. Salcido, A. Calles, EPL **12** 27 (1991).

# Introduction



X. P. Kong, E. G. D. Cohen, Phys. Rev. B **40**, 4838 (1989). Th. Ruijgrok, E. G. D. Cohen, Phys. Lett. A, **133** 415 (1988). H-F. Meng, E. G. D. Cohen, Phys. Rev. E **50** 2482 (1994).

# Introduction



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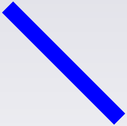
**flipping mirror landscape**

right mirror

left mirror

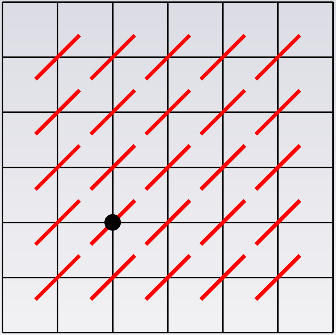


**flipping rotor landscape**

right rotor

left rotor

**flipping mirror landscape**



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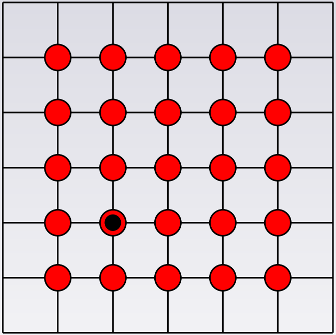

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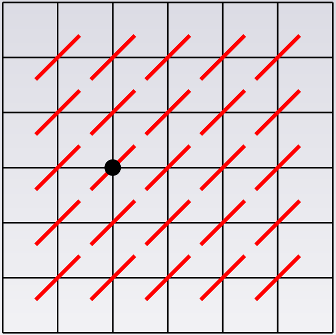

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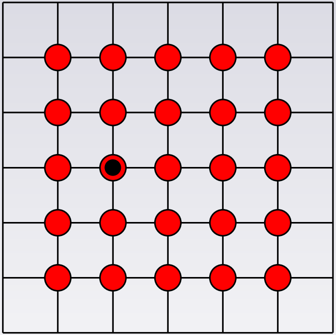

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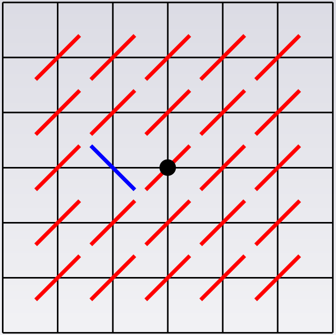

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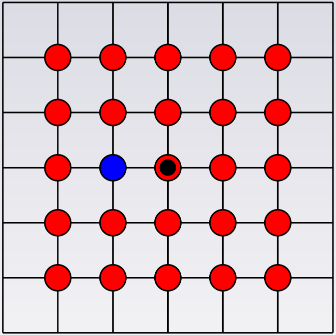

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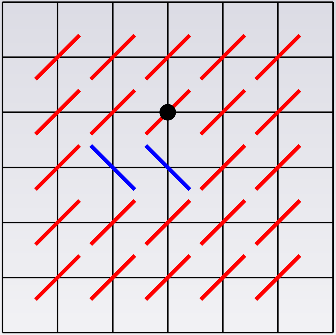

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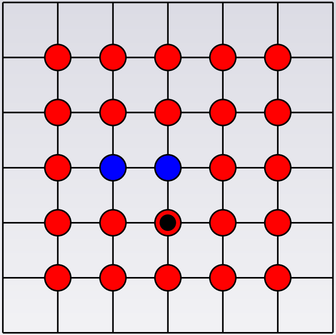

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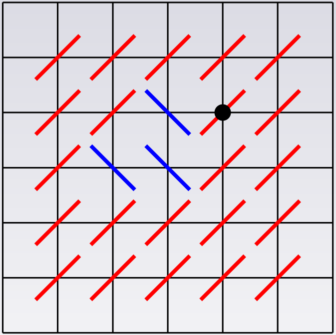

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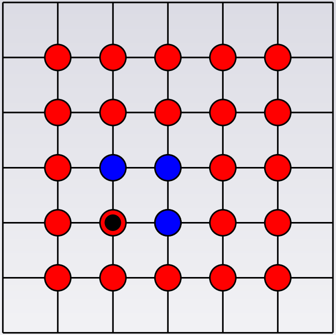

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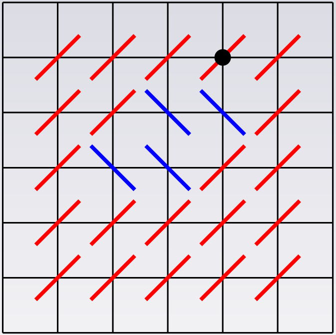

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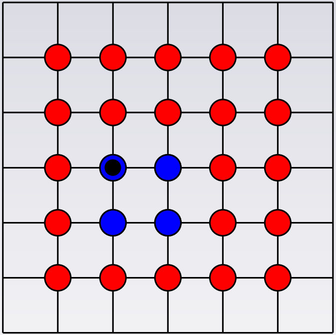
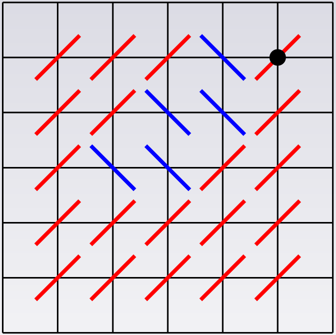
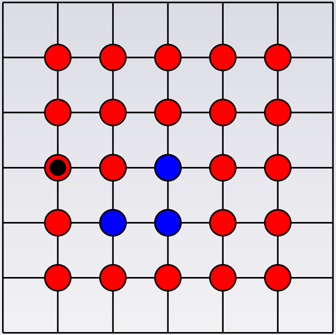


