

# INFRARED DIVERGENCES AT HIGH ORDERS

Lorenzo Magnea

University of Torino - INFN Torino

HP<sup>2</sup> 2014 - Firenze - GGI - 05/09/14



# Outline

- On infrared divergences
- Dipoles and multipoles
- The high-energy limit
- Weaving multi-particle webs
- Outlook

# ON INFRARED DIVERGENCES



# Brief motivation

Why worry about stuff that  **cancels**  in physical observables?

# Brief motivation

Why worry about stuff that **cancels** in physical observables?



**You still have to actually cancel it.**

- Cancellation must be performed **analytically** before numerical integrations.
- One needs **local counterterms** for matrix elements in all singular regions.
- State of the art: NLO multileg, NNLO for a few processes.

# Brief motivation

Why worry about stuff that **cancel**s in physical observables?



**You still have to actually cancel it.**

- Cancellation must be performed **analytically** before numerical integrations.
- One needs **local counterterms** for matrix elements in all singular regions.
- State of the art: NLO multileg, NNLO for a few processes.



**The cancellation is incomplete.**

- Singularities leave behind **finite** but potentially **large logarithms**.
- For **inclusive** observables: **analytic resummation** to high logarithmic accuracy.
- For **exclusive** final states: **parton shower** event generators, (N)LL accuracy.

# Brief motivation

Why worry about stuff that **cancel**s in physical observables?

📌 You still have to actually cancel it.

- Cancellation must be performed **analytically** before numerical integrations.
- One needs **local counterterms** for matrix elements in all singular regions.
- State of the art: NLO multileg, NNLO for a few processes.

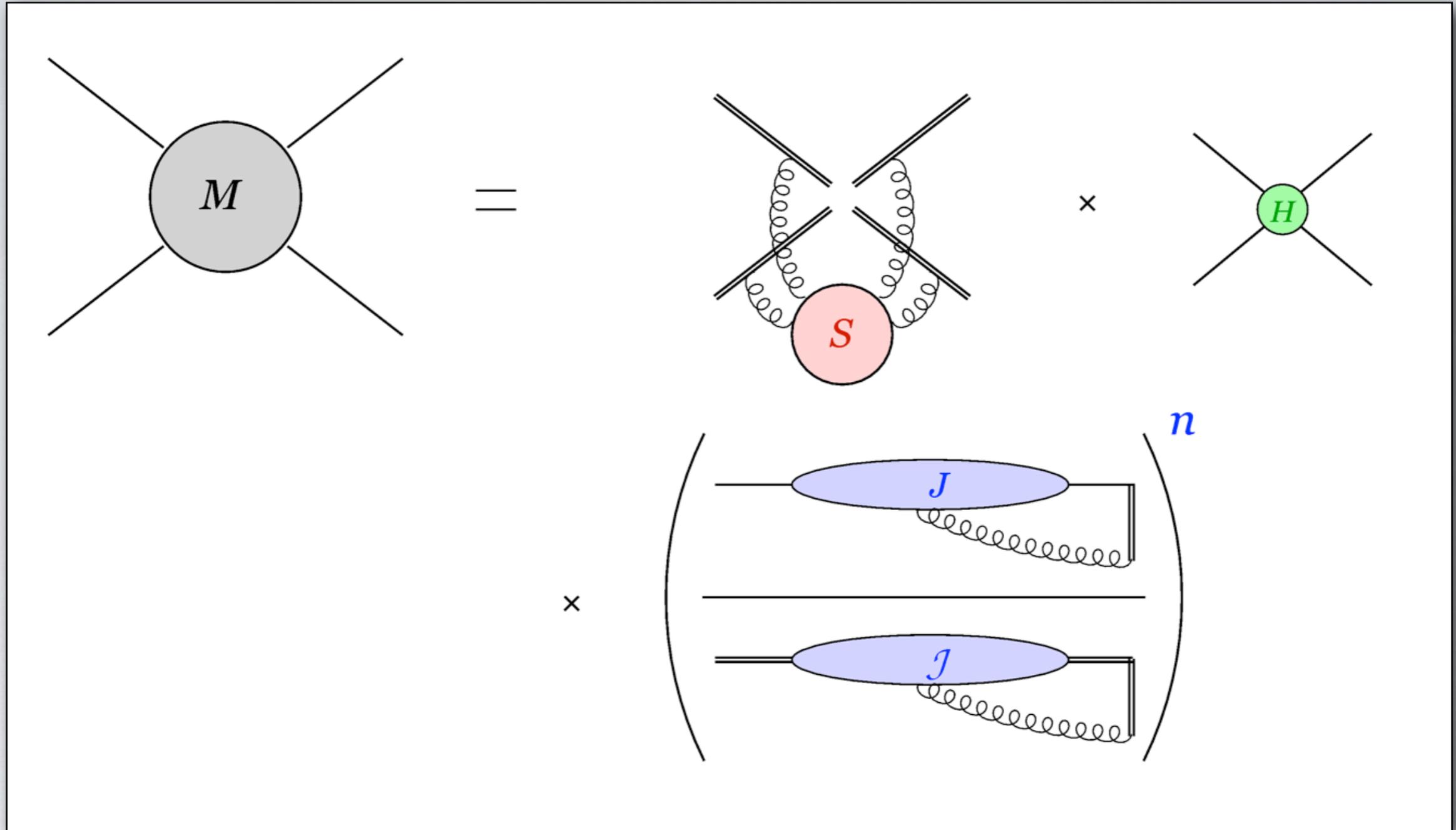
📌 The cancellation is incomplete.

