Four Dimensional Renormalization

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HP2.5@GGI, September 5th 2014

The Four Dimensional Renormalization philosophy

1. The **Ward-Identities (WI)** of a **QFT** can be written down in terms of **graphical identities among Green's functions**

   $\Longrightarrow$ relations among amplitudes can be demonstrated **algebraically** directly at the level of the Feynman rules and diagrams of the **QFT**

2. When loops are present, **WI** work at the **integrand** level

   provided the loop integration is shift invariant and *algebraic manipulations of the integrands hold*

3. If **algebra** and **shift invariance** are preserved

   $\Longleftrightarrow$ **WI** (and all symmetries) of the **QFT** are preserved

4. The **FDR** approach to **QFT** **defines** a **four-dimensional and finite (UV-free) loop-integration** in a way compatible with shift and gauge invariance

5. This is achieved by

encoding the UV subtraction
in the definition of the loop integrals

6. Then, correct gauge invariant results emerge once the theory is fixed in terms of physical observables by means of a **finite global** renormalization (which relates the parameters in the Lagrangian $\mathcal{L}$ to measured quantities)

Advantages of FDR (versus DR)

1. Four-dimensional and finite (suitable for numerical approaches)
2. Order-by-order renormalization avoided (**No counterterms** because $\mathcal{L}$ **untouched**)
3. ℓ-loop integrals are directly re-usable in $(\ell{+}1)$-loop calculations, with no need of further expanding in ϵ
4. **Soft and collinear** divergences dealt with within the same **four-dimensional** framework used to cope with the UV infinities
5. It allows an interpretation of **non-renormalizable theories** in which **predictivity** is restored at arbitrarily large loop orders

R. P., **arXiv:1208.5457** (first paper)

A. M. Donati and R. P., **arXiv:1302.5668** (1-loop EW)

R. P., **arXiv:1305.0419** (non-renormalizable theories)

R. P., **arXiv:1307.0705** (massless QCD)

A. M. Donati and R. P., **arXiv:1311.5500** (2-loop)

R. P., **arXiv:1408.5345** (integration-by-parts identities)

Outline

1. The FDR integration
2. **Bottom-up**: Use of FDR in renormalizable QFTs
3. Physical interpretation
4. **Top-down**: Non-renormalizable QFTs

- Take the **integrand** of an ℓ-loop function

$$J(q_1,\ldots,q_\ell) = [J_{\rm INF}(q_1,\ldots,q_\ell)] + J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- To avoid the occurrence of infrared divergences due to this separation

$$+i0 = -\mu^2$$

 in propagators and $\mu \to 0$ **outside** integration

- The divergent loop **integrands** in $[J_{\rm INF}(q_1,\ldots,q_\ell)]$ allowed to depend on μ, **but not on physical scales**

$$\Rightarrow \textbf{physics in } J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- The FDR integral over $J(q_1,\ldots,q_\ell)$ is **defined** as

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1,\ldots,q_\ell) \equiv \lim_{\mu\to 0} \int d^4q_1 \ldots d^4q_\ell\, J_{{\rm F},\ell}(q_1,\ldots q_\ell)$$

An example

$$\int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \equiv \lim_{\mu \to 0} \int_{\mathrm{R}} d^4q \left(\frac{1}{(\bar{q}^2 - M^2)^2} - \left[\frac{1}{\bar{q}^4} \right] \right)$$

$$= \lim_{\mu \to 0} \int d^4q \left(\frac{M^2}{\bar{q}^4 (\bar{q}^2 - M^2)} + \frac{M^2}{\bar{q}^2 (\bar{q}^2 - M^2)^2} \right), \quad \bar{q}^2 = q^2 - \mu^2$$

Dependence on UV regulator R canceled by partial fractioning

$$\frac{1}{\bar{q}^2 - M^2} = \frac{1}{\bar{q}^2} + \frac{M^2}{\bar{q}^2 (\bar{q}^2 - M^2)}$$

Tensors defined likewise

$$\int [d^4q] \frac{q^\alpha q^\beta}{(\bar{q}^2 - M^2)^3} \equiv \lim_{\mu \to 0} \int_{\mathrm{R}} d^4q \left(\frac{q^\alpha q^\beta}{(\bar{q}^2 - M^2)^3} - \left[\frac{q^\alpha q^\beta}{\bar{q}^6} \right] \right)$$

$$= \lim_{\mu \to 0} \int d^4q \, q^\alpha q^\beta \left(\frac{M^2}{\bar{q}^6 (\bar{q}^2 - M^2)} + \frac{M^2}{\bar{q}^4 (\bar{q}^2 - M^2)^2} + \frac{M^2}{\bar{q}^2 (\bar{q}^2 - M^2)^3} \right)$$

Formal properties of the FDR integration

i) Invariance under shift of any integration variable

$$\int [d^4q_1] \dots [d^4q_\ell]\, J(q_1, \dots, q_\ell)$$
$$= \int [d^4q_1] \dots [d^4q_\ell]\, J(q_1 + p_1, \dots, q_\ell + p_\ell)$$

ii) Simplifications among numerators and denominators

$$\int [d^4q_1] \dots [d^4q_\ell]\, \frac{\bar{q}_i^2 - m_i^2}{(\bar{q}_i^2 - m_i^2)^m \dots}$$
$$= \int [d^4q_1] \dots [d^4q_\ell]\, \frac{1}{(\bar{q}_i^2 - m_i^2)^{m-1} \dots}$$

i) + ii) guarantee Gauge Invariance: usual manipulations hold at the integrand level (**any graphical proof of WI holds!**)