**flipping mirror landscape** **flipping rotor landscape**





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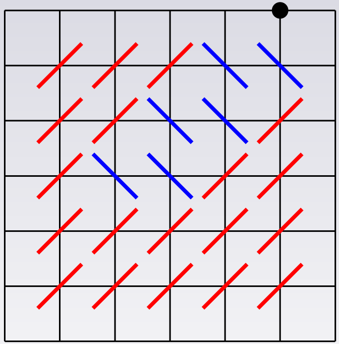
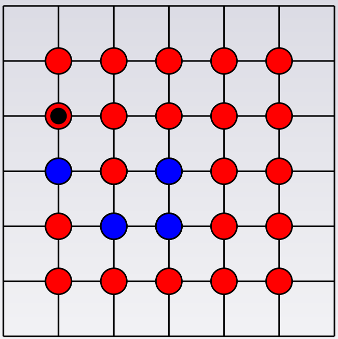
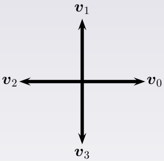


**flipping rotor landscape**



# Introduction

A walker moves on a $2D$ square lattice, the landscape, in discrete time steps to a nearest neighbor site according to the landscape. In so doing, he alters the landscape locally. At time $t$ the walker is at $(x, y)$ with one of four velocities $\mathbf{v}_0 = (1, 0)$, $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (-1, 0)$, or $\mathbf{v}_3 = (0, -1)$. The state of the landscape, $m(x, y)$, is either 1 or -1 and after the walker passes, $m$ changes sign. The landscape is made of flipping rotors or flipping mirrors. In the first case, the particle turns right or left according to $m(x, y)$, and in the second one, the particle is reflected by a "mirror" with an inclination of $45^\circ$ or $135^\circ$.




**flipping mirror landscape**

$$m(x, y) = \begin{cases} 1 & \text{walker reflects from a mirror at } 45^\circ \\ -1 & \text{walker reflects from a mirror at } 135^\circ \end{cases}$$

**flipping rotor landscape**

$$m(x, y) = \begin{cases} 1 & \text{walker rotates } 90^\circ \text{ to the right} \\ -1 & \text{walker rotates } 90^\circ \text{ to the left} \end{cases}$$

**flipping mirror landscape**

$$
\begin{aligned}
v_x' &= m v_y \\
v_y' &= +m v_x \\
m' &= -m \\
x' &= x + v_x' \\
y' &= y + v_y'
\end{aligned}
$$

**flipping rotor landscape**

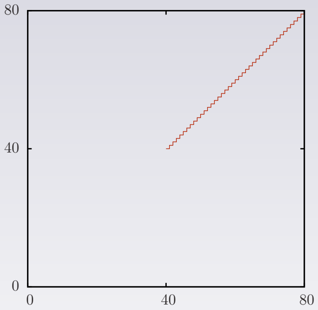
$$
\begin{aligned}
v_x' &= m v_y \\
v_y' &= -m v_x \\
m' &= -m \\
x' &= x + v_x' \\
y' &= y + v_y'
\end{aligned}
$$

The primed (unprimed) quantities refer to $t+1$ ($t$).

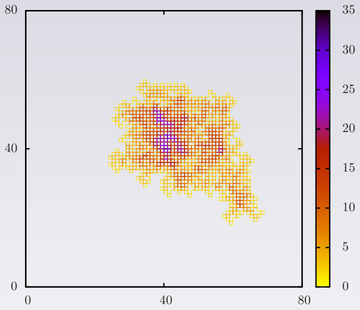
## Introduction

At $t = 0$, $m(x, y) = 1 \; \forall \, x, y$ and the walker is in the center of the lattice with $\mathbf{v} = \mathbf{v}_1$.

**flipping mirror landscape**




**flipping rotor landscape**

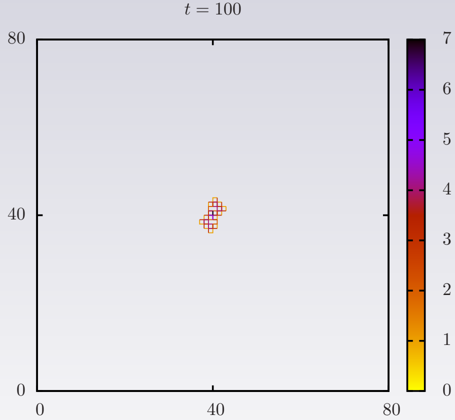



The walker moves alternatively one step vertically, one horizontally.

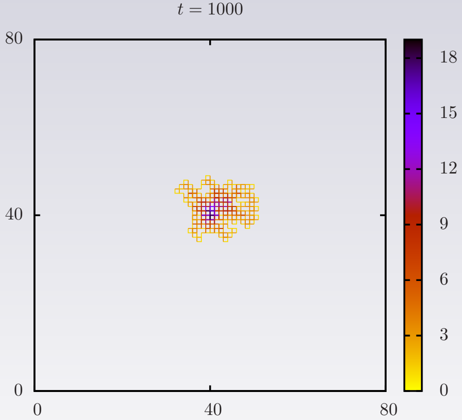
The walker has moved during 9,000 time steps. The colors show the number of times each site has been visited.

# Contents

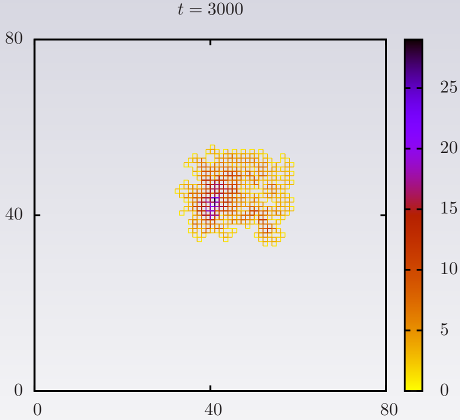
# Initially ordered flipping rotor landscape



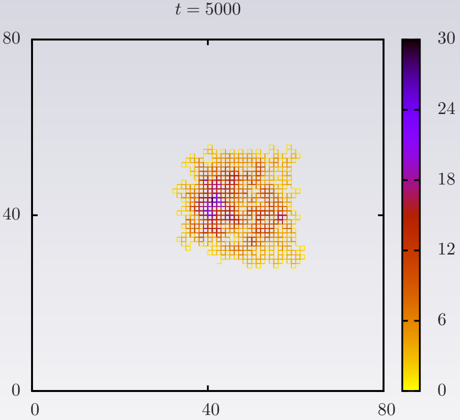
# Initially ordered flipping rotor landscape