- Singularities leave behind **finite** but potentially **large logarithms**.
- For **inclusive** observables: **analytic resummation** to high logarithmic accuracy.
- For **exclusive** final states: **parton shower** event generators, (N)LL accuracy.

📌 There is actual (non-perturbative) physics in the IR.

- We understand infrared radiation to **all orders** in any gauge theory.
- Power-suppressed **non-perturbative corrections** to QCD cross sections can be modeled.
- Links to the **strong coupling** regime can be established for **SUSY** gauge theories.
  - ➔ **N = 4** Super Yang-Mills **planar** amplitudes: **ABDKS ansatz**.
  - ➔ **Non planar** amplitudes: **awaiting** string theory input.

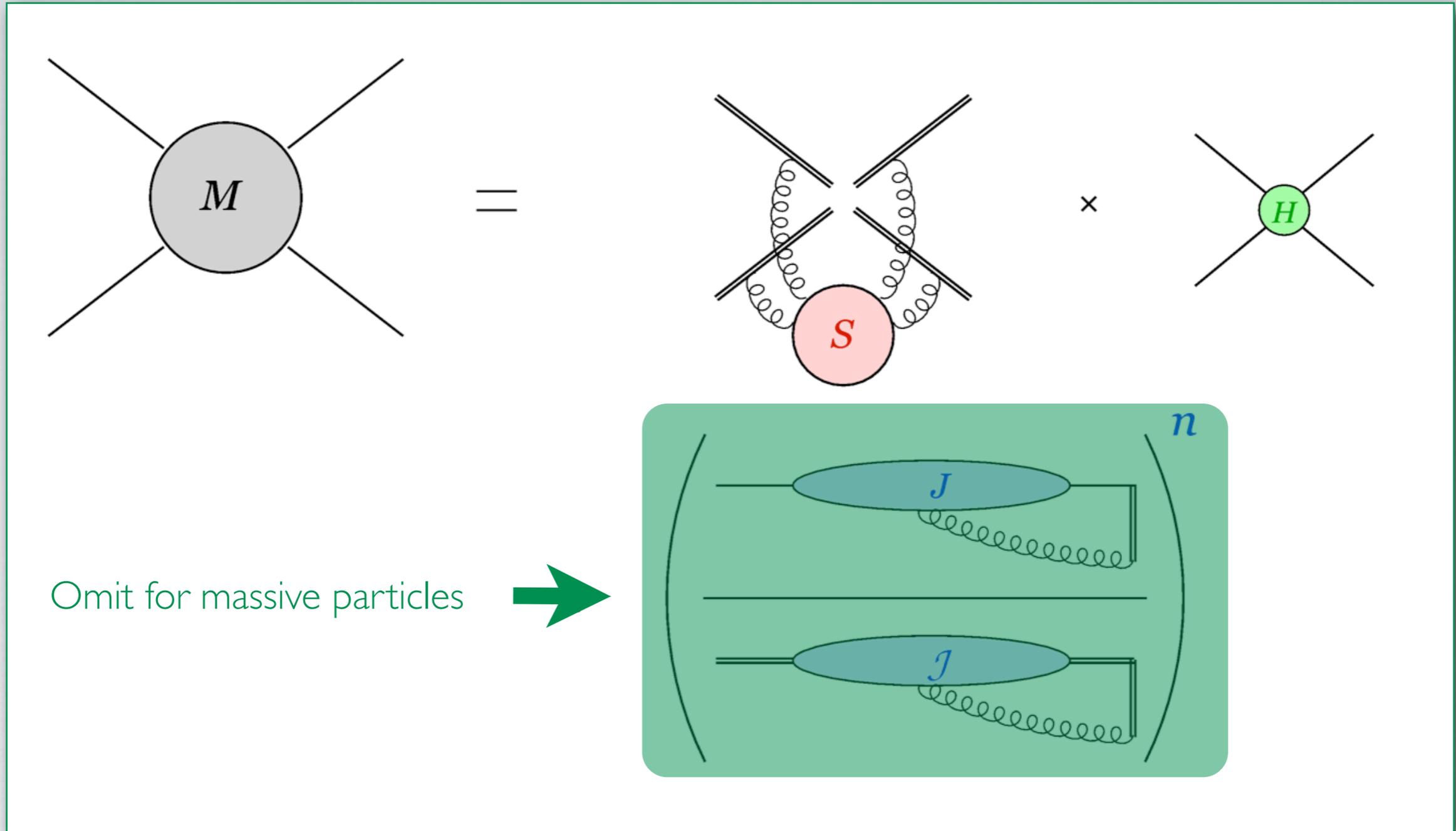
# Factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes



# Factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

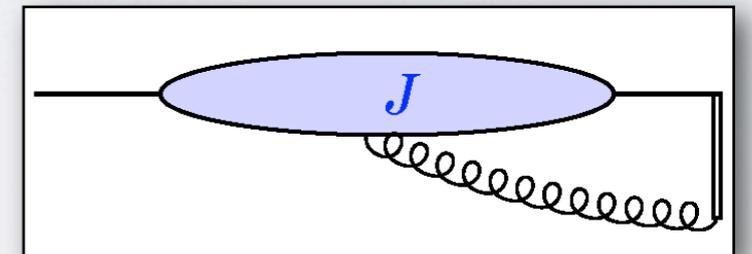
# Operator Definitions

The precise **functional form** of this graphical factorization is

$$\mathcal{M}_L(p_i/\mu, \alpha_s(\mu^2), \epsilon) = \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \\ \times \prod_{i=1}^n \left[ J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) / \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) \right],$$

Here we introduced dimensionless **four-velocities**  $p_i^\mu = Q \beta_i^\mu$ , (for massless particles  $\beta_i^2 = 0$ ), and **factorization vectors**  $n_i^\mu$ ,  $n_i^2 \neq 0$  to define, if needed, the jets.

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle.$$



where  $\Phi_n$  is the **Wilson line** operator along the direction  $n^\mu$ ,

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right].$$

**Note:** Wilson lines represent **fast** or **very massive** particles, **not recoiling** against **soft** radiation.

- The vectors  $n^\mu$ :
- Ensure **gauge invariance** of the jets.
  - **Separate** collinear gluons from wide-angle soft ones.
  - **Replace** other hard partons with a **collinear-safe** absorber.

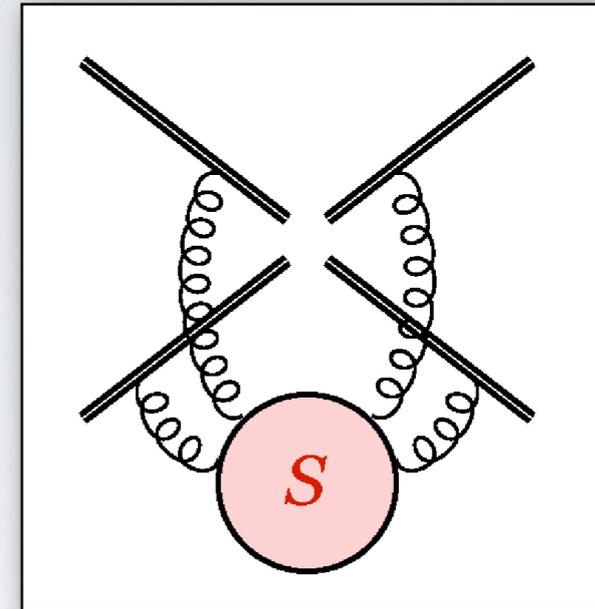
# Soft Matrices

The **soft function**  $\mathcal{S}$  is a **matrix**, mixing the available color tensors. It is defined by a correlator of **Wilson lines**.

$$(c_L)_{\{a_k\}} \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \epsilon) = \langle 0 | \prod_{k=1}^n [\Phi_{\beta_k}(\infty, 0)]_{a_k}^{b_k} | 0 \rangle (c_K)_{\{b_k\}},$$

The soft function  $\mathcal{S}$  obeys a **matrix** RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \epsilon) = -\mathcal{S}_{LJ}(\beta_i \cdot \beta_j, \epsilon) \Gamma_{JK}^{\mathcal{S}}(\beta_i \cdot \beta_j, \epsilon)$$



**NOTE:**  $\Gamma^{\mathcal{S}}$  is **singular** for **massless** theories, due to overlapping **UV** and **collinear** poles.

$\mathcal{S}$  is a **pure counterterm**. In dimensional regularization, using  $\alpha_s(\mu^2 = 0, \epsilon < 0) = 0$ ,

$$\mathcal{S}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) = P \exp \left[ -\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}}(\beta_i \cdot \beta_j, \alpha_s(\xi^2), \epsilon), \epsilon \right].$$

The determination of the **soft anomalous dimension matrix**  $\Gamma^{\mathcal{S}}$  is the **keystone** of the resummation program for multiparton **amplitudes** and **cross sections**.

- It **governs** the interplay of **color** exchange with **kinematics** in multiparton processes.
- It is the only **source** of multiparton **correlations** for singular contributions.
- **Collinear** effects are '**color singlet**' and can be extracted from **two-parton** scatterings.

# DIPOLES AND MULTIPOLES



# The Dipole Formula

**For massless partons**, the soft anomalous dimension matrix obeys an **exact equation** based on a **'conformal anomaly'**, which correlates color exchange with kinematics.

The **simplest solution** to this equation is a **sum over color dipoles** (Becher, Neubert; Gardi, LM, 09). It gives an **ansatz** for the all-order singularity structure of **all** multiparton fixed-angle **massless** scattering amplitudes: the **dipole formula**.