No reference to $[J_{\rm INF}]$ $\Rightarrow$ subtracted integrands **irrelevant!**

i)

FDR integrals as finite differences of shift invariant UV divergent integrals

$$\int [d^4q_1]\dots[d^4q_\ell]\, J(\{q_i\})$$
$$= \lim_{\mu\to 0} \mu_R^{-\ell\epsilon} \int d^nq_1\dots d^nq_\ell \Big(J(\{q_i\}) - [J_{\mathrm{INF}}(\{q_i\})] \Big)$$

r.h.s. regulated in DR *(but any regulator* R *would give same result)* ii)

Provided any q_i^2 appearing in the numerator from Feynman rules is also shifted $q_i^2 \to \bar{q}_i^2$ *(Global Prescription)*. For instance

$$\int [d^4q] \frac{\bar{q}^2 - M^2}{(\bar{q}^2 - M^2)^3} = \int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \qquad (1)$$

Extra integrals containing μ^2 appear, and Eq. (1) holds **only if** the same subtraction is performed in front of μ^2 **as if it was** $q^\alpha q^\beta$

$$\int [d^4q] \frac{\mu^2}{(\bar{q}^2 - M^2)^3} \equiv \lim_{\mu \to 0} \int_{\mathrm{R}} d^4q \left(\frac{\mu^2}{(\bar{q}^2 - M^2)^3} - \left[\frac{\mu^2}{\bar{q}^6} \right] \right) = \frac{i\pi^2}{2}$$

Dependence on μ of FDR integrals

$$\int [d^4 q_1] \ldots [d^4 q_\ell]\, J(\{q_i\})$$

$$= \lim_{\mu \to 0} \mu_R^{-\ell\epsilon} \int d^n q_1 \ldots d^n q_\ell \Big(J(\{q_i\}) - [J_{\rm INF}(\{q_i\})] \Big)$$

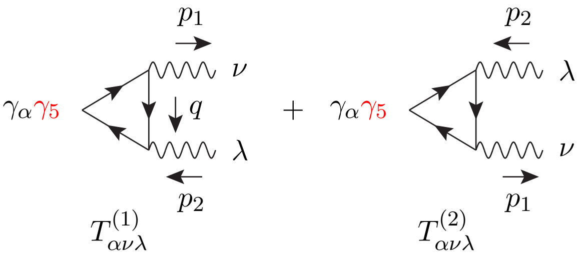
1. First term in r.h.s. independent of μ ($\mu \to 0$ in *integrand*)
2. Polynomially divergent integrals in $[J_{\rm INF}]$ cannot contribute either, being proportional to positive powers of μ
3. μ dependence of the l.h.s. entirely due to powers of $\ln(\mu/\mu_R)$ **generated by log divergent integrals in** $[J_{\rm INF}]$

a) FDR integrals depend on μ *logarithmically*

b) By sidestepping the subtraction of the $\ln(\mu/\mu_R)$s, $\lim_{\mu\to 0}$ can be formally taken by trading $\ln(\mu)$ for $\ln(\mu_R)$

FDR integrals do not depend on any cutoff but only on the renormalization scale μ_R (UV separation scale)

A 1-loop warm-up: The ABJ anomaly

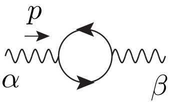


$$p^\alpha T_{\alpha\nu\lambda} = -i\frac{e^2}{4\pi^4}\mathrm{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1] \int [d^4q]\, \mu^2 \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2}$$

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2}\mathrm{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1]$$

A 2-loop warm-up: LL γ self-energy in QED

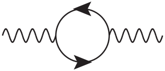
It is obtained by squaring the diagram



$$= i\,T_{\alpha\beta}\,\Pi(p^2) \qquad T_{\alpha\beta} = g_{\alpha\beta}p^2 - p_\alpha p_\beta$$

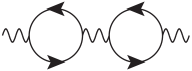
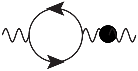
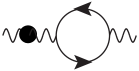
$$\Pi(p^2) = \frac{1}{\epsilon}\,\Pi_{-1} + \Pi_0 + \epsilon\,\Pi_1$$

In DR, one-loop counterterms are needed to avoid $\Pi_{-1}\Pi_1$

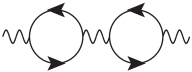
 $+$ 

$$= i\,T_{\alpha\beta}\,\Pi_0 + \mathcal{O}(\epsilon)$$

Therefore, up to terms $\mathcal{O}(\epsilon)$

 $+$  $+$  $+$  $= iT_{\alpha\beta}\,\Pi_0^2$

In FDR, the product of two one-loop diagrams **is the product of the two finite parts**, so that one obtains **without counterterms**

 $= i\,T_{\alpha\beta}\,\Pi^2_{\rm FDR}(p^2)$

with $\Pi_{\rm FDR}(p^2) = \Pi_0 = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\,\ln\frac{m^2-p^2x(1-x)}{\mu_R^2}$