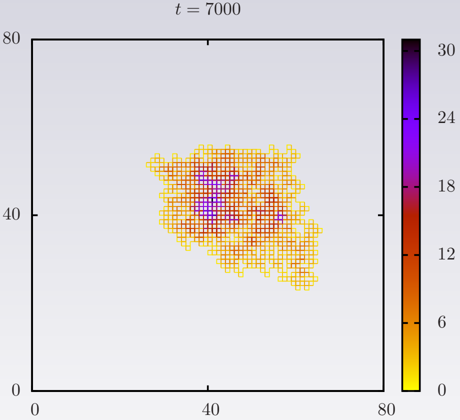
# Initially ordered flipping rotor landscape



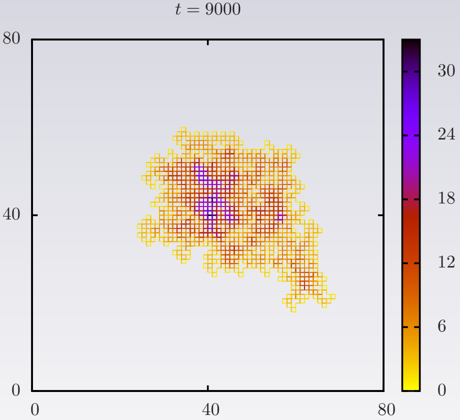
# Initially ordered flipping rotor landscape



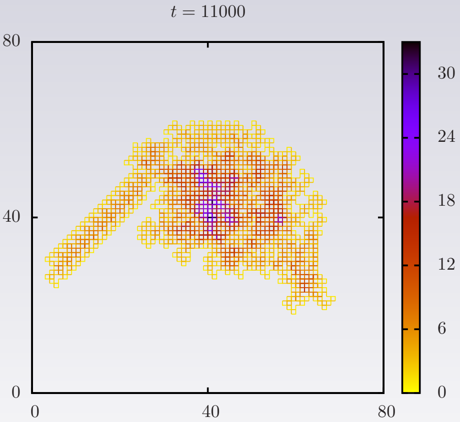
# Initially ordered flipping rotor landscape



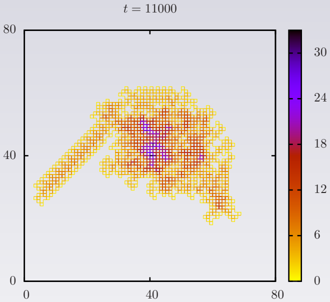
# Initially ordered flipping rotor landscape



# Initially ordered flipping rotor landscape



At $t = 0$, $m(x, y) = 1\ \forall(x, y)$ and the walker is in the center of the lattice with $\mathbf{v} = \mathbf{v}_1$.

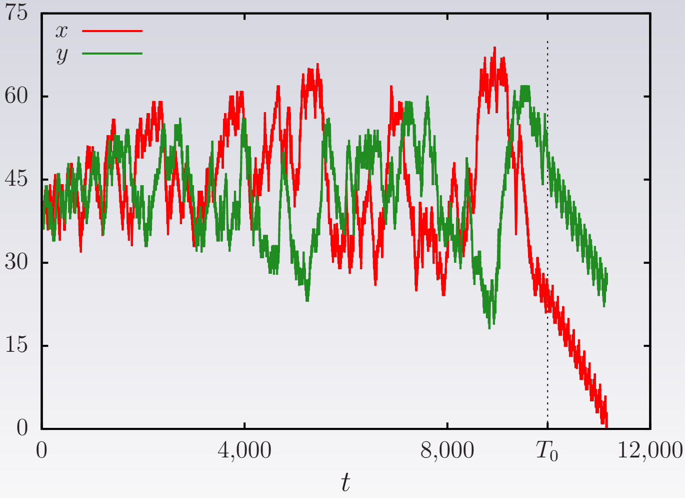



$$
\begin{aligned}
v_x' &= m v_y \\
v_y' &= -m v_x \\
m' &= -m \\
x' &= x + v_x' \\
y' &= y + v_y'
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}' &= \mathbf{v}_{k-m} \\
m' &= -m \\
\mathbf{r}' &= \mathbf{r} + \mathbf{v}'
\end{aligned}
$$

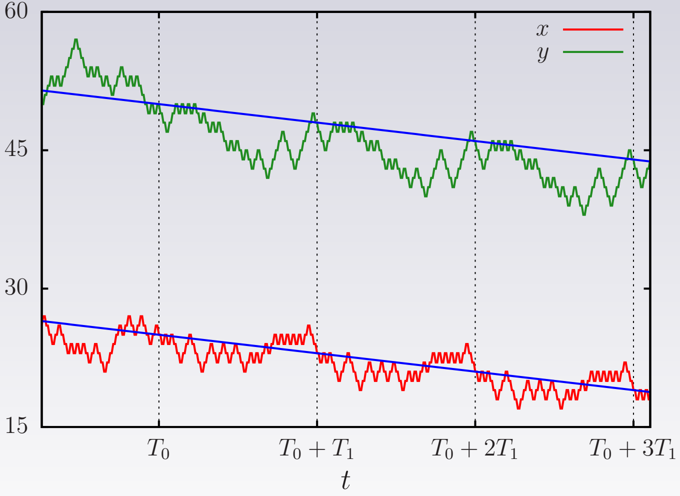
After almost 10,000 time steps, $T_0$, the walker begins to move periodically. Every 100 or so time steps, $T_1$, it moves 2 sites horizontally and 2 vertically.

$T_0 = 9,977$. For $t > T_0$ the particle moves periodically with period $T_1 = 104$.

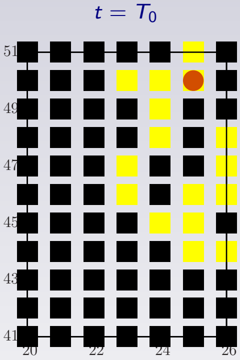


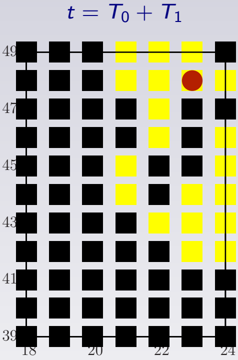
# Initially ordered flipping rotor landscape

For $t > T_0$, the walker moves periodically with period $T_1 = 104$ and $x$ and $y$ diminish by 2 with a speed $u = 2\sqrt{2}/104$.