📌 All **soft** and **collinear** singularities can be **collected** in a multiplicative operator **Z**

$$\mathcal{M} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = Z \left( \frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) \mathcal{H} \left( \frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon \right),$$

📌 **Z** contains both soft singularities from **S**, and collinear ones from the jet functions. It must **satisfy** its own matrix **RG equation**

$$\frac{d}{d \ln \mu} Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = - Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \Gamma \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right).$$

The matrix **Γ** has a surprisingly simple **dipole structure**. It reads

$$\Gamma_{\text{dip}} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right) = -\frac{1}{4} \hat{\gamma}_K(\alpha_s(\mu^2)) \sum_{j \neq i} \ln \left( \frac{-2 p_i \cdot p_j}{\mu^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^n \gamma_{J_i}(\alpha_s(\mu^2)).$$

Note that **all singularities** are **generated by integration** over the scale of the coupling.

# The Dipole Formula

**For massless partons**, the soft anomalous dimension matrix obeys an **exact equation** based on a **'conformal anomaly'**, which correlates color exchange with kinematics.

The **simplest solution** to this equation is a **sum over color dipoles** (Becher, Neubert; Gardi, LM, 09). It gives an **ansatz** for the all-order singularity structure of **all** multiparton fixed-angle **massless** scattering amplitudes: the **dipole formula**.

📌 All **soft** and **collinear** singularities can be **collected** in a multiplicative operator **Z**

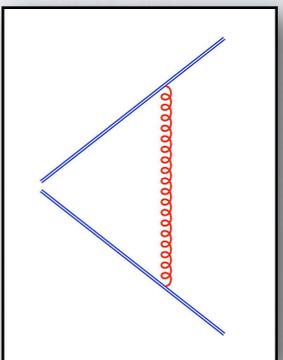
$$\mathcal{M} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = Z \left( \frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) \mathcal{H} \left( \frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon \right),$$

📌 **Z** contains both soft singularities from **S**, and collinear ones from the jet functions. It must **satisfy** its own matrix **RG equation**

$$\frac{d}{d \ln \mu} Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = - Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \Gamma \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right).$$

The matrix **Γ** has a surprisingly simple **dipole structure**. It reads

$$\Gamma_{\text{dip}} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right) = -\frac{1}{4} \hat{\gamma}_K(\alpha_s(\mu^2)) \sum_{j \neq i} \ln \left( \frac{-2 p_i \cdot p_j}{\mu^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^n \gamma_{J_i}(\alpha_s(\mu^2)).$$



Note that **all singularities** are **generated by integration** over the scale of the coupling.

# Features of the dipole formula

- **Known results** for IR divergences of **massless** gauge theory amplitudes **are recovered**.
- The **absence of multiparton correlations** implies remarkable diagrammatic cancellations.
- The color **matrix structure** is **fixed at one loop**: path-ordering is not needed.
- **All divergences** are determined by a **handful** of anomalous dimensions.
- The **cusp anomalous dimension** plays a very special role: a **universal** IR coupling.

Can this be the **definitive answer** for IR divergences in massless non-abelian gauge theories?

► There are **precisely two** sources of possible **corrections**.

- **Quadrupole** correlations may enter starting at **three loops**: they must be tightly constrained functions of **conformal cross ratios** of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\text{dip}}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta(\rho_{ijkl}, \alpha_s(\mu^2)) \quad , \quad \rho_{ijkl} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_k p_j \cdot p_l}$$

- The **cusp** anomalous dimension may **violate Casimir scaling** beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s)$$

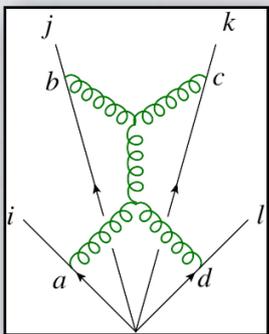
- The functional form of  $\Delta$  is further constrained by: **collinear limits**, **Bose symmetry**, **bounds on weights**, **high-energy constraints**. (Becher, Neubert; Dixon, Gardi, LM, 09).
- Recent evidence points to **non-vanishing  $\Delta$**  at **four loops** (Caron-Huot, Grozin et al.).

# Features of the dipole formula

- **Known results** for IR divergences of **massless** gauge theory amplitudes **are recovered**.
- The **absence of multiparton correlations** implies remarkable diagrammatic cancellations.
- The color **matrix structure** is **fixed at one loop**: path-ordering is not needed.
- **All divergences** are determined by a **handful** of anomalous dimensions.
- The **cusp anomalous dimension** plays a very special role: a **universal** IR coupling.

Can this be the **definitive answer** for IR divergences in massless non-abelian gauge theories?

► There are **precisely two** sources of possible **corrections**.