$\Longrightarrow$ No order-by-order renormalization in FDR

- The previous example also shows that **ℓ-loop integrals are directly re-usable in $(\ell{+}1)$-loop calculations**

- For instance, the two-loop factorizable FDR integral

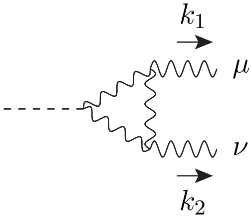
$$\int \frac{[d^4q_1]}{(\bar{q}_1^2 - m_1^2)^\alpha} \times \int \frac{[d^4q_2]}{(\bar{q}_2^2 - m_2^2)^\beta}$$

is simply the product of two one-loop FDR integrals

- That **is not** the case in DR, where further expanding in ϵ is required

Example 1: $H \to \gamma(k_1^\mu)\,\gamma(k_2^\nu)$ (generic R_ξ gauge)

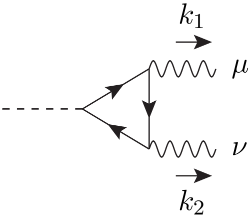
Alice M. Donati and R.P., arXiv:1302.5668 [hep-ph]



$\widetilde{\mathcal{M}}_W(\beta)$

26 diagrams

$$\beta = \frac{4\,M_W^2}{M_H^2}$$



$\widetilde{\mathcal{M}}_f(\eta)$

2 diagrams

$$\eta = \frac{4\,m_f^2}{M_H^2}$$

$$
\begin{aligned}
\mathcal{M}^{\mu\nu}(\beta,\eta) &= \left(\widetilde{\mathcal{M}}_{W}(\beta) + \sum_f N_c Q_f^2\, \widetilde{\mathcal{M}}_{f}(\eta)\right) T^{\mu\nu} \\
T^{\mu\nu} &= k_1^\nu k_2^\mu - (k_1 \cdot k_2)\, g^{\mu\nu} \\
\widetilde{\mathcal{M}}_{W}(\beta) &= \frac{i\, e^3}{(4\pi)^2 s_W M_W} \left[2 + 3\beta + 3\beta(2-\beta) f(\beta) \right] \\
\widetilde{\mathcal{M}}_{f}(\eta) &= \frac{-i\, e^3}{(4\pi)^2 s_W M_W}\, 2\eta \left[1 + (1-\eta) f(\eta) \right]
\end{aligned}
$$

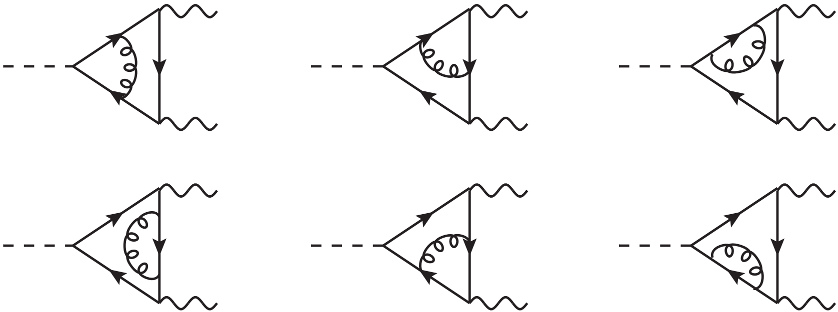
$$
f(x) = -\frac{1}{4} \ln^2 \left(\frac{1+\sqrt{1-x+i\varepsilon}}{-1+\sqrt{1-x+i\varepsilon}} \right)
$$

NOTE:

$$
\int [d^4q] \frac{\bar{q}^2 g_{\mu\nu} - 4 q_\mu q_\nu}{(\bar{q}^2 - M^2)^3} = \int [d^4q] \frac{-\mu^2}{(\bar{q}^2 - M^2)^3} g_{\mu\nu} = -\frac{i\pi^2}{2} g_{\mu\nu}
$$

Example 2: gluonic corrections to $\Gamma(\mathbf{H} \to \gamma\gamma)$

Alice M. Donati and R.P., arXiv:1311.5500



12 diagrams

Important facts

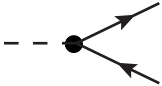
- $$\mathcal{M}^{(2-loop)} = \underbrace{\mathcal{M}^{(1-loop)}}_{\frac{i\alpha}{3\pi v}} \left(1 - \frac{\alpha_S}{\pi}\right) \quad (\text{when } m_{\text{top}} \to \infty)$$

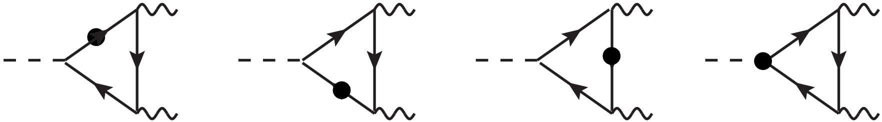
- **No** integral by integral correspondence between DR and FDR and results coincide only at the very end

- If $m_{\text{top}} \to \infty$ **no** finite renormalization needed in FDR

- In DR no renormalization (of sub-divergences) with counterterms gives a **wrong** result