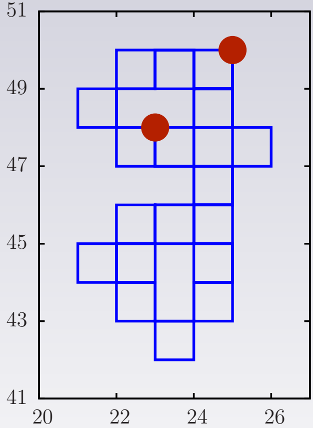

The two straight lines have slope $-2/104$.



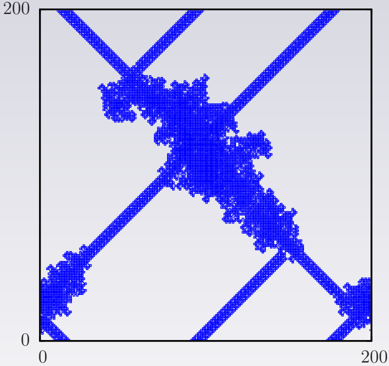

At $t = T_0$ the walker is at the site marked by the red circle (left Fig.) with $\mathbf{v} = \mathbf{v}_2$. At $t = T_0 + T_1$ the walker is at the site marked by the red circle (right Fig.) with $\mathbf{v} = \mathbf{v}_2$. The walker moved two sites to the left and two down. In doing so the walker prepared the landscape in such a way that its motion becomes periodic. The state of the rotors of the two Figs. are the same, except on the top row and the right column, but these sites are not visited by the particle as shown in the next Fig.

# Initially ordered flipping rotor landscape




Trajectory of the walker between $t = T_0$ and $t = T_1$ to be compared with the previous Figs. At $t = T_0$ the walker is in $(25, 50)$, the upper right red circle, and at $t = T_0 + T_1$, the walker is in $(23, 48)$, the lower left circle.

# Initially ordered flipping rotor landscape with periodic boundary conditions



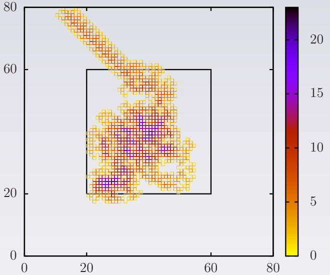

After $T_0$ time steps the walker moves periodically along a diagonal,reaches a border, enters on the opposite one. It eventually goes back to the central part of the lattice and after some time it again moves periodically. This goes on and on. The total time is $T = 80,000$.
This behavior suggests that the walker will move periodically if there is a sufficiently large region with ordered rotors.

# Contents

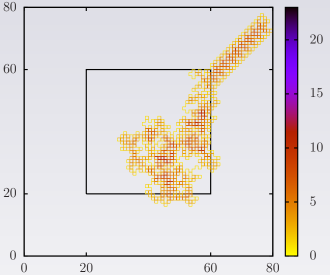
# Partially ordered flipping rotor landscape

At $t = 0$, $m(x, y) = 1$ (right rotor), with $0 \leq x < 80$, $0 \leq y < 80$. Inside the small box, $20 \leq x < 60$, $20 \leq y < 60$, $m(x, y) = -1$ (left rotor) with probability $q$. The landscape is initially disordered inside the small box and ordered outside of it.

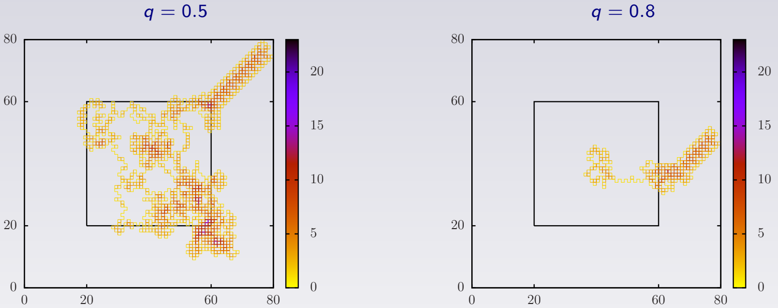
$q = 0.01$



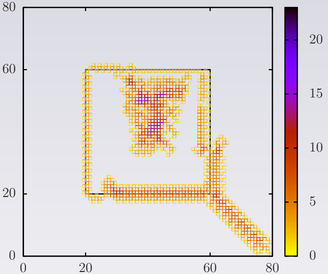

$q = 0.2$



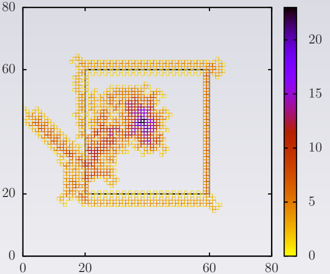
# Partially ordered flipping rotor landscape



$q = 0.99$




$q = 1.0$

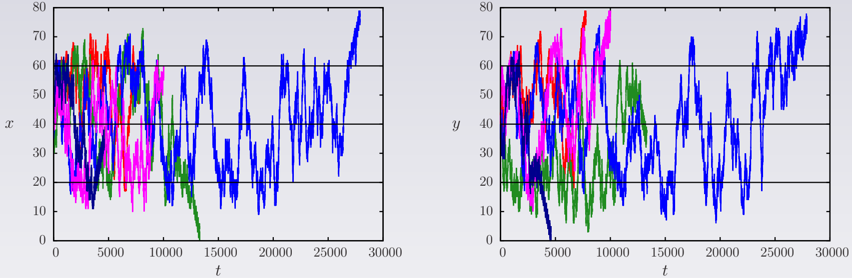



As

long as the walker finds an ordered landscape he will move periodically with period $T_1$.