- **Quadrupole** correlations may enter starting at **three loops**: they must be tightly constrained functions of **conformal cross ratios** of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\text{dip}}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta(\rho_{ijkl}, \alpha_s(\mu^2)) \quad , \quad \rho_{ijkl} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_k p_j \cdot p_l}$$

- The **cusp** anomalous dimension may **violate Casimir scaling** beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s)$$

- The functional form of  $\Delta$  is further constrained by: **collinear limits**, **Bose symmetry**, **bounds on weights**, **high-energy constraints**. (Becher, Neubert; Dixon, Gardi, LM, 09).
- Recent evidence points to **non-vanishing  $\Delta$**  at **four loops** (Caron-Huot, Grozin et al.).



# Features of the dipole formula

- **Known results** for IR divergences of **massless** gauge theory amplitudes **are recovered**.
- The **absence of multiparton correlations** implies remarkable diagrammatic cancellations.
- The color **matrix structure** is **fixed at one loop**: path-ordering is not needed.
- **All divergences** are determined by a **handful** of anomalous dimensions.
- The **cusp anomalous dimension** plays a very special role: a **universal** IR coupling.

Can this be the **definitive answer** for IR divergences in massless non-abelian gauge theories?

► There are **precisely two** sources of possible **corrections**.

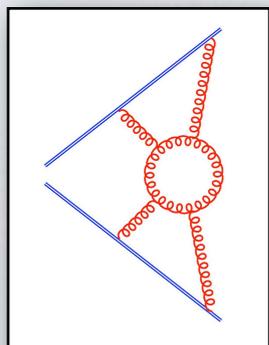
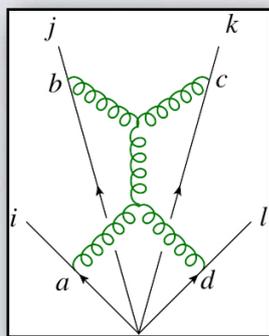
- **Quadrupole** correlations may enter starting at **three loops**: they must be tightly constrained functions of **conformal cross ratios** of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\text{dip}}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta(\rho_{ijkl}, \alpha_s(\mu^2)) \quad , \quad \rho_{ijkl} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_k p_j \cdot p_l}$$

- The **cusp** anomalous dimension may **violate Casimir scaling** beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s)$$

- The functional form of  $\Delta$  is further constrained by: **collinear limits**, **Bose symmetry**, **bounds on weights**, **high-energy constraints**. (Becher, Neubert; Dixon, Gardi, LM, 09).
- Recent evidence points to **non-vanishing  $\Delta$**  at **four loops** (Caron-Huot, Grozin et al.).

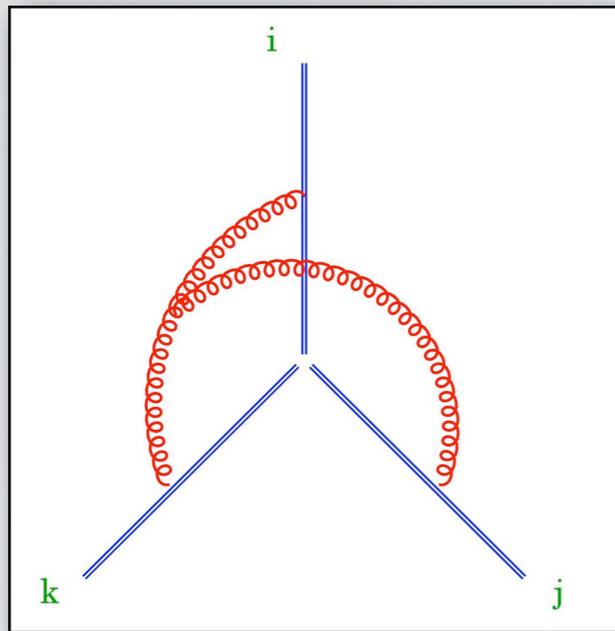


Ferrogia, Neubert, Pecjak, Yang 2009;  
 Kidonakis 2009; Mitov, Sterman, Sung 2010;  
 Chien, Schwartz, Simons-Duffin, Stewart 2011;

# Tripoles at two loops

**For massive partons**, the conformal invariance of semi-infinite Wilson line correlators is **not anomalous**, and the soft anomalous dimension is **unconstrained**.

An explicit calculation shows that **tripole color correlations arise** (Ferrogia et al. 09). They are expressed in terms of the **cusp angles** between massive partons



The connected two-loop tripole diagram

$$\gamma_{ij} = \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}} \equiv -\alpha_{ij} - \frac{1}{\alpha_{ij}}$$

$$\xi_{ij} = \cosh^{-1} \left( -\frac{\gamma_{ij}}{2} \right) = \ln(\alpha_{ij})$$

$$\Gamma_{\text{cusp}}^{(1)}(\xi) = \xi \coth(\xi) \gamma_K^{(1)} = -\frac{1 + \alpha^2}{1 - \alpha^2} \ln(\alpha) \gamma_K^{(1)}$$