 $= -i\,\delta m$

 $= -i\,\dfrac{\delta m}{v}$



$$= \begin{cases} 0 \times \delta m & \text{in FDR} \quad \text{with } \delta m \propto \ln \mu_R \\ \mathcal{O}(\epsilon) \times \delta m & \text{in DR} \quad \text{with } \delta m \propto 1/\epsilon \end{cases}$$

Example 3: $\Gamma(\mathbf{H} \to \mathbf{gg})$

R. P., arXiv:1307.0705 [hep-ph]

- FDR is used to compute the NLO QCD corrections to $\mathbf{H} \to \mathbf{gg}$ in the large top mass limit
- The well known fully inclusive result

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2)) \left[1 + \frac{95}{4}\frac{\alpha_S}{\pi}\right]$$

is re-derived **both analytically and numerically**, where

$$\Gamma^{(0)}(\alpha_S(M_H^2)) = \frac{G_F \alpha_S^2(M_H^2)}{36\sqrt{2}\pi^3} M_H^3$$

- **UV**, **SOFT** and **CL** divergences, besides α_S **renormalization**

FDR vs CL/UV Virtual Infinities

- CL/UV singularities regulated by μ^2, e.g.

$$B^{\rm FDR}(p^2=0,0,0) = \int [d^4q] \frac{1}{\bar{q}^2((q+p)^2-\mu^2)} = \mathbf{0}$$

- **Due to a cancellation between** CL **and** UV **regulators**

$$B^{\rm FDR}(p^2,0,0) = -i\pi^2 \lim_{\mu\to 0}\int_0^1 dx\ \left[\ln(\mu^2 - p^2x(1-x)) - \ln(\mu^2)\right]$$

- As in DR, **FDR scaleless integrals vanish!**

SOFT/CL Virtual infinities and the Virtual Part

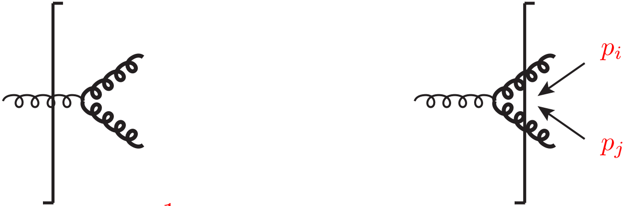
- Overlapping SOFT/CL infinities **also regulated by μ^2**.

If $\bar{D}_i = (q+p_i)^2 - \mu^2$ with $p_i^2 = 0$:

$$
\begin{aligned}
C(s) &= \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} = \lim_{\mu \to 0} \int d^4q \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \\
&= \frac{i\pi^2}{s} \left[\frac{\ln^2(\mu_0) - \pi^2}{2} + i\,\pi \ln(\mu_0) \right] \\
s &= M_H^2 = -2(p_1 \cdot p_2) \quad \text{with} \quad \mu_0 = \mu^2/s
\end{aligned}
$$

$$
\Gamma_V(\mathbf{H} \to \mathbf{gg}) = -3 \frac{\alpha_S}{\pi} \Gamma^{(0)}(\alpha_S)\, M_H^2\, \mathcal{R}e \left[\frac{C(M_H^2)}{i\pi^2} \right]
$$

The Real Part



$$\frac{1}{q^2-\mu^2} \leftrightarrow \delta(q^2-\mu^2)\,\theta(q(0))$$

$p_{i,j}^2 = \mu^2 \to 0$ in the Phase–Space boundaries ($\mu-$massive PS)

The massless (gauge invariant!) $|M|^2$ has to be integrated over a μ-massive PS (with massive boundaries on $s_{ij} = (p_i+p_j)^2$)

$$\int d\Phi_3 = \frac{\pi^2}{4s}\int_{\mu-\text{massive}} ds_{12}ds_{13}ds_{23}\,\delta(s-s_{12}-s_{13}-s_{23})$$

- One obtains

$$
\begin{aligned}
\Gamma_R(\mathbf{H} \to \mathbf{ggg}) &= \frac{3}{2}\frac{\alpha_S}{\pi}\,\Gamma^{(0)}(\alpha_S) \\
&\times \left[\ln^2 \frac{M_H^2}{\mu^2} - \pi^2 + \frac{73}{6} - \frac{11}{3}\ln\frac{M_H^2}{\mu^2}\right]
\end{aligned}
$$

and, accounting for the finite renormalization term $\left(1 + \frac{11}{4}\frac{\alpha_S}{\pi}\right)$ in A

$$
\begin{aligned}
\Gamma(\mathbf{H} \to \mathbf{gg}) &= \Gamma_V(\mathbf{H} \to \mathbf{gg}) + \Gamma_R(\mathbf{H} \to \mathbf{ggg}) \\
&= \Gamma^{(0)}(\alpha_S)\left[1 + \frac{\alpha_S}{\pi}\left(\frac{95}{4} - \frac{11}{2}\ln\frac{M_H^2}{\mu^2}\right)\right]
\end{aligned}
$$

α_S Renormalization

- The residual μ^2 is a universal dependence on the renormalization scale $(\mu = \mu_R)$

- $\ln(\mu_R^2)$ can be reabsorbed in the gluonic running of the strong coupling constant **(Finite Global Renormalization)**