# Partially ordered flipping rotor landscape

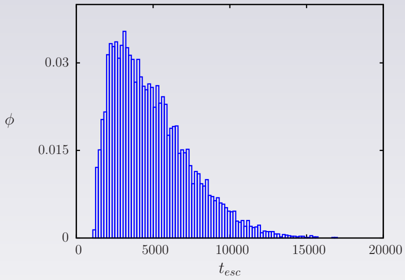
$p = q = 0.5$




The escape time $t_{esc}$ is the time when $x$ or $y$ cross one of the red lines.

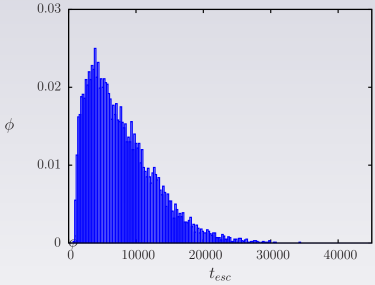
Distribution of escape times, $\phi$ for diffferent values of $p$. For every value of $q$, $\phi$ is the result of 10,000 simulations.

$q = 0.01$




$M(t_{esc}) = 6,954, \quad D(t_{esc}) = 3,922$

$q = 0.20$



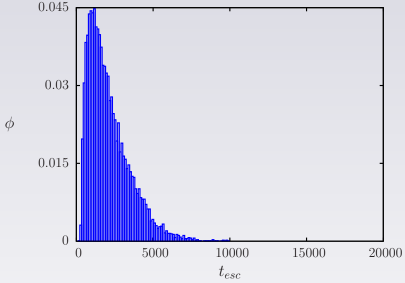

$M(t_{esc}) = 7,200, \quad D(t_{esc}) = 3,876$

$M$ is the median and $D$ the average absolute deviation.

# Partially ordered flipping rotor landscape

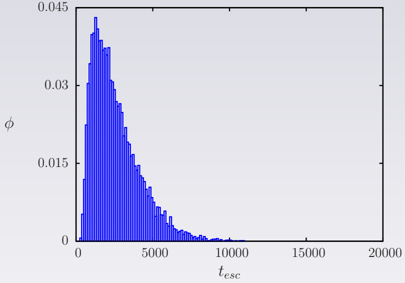
Distribution of escape times, $\phi$ for diffferent values of $p$. For every value of $p$, $\phi$ is the result of 10,000 simulations.

$q = 0.50$




$M(t_{esc}) = 7,584, \quad D(t_{esc}) = 4,080$

$q = 0.80$




$M(t_{esc}) = 8,874, \quad D(t_{esc}) = 4,088$

# Contents

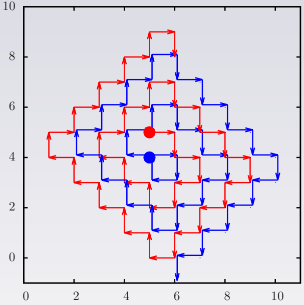
# Two walkers on an initially ordered flipping rotor landscape

Antonio Prohias, Spy *vs* Spy, Mad magazine, January 1960 to March 1987.

# Two walkers on an initially ordered flipping rotor landscape

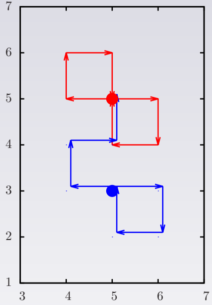
Walker Red is chased by walker Blue. Initially they are a distance $d$ apart, both with the same velocity, $\mathbf{v}_0$. The initial positions of the walkers are marked by the filled circles.

$d = 1$




Blue follows Red in square spirals

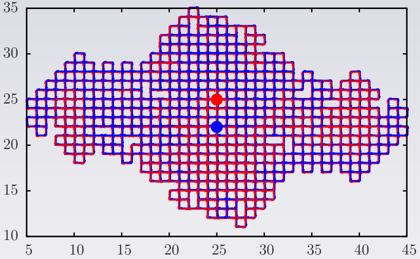
$d = 2$




At $t = 8$ Blue catches Red in the red circle

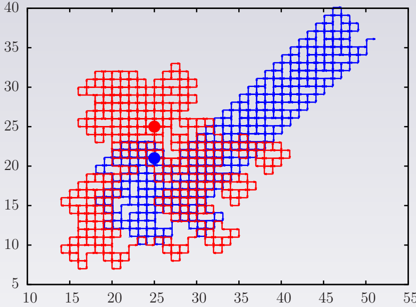
# Two walkers on an initially ordered flipping rotor landscape

$d = 3$




Blue never catches Red

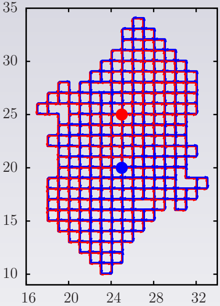
$d = 4$




Blue forgets about Red

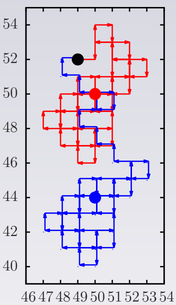
# Two walkers on an initially ordered flipping rotor landscape

$d = 5$




Blue never catches Red

$d = 6$




At $t = 84$ Blue catches Red in the black circle

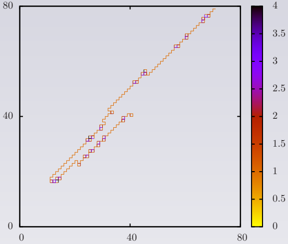
- If Blue is after Red, his best strategy is to be two sites away, the next best one is to be 6 sites away.
- For $d$ odd Blue "never" catches Red. The patterns have some symmetry.

# Contents

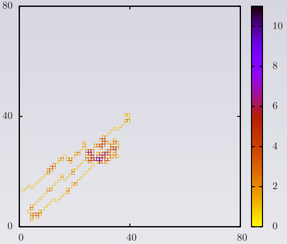
# Iinitially disordered flipping mirror landscape

Initially $m(\mathbf{r}) = 1$ (right mirror) with probability $p$ and $m(\mathbf{r}) = -1$ (left mirror) with probability $q = 1 - p$, with $0 \leq x < L$, and $0 \leq y < L$.