**Note:**

$$\Gamma_{\text{trip}}^{(2)}(\xi_{mn}) = if_{abc} \sum_{ijk} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathcal{F}^{(2)}(\xi_{ij}, \xi_{jk}, \xi_{ki})$$

$$\mathcal{F}^{(2)}(\xi_{ij}, \xi_{jk}, \xi_{ki}) = \frac{4}{3} \sum_{ijk} \epsilon_{ijk} g(\xi_{ij}) \xi_{jk} \coth(\xi_{jk})$$

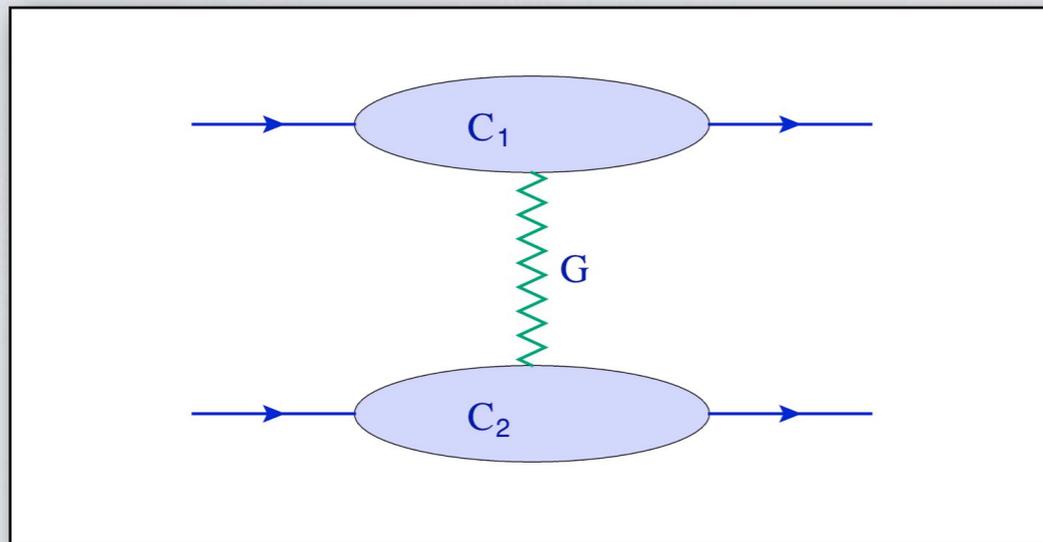
- Non-trivial **factorized** form in terms of **cusp angles**
- Non-trivial **massless limit**, the connected tripole  **cancels**  against 'planar' graphs.

# THE HIGH-ENERGY LIMIT



# Regge factorization

- In **perturbative** QCD the **high-energy limit** is governed by **t-channel** gluon exchange.
- In the  $t/s \rightarrow 0$  limit **gluons** in the **t-channel** 'Reggeize' with a computable **Regge trajectory**.
- The amplitude **factorizes** with **universal t-channel** exchange connecting two **impact factors**.



Quark-quark scattering: the t-channel gluon Reggeizes

- Large logarithms** of  $s/t$  are **generated** by a simple **replacement** of the **t-channel propagator**,

$$\frac{1}{t} \longrightarrow \frac{1}{t} \left( \frac{s}{-t} \right)^{\alpha(t)}$$

- The **Regge trajectory** has a perturbative expansion, with **IR divergent** coefficients

$$\alpha(t) = \frac{\alpha_s(-t, \epsilon)}{4\pi} \alpha^{(1)} + \left( \frac{\alpha_s(-t, \epsilon)}{4\pi} \right)^2 \alpha^{(2)} + \mathcal{O}(\alpha_s^3)$$

- The **gluon** has been shown to **Reggeize** at **NLL**, the **two-loop** Regge trajectory is known.

$$\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A \frac{2}{\epsilon} \quad \alpha^{(2)} = C_A \left[ -\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)} \frac{2}{\epsilon} + C_A \left( \frac{404}{27} - 2\zeta_3 \right) + n_f \left( -\frac{56}{27} \right) \right]$$

- Gluon reggeization is **exact** in the **planar limit** for the **N=4 SYM** four-point amplitude.

# Regge master formula

- Regge factorization, under the assumption of 'only poles' in the  $J$  plane, and including crossing information, leads to a 'master formula' for color-octet  $t$ -channel exchange.

$$\mathcal{M}_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon \right) = 2\pi\alpha_s H_{ab}^{(0),[8]} \left[ C_a \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) A_+ \left( \frac{s}{t}, \alpha_s, \epsilon \right) C_b \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \kappa C_a \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) A_- \left( \frac{s}{t}, \alpha_s, \epsilon \right) C_b \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \mathcal{R}_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \mathcal{O} \left( \frac{t}{s} \right) \right],$$

- Here the signature factor  $\kappa = \frac{4 - N_c^2}{N_c^2}$  for quarks, while  $\kappa = 1$  for gluons.

- The Regge trajectory appears in the (anti)symmetrized factors

$$A_{\pm} \left( \frac{s}{t}, \alpha_s, \epsilon \right) = \left( \frac{-s}{-t} \right)^{\alpha(t)} \pm \left( \frac{s}{-t} \right)^{\alpha(t)},$$

- Regge factorization is proved only at LL and at NLL for the real part of the amplitude, but it is valid for finite terms as well.