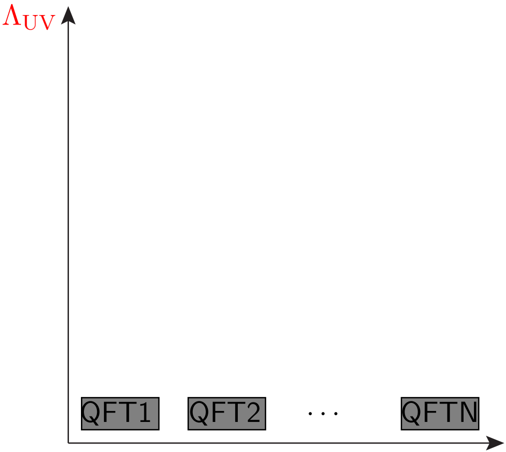
$$
\begin{aligned}
\Gamma^{(0)}(\alpha_S) \quad &\rightarrow \quad \Gamma^{(0)}(\alpha_S(\mu_R^2)) \\
\alpha_S(M_H^2) \quad &= \quad \frac{\alpha_S(\mu_R^2)}{1 + \frac{\alpha_S}{2\pi}\frac{11}{2}\ln\frac{M_H^2}{\mu_R^2}}
\end{aligned}
$$

$$
\Gamma(\mathbf{H} \rightarrow \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2))\left[1 + \frac{95}{4}\,\frac{\alpha_S}{\pi}\right]
$$

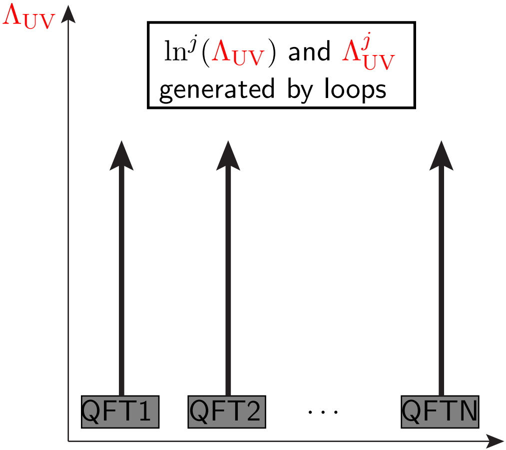
quod erat demostrandum

Physical Interpretation

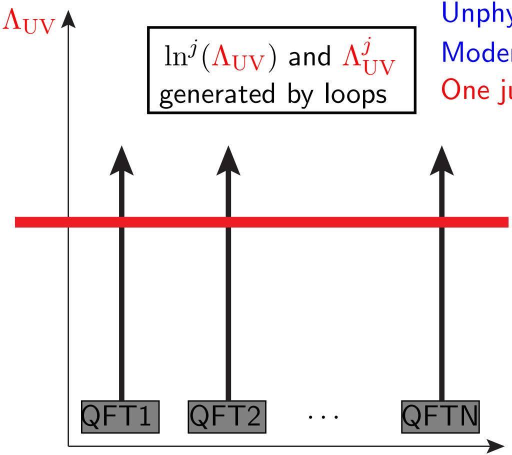
QFTs vs UV cutoff (I)



QFTs vs UV cutoff (II)



QFTs vs UV cutoff (III)



Unphysical if $\Lambda_{\rm UV} \to \infty$

Modern version of *aether*?

One just **ignores** them

The real question is not "Where do the infinities go?" but

What is the cost of **ignoring** infinities?

- No cost for polynomially divergent infinities (decoupling)
- Only logarithmic infinities influence the physical spectrum ($\ln \mu_R$ pops up in $J_{\mathrm{F},\ell}(q_1, \ldots, q_\ell)$ when separating them)
- Physics at Λ_{UV} scale manifests itself only logarithmically at lower energies

Polynomial divergences are unobservable!

Classification

independent of the number of external legs!

1. $\left[\frac{1}{\bar{q}^4}\right]$ is the only possible **subtracted** 1-loop log divergent scalar

   Vacuum Integrand $\Leftrightarrow$ Vacuum Bubble

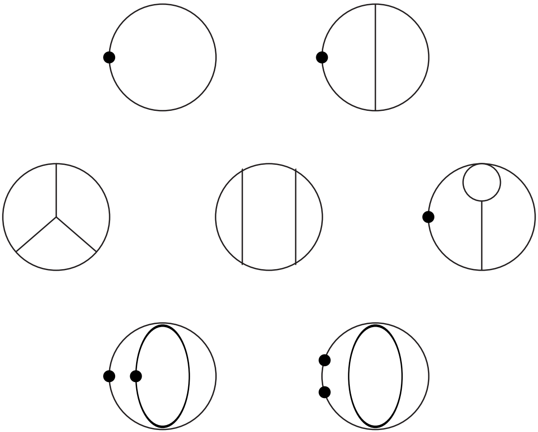
2. At 2 loops $\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right]$ is log divergent

3. Five additional log divergent vacuum integrands at 3 loops

$$\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{13}^2((q_2-q_3)^2-\mu^2)}\right] \quad \left[\frac{1}{\bar{q}_1^2\bar{q}_3^2\bar{q}_2^4\bar{q}_{12}^2\bar{q}_{23}^2}\right]$$