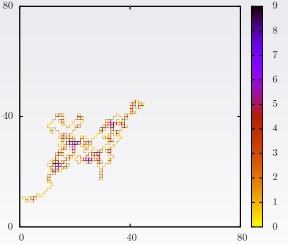
$p = 0.9, \ T = 334$



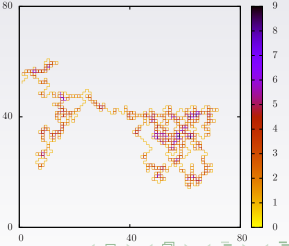

$p = 0.8, \ T = 659$




$p = 0.7, \ T = 1,047$



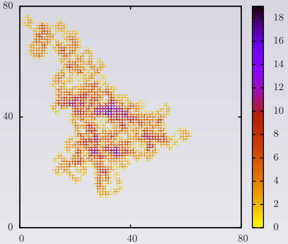

$p = 0.5, \ T = 2,319$



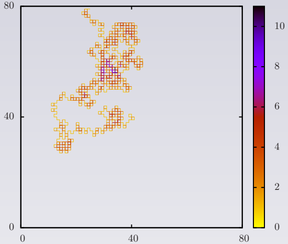
# Iinitially disordered flipping rotor landscape

A landscape of side $L$. Initially $m(\mathbf{r}) = 1$ (right rotor) with probability $p$ and $m(\mathbf{r}) = -1$ (left rotor) with probability $0 \leq x < L$, and $0 \leq y < L$, with $\mathbf{r} = (x, y)$, $x, y \in \mathbb{N}$,

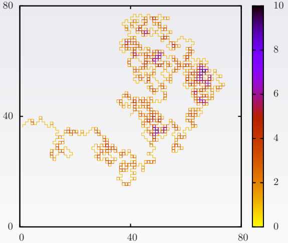
$p = 0.9, \ T = 7,675$



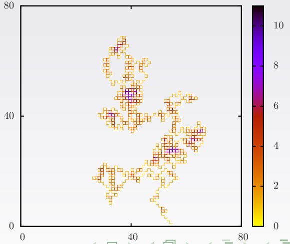

$p = 0.8, \ T = 7,174$




$p = 0.7, \ T = 3,011$




$p = 0.5, \ T = 2,238$

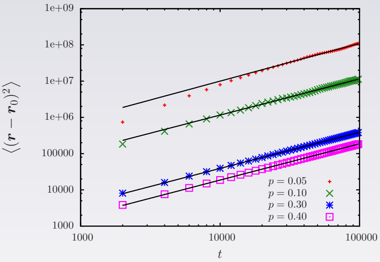


# Initially disordered landscape

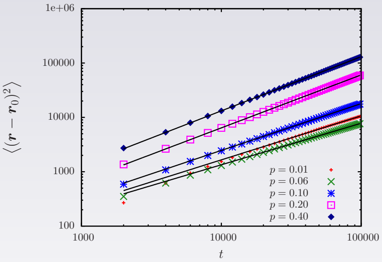
$$\left\langle (\mathbf{r}-\mathbf{r}_0)^2 \right\rangle_p = 2dDt^{\alpha}, \qquad d = 2 \tag{1}$$

$$\left\langle (\mathbf{r}-\mathbf{r}_0)^2 \right\rangle_p = \left\langle (\mathbf{r}-\mathbf{r}_0)^2 \right\rangle_{1-p}$$

**flipping mirror landscape**




**flipping rotor landscape**

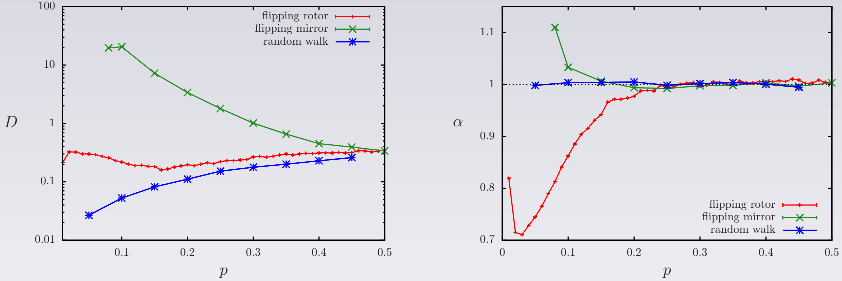



$\langle \cdot \rangle_p$ is the average over $N$ samples of initial landscapes with a fraction $p$ of right mirrors. $N = 100,000$, $T = 100,000$, and $L$ suficiently large. Two exceptions: in the flipping mirror landscape, for $p = 0.05$ and $p = 0.10$, $N = 1,000$. The fit of Eq. (1) to the data is for $10,000 \leq t \leq 100,000$.

# Initially disordered landscape

For comparison we also consider a random walker that at every site turns right with probability $p$ and left with probability $q$.

$$\left\langle (\mathbf{r} - \mathbf{r}_0)^2 \right\rangle_p = 2dDt^{\alpha}, \qquad d = 2$$




- $D$ and $\alpha$ are taken from the best fits for $t > 10,000$.
- Note logarithmic vertical scale for $D$.
- Subdiffusion ($\alpha < 1$) on the flipping rotor landscape for $0 < p <\lesssim 0.3$ and $0.7 \lesssim p < 1.0$.
- Superdiffusion ($\alpha > 1$) on the flipping mirror landscape for $0 < p \lesssim 0.15$ and $0.85 \lesssim p < 1$.

The error of the fits is smaller than the size of the points of the graphs

# Initially disordered landscape

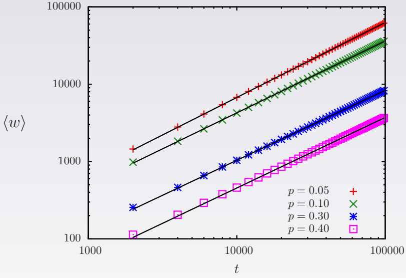
$$w(t) = \sum_{s=0}^{t} m(x(s), y(s)) = n_l(t) - n_r(t)$$

$$\langle w(t) \rangle_p = Bt^{\beta} \tag{2}$$

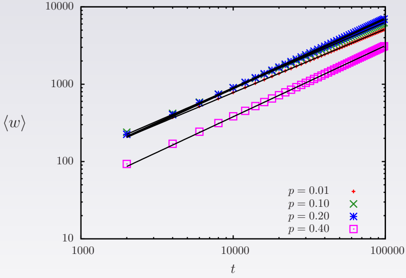
$$\langle w(t) \rangle_p = -\langle w(t) \rangle_{1-p}$$

$n_l(t)$ ($n_r(t)$) are the number of left (right) turns of the walker after $t$ time steps.