# Regge master formula

- Regge factorization, under the assumption of 'only poles' in the  $J$  plane, and including crossing information, leads to a 'master formula' for color-octet  $t$ -channel exchange.

$$\mathcal{M}_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon \right) = 2\pi\alpha_s H_{ab}^{(0),[8]} \left[ C_a \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) A_+ \left( \frac{s}{t}, \alpha_s, \epsilon \right) C_b \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \kappa C_a \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) A_- \left( \frac{s}{t}, \alpha_s, \epsilon \right) C_b \left( \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \mathcal{R}_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon \right) + \mathcal{O} \left( \frac{t}{s} \right) \right],$$

- Here the signature factor  $\kappa = \frac{4 - N_c^2}{N_c^2}$  for quarks, while  $\kappa = 1$  for gluons.

- The Regge trajectory appears in the (anti)symmetrized factors

$$A_{\pm} \left( \frac{s}{t}, \alpha_s, \epsilon \right) = \left( \frac{-s}{-t} \right)^{\alpha(t)} \pm \left( \frac{s}{-t} \right)^{\alpha(t)},$$

- Regge factorization is proved only at LL and at NLL for the real part of the amplitude, but it is valid for finite terms as well.

- We have introduced a color-octet non-factorizing remainder function  $\mathcal{R}_{ab}^{[8]}$ .

- The remainder function starts at NNLL for the real part, but could in principle have NLL imaginary parts. They will turn out to vanish.

- We consider **quark** and **gluon four-point** amplitudes in **QCD**. **Soft-collinear factorization** leads to a '**master formula**' for these amplitudes valid to **leading power** in **t/s**.

$$\begin{aligned} \mathcal{M} \left( \frac{s}{t}, \alpha_s \right) &= \left[ \prod_{i=1}^4 \mathcal{Z}_{1,\mathbf{R}}^{(i)} \left( \frac{t}{\mu^2}, \alpha_s \right) \right] \tilde{\mathcal{Z}}_S \left( \frac{s}{t}, \alpha_s \right) \mathcal{H} \left( \frac{s}{t}, \alpha_s \right) \\ \tilde{\mathcal{Z}}_S \left( \frac{s}{t}, \alpha_s \right) &= \exp \left\{ K(\alpha_s) \left[ \left( \log \left( \frac{s}{-t} \right) - i \frac{\pi}{2} (1 + \kappa_{ab}) \right) \mathbf{T}_t^2 + i \frac{\pi}{2} (\mathbf{T}_s^2 - \mathbf{T}_u^2 + \kappa_{ab} \mathbf{T}_t^2) \right] \right\} \\ \mathcal{Z}_{1,\mathbf{R}}^{(i)} \left( \frac{t}{\mu^2}, \alpha_s \right) &= \exp \left\{ \frac{1}{2} \left[ K(\alpha_s) \log \left( \frac{-t}{\mu^2} \right) + D(\alpha_s) \right] \mathcal{C}^{(i)} + B^{(i)}(\alpha_s) \right\} \end{aligned}$$

- We introduced '**Mandelstam**' **color** operators, and used **color** and **momentum** conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4), & \mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 &= \sum_{i=1}^4 C_i \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4), & s + t + u &= 0 \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3) \end{aligned}$$

- Coupling** dependence for **leading** logarithms is completely **determined** by the **cusp** anomalous dimension and by the  **$\beta$  function**, through the integral (Korchinsky 94-96)

$$K(\alpha_s(\mu^2), \epsilon) = -\frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \hat{\gamma}_K(\alpha_s(\lambda^2), \epsilon) = \frac{\alpha_s}{\pi} \frac{\hat{\gamma}_K^{(1)}}{4\epsilon} + \left( \frac{\alpha_s}{\pi} \right)^2 \left( \frac{\hat{\gamma}_K^{(2)}}{8\epsilon} - \frac{b_0 \hat{\gamma}_K^{(1)}}{32\epsilon^2} \right) + \mathcal{O}(\alpha_s^3)$$

# The rise and fall of Reggeization

- At **leading logarithmic** accuracy, only the **t**-channel contribution survives. If, at **LO** and at **leading power** in **t/s**, the scattering is **dominated** by **t**-channel exchange, then the **hard function** is an **eigenstate** of the color operator  $\mathbf{T}_t^2$ ,

$$\mathbf{T}_t^2 \mathcal{H}^{gg \rightarrow gg} \xrightarrow{|t/s| \rightarrow 0} C_t \mathcal{H}_t^{gg \rightarrow gg}$$

- Leading-logarithmic **Reggeization** for **arbitrary t**-channel color representations **follows**, with the Regge trajectory given by the integral **K** times the appropriate Casimir eigenvalue.

$$\mathcal{M}^{gg \rightarrow gg} = \left( \frac{s}{-t} \right)^{C_A K(\alpha_s(\mu^2), \epsilon)} Z_1 \mathcal{H}_t^{gg \rightarrow gg}$$