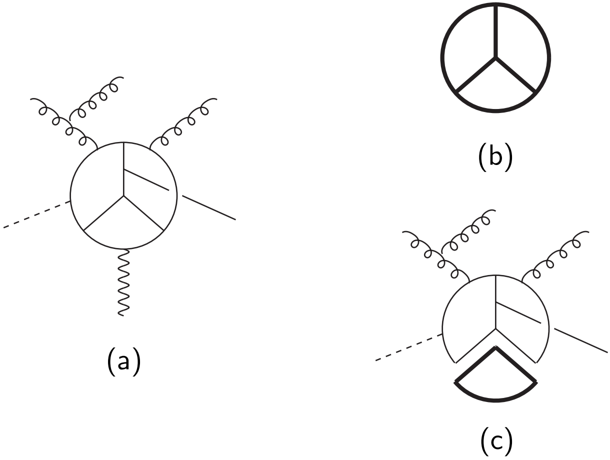
$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^4\bar{q}_2^4\bar{q}_3^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^6\bar{q}_2^2\bar{q}_3^2\bar{q}_{123}^2}\right]$$

Corresponding 1-, 2- and 3-loop log topologies



By tensor reduction divergent tensors are reducible to combinations of those scalar topologies plus finite constants

Vacuum inside loops (pictorially)



(b) *and* (c) *are* **Vacuum Bubbles** *generated by the generic diagram* (a). *They do not contribute to the interaction and are discarded (irrelevant!)*

- Infinities are put back into the vacuum, rather than absorbed in the parameter of the Lagrangian $\mathcal{L}$

The vacuum is by far more efficient in accommodating infinities than $\mathcal{L}$

- This is possible because no cutoff is left in FDR integrals to be compensated by counterterms in $\mathcal{L}$

Order-by-order **vacuum redefinition** dubbed
Topological Renormalization

- The vacuum back-reacts by trading the cutoff μ for an arbitrary UV separation scale μ_R, which, however, drops after fixing the **QFT** by means of a

Global Finite Renormalization

Global Finite Renormalization

Consider the Lagrangian of a renormalizable QFT dependent on m parameters p_i $(i = 1 : m)$

$$\mathcal{L}(p_1, \ldots, p_m)$$

Before an observable $\mathcal{O}^{\rm TH}_{m+1}$ can be calculated, p_i must be fixed by means of m measurements

$$\mathcal{O}^{\rm TH}_i(p_1, \ldots, p_m) = \mathcal{O}^{\rm EXP}_i$$

which determine p_i in terms of observables $\mathcal{O}^{\rm EXP}_i$ and corrections computed at the loop level ℓ one is working:

$$p_i = p_i^{\ell-loop}(\mathcal{O}^{\rm EXP}_1, \ldots, \mathcal{O}^{\rm EXP}_m) \equiv \bar{p}_i$$

Then

$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m) \quad \text{with} \quad \frac{\partial\, \mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m)}{\partial \mu_R} = 0$$

is a **prediction** of the QFT

Non-renormalizable QFTs

Extending the FDR framework to a non-renormalizable QFT described by a Lagrangian $\mathcal{L}_{NR}$

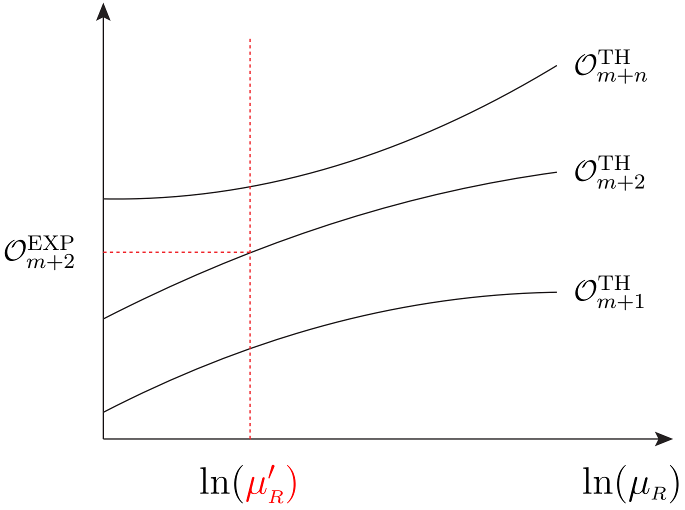
1. Now $\ln(\mu_R)$ *might* appear when computing observables

$$\mathcal{O}^{\mathrm{TH}}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu_R))$$

2. However, combinations of observables in which μ_R disappears can be unambiguously predicted by $\mathcal{L}_{NR}$. E. g. (at one loop)

$$\begin{aligned}
\mathcal{O}^{\mathrm{TH}}_{m+1} &= \alpha \ln(\mu_R) + k_1 \\
\mathcal{O}^{\mathrm{TH}}_{m+2} &= \beta \ln(\mu_R) + k_2 \\
\mathcal{O}^{\mathrm{EXP}}_{\mathrm{Predictable}} &= \frac{\mathcal{O}^{\mathrm{EXP}}_{m+1}}{\alpha} - \frac{\mathcal{O}^{\mathrm{EXP}}_{m+2}}{\beta} = \frac{k_1}{\alpha} - \frac{k_2}{\beta}
\end{aligned}$$