**flipping mirror landscape**

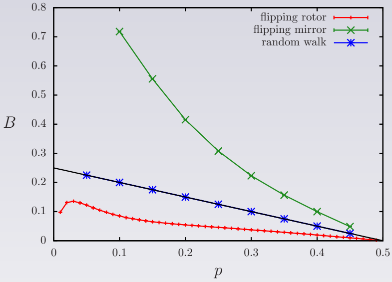


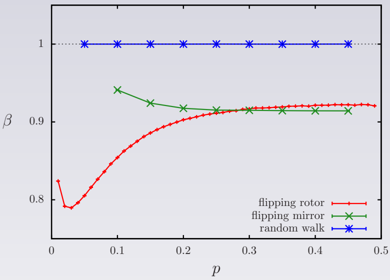

**flipping rotor landscape**



# Initially disordered landscape

$$\langle w(t) \rangle_p = Bt^{\beta}$$
$$\langle w(t) \rangle_p = -\langle w(t) \rangle_{1-p}$$



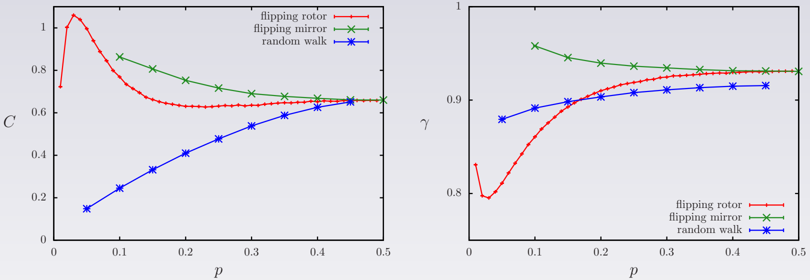

For random walks $B = (1-2p)/4$, shown in black in the Fig. on the left, and $\beta = 1$, Fig. on the right. $B \to 0$ as $p \to 1/2$ due to the symmetry of $\langle w \rangle$. $T = 100,000$, $N = 100,000$, and $L$ sufficiently large. The fit of Eq. (2) to the data is for $0 \leq t \leq 100,000$.

# Initially disordered landscape

At time $t$, a walker has visited $N_s$ sites.

$$\langle N_s(t) \rangle_p = Ct^{\gamma} \tag{3}$$

$$\langle N_s(t) \rangle_p = \langle N_s(t) \rangle_{1-p}$$



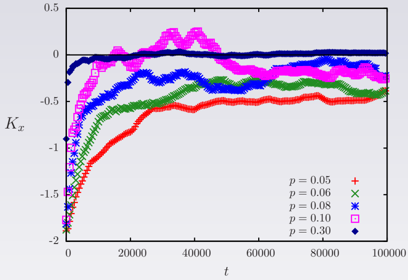

$T = 100,000$, $N = 100,000$, and $L$ sufficiently large. The fit of Eq. (4) to the data is for $10,000 \le t \le 100,000$.

# Initially disordered landscape

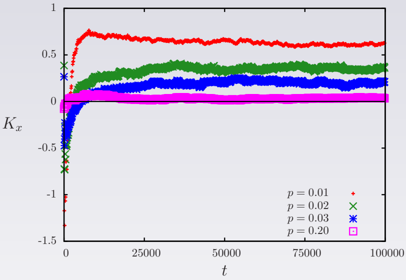
The one dimensional kurtosis $K_x$ is defined by

$$K_x = \frac{\langle (x - x_0)^4 \rangle - 3\langle (x - x_0)^2 \rangle}{\langle (x - x_0)^2 \rangle^2}$$

**flipping mirror landscape**



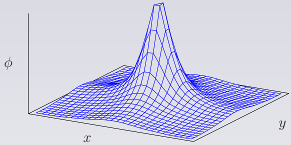

**flipping rotor landscape**



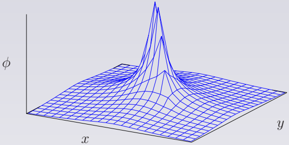

$T = 100,000$, $N = 100,000$, and $L$ sufficiently large. The fit of Eq. (4) to the data is for $10,000 \leq t \leq 100,000$.

$\phi(x, y, t)\Delta x \Delta y$ is the probability of finding a walker at $(X, Y)$ with $x < X < x + \Delta x$ and $y < Y < y + \Delta y$ at time $t$.

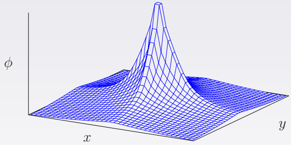
**flipping mirror landscape**




**flipping rotor landscape**



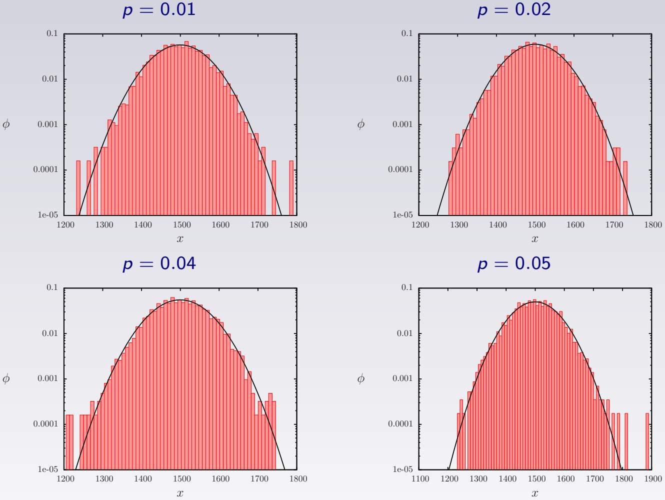

**random walk**




$p = 0.20$, $T = 10,000$, $N = 20,000$ and $L$ suficiently large.