3. This is equivalent to extracting $\ln(\mu_R)$ from $\mathcal{O}^{\rm EXP}_{m+2} = \mathcal{O}^{\rm TH}_{m+2}$ and inserting it in $\mathcal{O}^{\rm TH}_{m+1}$



4. **At any loop order just one** additional measurement needed to fix μ_R by solving

$$\mathcal{O}^{\rm EXP}_{m+2} = \mathcal{O}^{\rm TH}_{m+2}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu'_R))$$

and setting $\mu_R = \mu'_R$ in $\mathcal{O}^{\rm TH}_{m+1}, \cdots, \mathcal{O}^{\rm TH}_{m+n}$

5. **Predictivity** restored in the infinite loop limit

6. **The physical meaning of the extra measurement is**

disentangling from the physical spectrum the effects of the unknown UV completion of $\mathcal{L}_{NR}$

Summary

1. **QFTs** renormalized by **defining** a new mathematical object (FDR integral) compatible with shift and gauge invariance
2. Results of renormalizable **QFTs** reproduced, only **finite** and **global** renormalization needed, $\mathcal{L}$ **untouched**, no order-by-order counterterms (and IR divergences naturally fit)
3. FDR integral $\Leftrightarrow$ order-by-order re-definition of the vacuum
4. In non-renormalizable **QFTs** **ONE** additional measurement fixes the theory which becomes predictive *without modifying* $\mathcal{L}$
5. Non-renormalizable **QFTs** rescued: focus moved from occurrence of UV infinities to consistency of the **QFT** at hand (**does** $\mathcal{L}$ **reproduce data?**)

Thank you!

Backup slides

FDR versus BPHZ

1. The FDR subtraction is obtained by a formal expansion of the original loop integrands **around poles in** $\bar{q}_i^2$, and not via a Taylor expansion in the external momenta
2. In FDR poles in $\bar{q}_i^2$ giving rise to UV divergences are subtracted **without any attempt of re-introducing them into the Lagrangian**
3. **Gauge invariance is automatically respected** in FDR, while it must be enforced by hand in BPHZ

Shift invariance of one-loop FDR integrals

Given

$$\begin{aligned}\bar{D} &= q^2 - M^2 - \mu^2 \\ \bar{D}_p &= (q+p)^2 - M^2 - \mu^2\end{aligned}$$

and

$$\begin{aligned}I^{(0)} &= \int [d^4q] \frac{1}{\bar{D}^2}\,, \qquad I_p^{(0)} = \int [d^4q] \frac{1}{\bar{D}_p^2} \\ I^{(2)} &= \int [d^4q] \frac{1}{\bar{D}}\,, \qquad I_p^{(2)} = \int [d^4q] \frac{1}{\bar{D}_p}\end{aligned}$$

I prove that

$$I^{(0)} = I_p^{(0)} \quad \text{and} \quad I^{(2)} = I_p^{(2)}$$

$$I^{(0)} = I_p^{(0)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}^2} = \left[\frac{1}{\bar{q}^4}\right] + J_F^{(0)}$$
$$\frac{1}{\bar{D}_p^2} = \left[\frac{1}{\bar{q}^4}\right] + J_{F,p}^{(0)}$$

Then

$$I^{(0)} = \lim_{\mu\to 0}\int d^n q \left(\frac{1}{\bar{D}^2} - \frac{1}{\bar{q}^4}\right) = \lim_{\mu\to 0}\int d^n q \left(\frac{1}{\bar{D}_p^2} - \frac{1}{\bar{q}^4}\right) = I_p^{(0)}$$

$$I^{(2)} = I_p^{(2)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}} = \left[\frac{1}{\bar{q}^2}\right] + M^2 \left[\frac{1}{\bar{q}^4}\right] + J_F^{(2)}$$

$$\frac{1}{\bar{D}_p} = \left[\frac{1}{\bar{q}^2}\right] + (M^2 - p^2) \left[\frac{1}{\bar{q}^4}\right] - 2p^\alpha \left[\frac{q_\alpha}{\bar{q}^4}\right] + 4p^\alpha p^\beta \left[\frac{q_\alpha q_\beta}{\bar{q}^6}\right] + J_{F,p}^{(2)}$$

Then

$$I^{(2)} = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}} - \frac{1}{\bar{q}^2} - \frac{M^2}{\bar{q}^4} \right)$$

and

$$I_p^{(2)} = I^{(2)} + \underbrace{\int d^n q \left(\frac{p^2}{\bar{q}^4} + 2\frac{(q \cdot p)}{\bar{q}^4} - 4\frac{(q \cdot p)^2}{\bar{q}^6} \right)}_{=0}$$

This is because

$$\int d^n q \frac{1}{q^2-\mu^2} = \int d^n q \frac{1}{(q+p)^2-\mu^2} =$$

$$\int d^n q \frac{1}{q^2-\mu^2}\left[1-\Big(\underbrace{\frac{p^2+2(q\cdot p)}{\bar{q}^2}-4\frac{(q\cdot p)^2}{\bar{q}^4}}_{\propto\ p^2\ \text{when integrated}}\Big)+\mathcal{O}(p^3)\right]$$

Then

$$\int d^n q \left(\frac{p^2}{\bar{q}^4}+2\frac{(q\cdot p)}{\bar{q}^4}-4\frac{(q\cdot p)^2}{\bar{q}^6}\right)=0$$

which can also be tested by a direct computation

Equivalence of FDR and DR (in $\overline{\mathrm{MS}}$) at one loop

When computed in DR in $n = 4 + \epsilon$ dimensions, the subtracted one-loop tensors obey *gauge preserving consistency relations*

$$\int d^n q \left[\frac{q^\mu q^\nu}{\bar{q}^6}\right] = \frac{g^{\mu\nu}}{4} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

$$\int d^n q \left[\frac{q^\mu q^\nu q^\rho q^\sigma}{\bar{q}^8}\right] = \frac{(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})}{24} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

For **both** scalars and tensors $J_{\mathrm{INF}}(q)$ is proportional to

$$\mu_R^{-\epsilon} \int d^n q \left[\frac{1}{\bar{q}^4}\right] = i\pi^2 \left(-\frac{2}{\epsilon} - \gamma_E - \ln \pi - \ln \frac{\mu^2}{\mu_R^2}\right)$$

In FDR all terms but $\ln \frac{\mu^2}{\mu_R^2}$ are subtracted, as in $\overline{\mathrm{MS}}$

UV divergences versus $\ln(\mu_R)$ in FDR integrals

The absence of UV infinities in $[J_{\rm INF}]$ is a sufficient **but not necessary** condition for the absence of $\ln(\mu_R)$ in $J_{{\rm F},\ell}$. For instance

$$\int [d^4q_1][d^4q_2]\left(\frac{2}{\bar D_1^2 \bar D_2 \bar D_{12}} - \frac{1}{\bar D_1^2 \bar D_2^2} + \frac{4m^2}{\bar D_1^3 \bar D_2^2}\right) = 2\pi^4 f$$

with $\bar D_i = \bar q_i^2 - m^2$ and $f = \frac{i}{\sqrt{3}}\left({\rm Li}_2(e^{i\frac{\pi}{3}}) - {\rm Li}_2(e^{-i\frac{\pi}{3}})\right)$. While

$$\mu_R^{-2\epsilon}\int d^nq_1 d^nq_2 \left[\frac{2}{\bar D_1^2 \bar D_2 \bar D_{12}} - \frac{1}{\bar D_1^2 \bar D_2^2} + \frac{4m^2}{\bar D_1^3 \bar D_2^2}\right]_{\rm INF}$$

$$= \pi^4\left\{-2\left(\frac{1}{\epsilon} + \ln\pi + \gamma_E + \ln\frac{m^2}{\mu_R^2}\right) - 3 + 2f\right\}$$

Examples of IBP identities in FDR

- At one loop, with $D_0 = q^2 - m_0^2$ and $D_1 = (q+p)^2 - m_1^2$

$$0 = \int [d^4q] \frac{\partial}{\partial q^\alpha} \frac{q^\alpha}{\bar{D}_0 \bar{D}_1} = \int [d^4q] \left\{ \frac{4}{\bar{D}_0 \bar{D}_1} - 2\frac{q^2}{\bar{D}_0^2 \bar{D}_1} - 2\frac{q^2 + (q \cdot p)}{\bar{D}_0 \bar{D}_1^2} \right\}$$

- At two loops, with $D_i = q_i^2 - m_i^2$ and $q_{12} = q_1 + q_2$

$$\begin{aligned} 0 &= \int [d^4q_1][d^4q_2] \frac{\partial}{\partial q_1^\alpha} \frac{q_1^\alpha q_1^\beta q_1^\gamma}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} \\ &= \int [d^4q_1][d^4q_2]\, q_1^\beta q_1^\gamma \left\{ \frac{6}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} - \frac{6q_1^2}{\bar{D}_1^4 \bar{D}_2 \bar{D}_{12}} - 2\frac{(q_1 \cdot q_{12})}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}^2} \right\} \end{aligned}$$

Naive treatment of scaleless integrals in DR

$$B^{\rm DR}(p^2,0,0) = \int d^nq \frac{1}{q^2(q+p)^2} \qquad (p^2=0)$$

$$\begin{aligned}\frac{1}{(q+p)^2} &= \frac{1}{q^2-M^2} - \left(\frac{1}{q^2-M^2} - \frac{1}{(q+p)^2}\right)\\ &= \frac{1}{q^2-M^2} - \frac{M^2+2(q\cdot p)}{(q^2-M^2)(q+p)^2}\end{aligned}$$

$$B^{\rm DR}(p^2,0,0) = \underbrace{\int d^nq \frac{1}{q^2(q^2-M^2)}}_{\text{defined if } \epsilon<0} - \underbrace{\int d^nq \frac{M^2+2(q\cdot p)}{q^2(q^2-M^2)(q+p)^2}}_{\text{defined if } \epsilon>0}$$

They cancel but **do they define $B^{\rm DR}(p^2,0,0)$?**

(**NO** ϵ can be found for which they simultaneously exist)