# Initially disordered flipping rotor landscape

$\phi = \phi(x, L/2, T)$

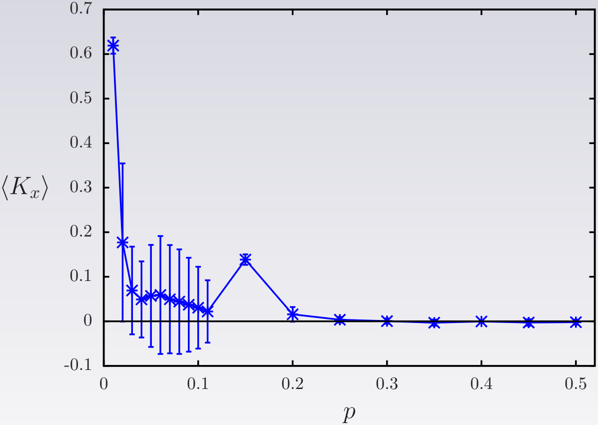



$T = 200,000$, $N = 100,000$, $L = 3,000$, and $\Delta x = 9$. Normal distribution $\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x - \langle x \rangle)^2}{2\sigma^2}\right)$ black.

# Initially disordered landscape

From the previous results, $\langle K_x \rangle$ is the average of $K_x$ after a transient that is taken as one half the final time.

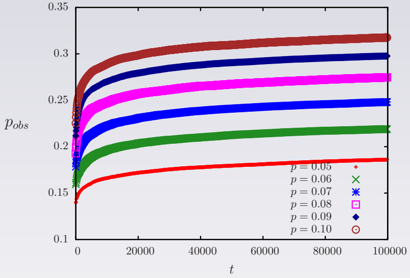
**flipping rotor landscape**




$T = 100,000$, $N = 100,000$, and $L$ sufficiently large.

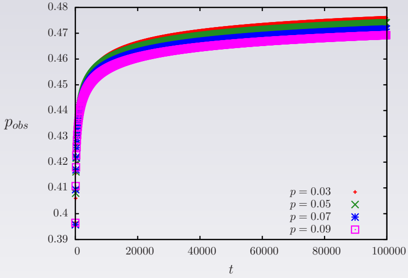
The "observed probability" $p_{obs}$ is the average fraction of right obstacles the wallkers encounter.

**flipping mirror landscape**




$N = 1,000$

**flipping rotor landscape**




$N = 100,000$

# Initially disordered landscape

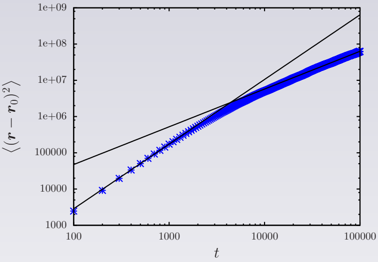
The "observed probability" $p_{obs}$ is the average fraction of right obstacles the wallkers encounter.

| **flipping mirror landscape** | | **flipping rotor landscape** | |
|---|---|---|---|
| $p_{obs}(t=0)$ | $p_{obs}(t=10^5)$ | $p_{obs}(t=0)$ | $p_{obs}(t=10^5)$ |
| 0.30 | 0.458106 | 0.01 | 0.474162 |
| 0.40 | 0.481395 | 0.10 | 0.468232 |
| | | 0.20 | 0.464660 |
| | | 0.30 | 0.471276 |
| | | 0.40 | 0.484226 |

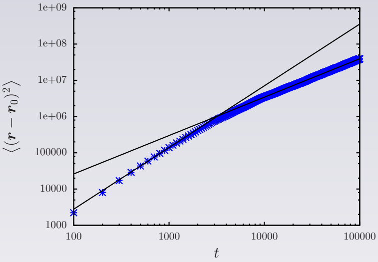
# Initially disordered landscape

**flipping mirror landscape**

$p = 0.06$



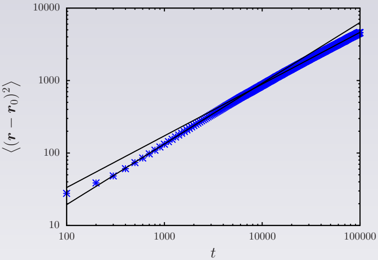

$p = 0.07$

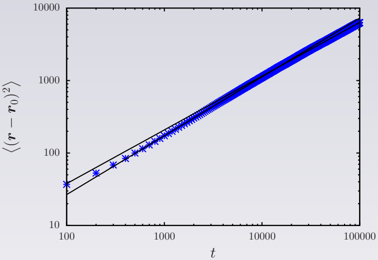



| $p$ | $t_0$ | $D$ | $\alpha$ | $t_1$ | $D$ | $\alpha$ |
|---|---|---|---|---|---|---|
| 0.06 | 10,000 | 0.193 | 1.784 | 10,000 | 97.669 | 1.042 |
| 0.07 | 1,000 | 0.273 | 1.702 | 10,000 | 49.559 | 1.06 |

Two scalings, one for $t < t_0$, the other one for $t_1 < t$. $T = 10,000$, $N = 1,000$. For $p = 0.06$, $L = 40,000$ and for $p = 0.07$, $L = 35,000$.

**flipping rotor landscape**




| $p$ | $t_0$ | $D$ | $\alpha$ | $t_1$ | $D$ | $\alpha$ |
|---|---|---|---|---|---|---|
| 0.03 | 3,000 | 0.102 | 0.838 | 10,000 | 0.311 | 0.714 |
| 0.05 | 3,000 | 0.159 | 0.812 | 10,000 | 0.311 | 0.741 |

Two scalings, one for $t < t_0$, the other one for $t_1 < t$. $T = 10,000$, $N = 100,000$, and $L = 5,000$.

# Contents

## Concluding remarks

### A simple example of a complex system.

- A walk on an initial ordered rotor landscape.
- A walk on a partially ordered rotor landscape.
- Two walkers on an initially ordered landscape.

### A model for anomalous transport

$$\left\langle (\mathbf{r} - \mathbf{r}_0)^2 \right\rangle = 2dDt^{\alpha}$$

- Crowded biological media.
- Polymeric networks.
- Porous materials.
- Cytoskeletal fibers and molecular motors.

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# Thanks!!

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