- The infrared **operator Z** can be **systematically expanded** beyond **LL**, using the **BCH** formula. The **real part** of the amplitude **Reggeizes** at **NLL** for **general t**-channel exchanges.
- At **NNLL** **Regge pole factorization** generically **breaks down**.
  - At **two loops**, terms that are **non-logarithmic** and **non-diagonal** in a **t**-channel basis arise.
  - At **three loops**, the first Reggeization-breaking **logarithms** of **s/t** arise.

$$\mathcal{E}_0(\alpha_s, \epsilon) \equiv -\frac{1}{2}\pi^2 K^2(\alpha_s, \epsilon) (\mathbf{T}_s^2)^2 \quad \mathcal{E}_1\left(\frac{s}{t}, \alpha_s, \epsilon\right) \equiv -\frac{\pi^2}{3} K^3(\alpha_s, \epsilon) \ln\left(\frac{s}{-t}\right) [\mathbf{T}_s^2, [\mathbf{T}_t^2, \mathbf{T}_s^2]]$$

- The **color commutator** terms are related to **Glauber** exchanges (Forshaw *et al.*, Catani *et al.*)



# Finite order expansions

📌 To proceed, we **expand** all factors **in powers** of the **coupling** and of the **high-energy logarithm**.

$$\tilde{Z}\left(\frac{s}{t}, \alpha_s, \epsilon\right) = \sum_{n=0}^{\infty} \sum_{i=0}^n \left(\frac{\alpha_s}{\pi}\right)^n \log^i\left(\frac{s}{-t}\right) \tilde{Z}^{(n),i}(\epsilon),$$

$$\mathcal{R}_{ab}^{[8]}\left(\frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon\right) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \left(\frac{\alpha_s}{\pi}\right)^n \log^k\left(\frac{s}{-t}\right) R_{ab}^{(n),i,[8]}\left(\frac{t}{\mu^2}, \epsilon\right),$$

# Finite order expansions

To proceed, we **expand** all factors **in powers** of the **coupling** and of the **high-energy logarithm**.

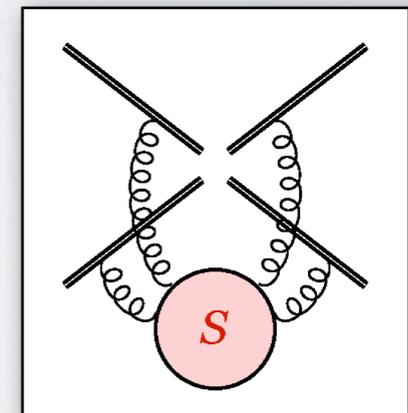
$$\tilde{Z}\left(\frac{s}{t}, \alpha_s, \epsilon\right) = \sum_{n=0}^{\infty} \sum_{i=0}^n \left(\frac{\alpha_s}{\pi}\right)^n \log^i\left(\frac{s}{-t}\right) \tilde{Z}^{(n),i}(\epsilon),$$

$$\mathcal{R}_{ab}^{[8]}\left(\frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon\right) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \left(\frac{\alpha_s}{\pi}\right)^n \log^k\left(\frac{s}{-t}\right) R_{ab}^{(n),i,[8]}\left(\frac{t}{\mu^2}, \epsilon\right),$$

At **one loop**, for example, **soft-collinear** factorization yields

$$M^{(1),0} = \left[ Z_{1,\mathbf{R}}^{(1)} + i\pi K_1 \left( \mathbf{T}_s^2 - \frac{1}{2} \mathcal{C}_{\text{tot}} \right) \right] H^{(0)} + H^{(1),0},$$

$$M^{(1),1} = K_1 \mathbf{T}_t^2 H^{(0)} + H^{(1),1},$$



# Finite order expansions

To proceed, we **expand** all factors **in powers** of the **coupling** and of the **high-energy logarithm**.

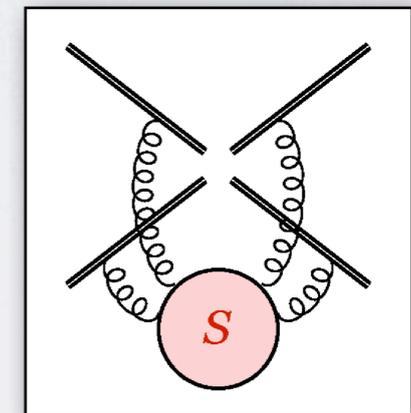
$$\tilde{Z}\left(\frac{s}{t}, \alpha_s, \epsilon\right) = \sum_{n=0}^{\infty} \sum_{i=0}^n \left(\frac{\alpha_s}{\pi}\right)^n \log^i\left(\frac{s}{-t}\right) \tilde{Z}^{(n),i}(\epsilon),$$

$$\mathcal{R}_{ab}^{[8]}\left(\frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon\right) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \left(\frac{\alpha_s}{\pi}\right)^n \log^k\left(\frac{s}{-t}\right) R_{ab}^{(n),i,[8]}\left(\frac{t}{\mu^2}, \epsilon\right),$$

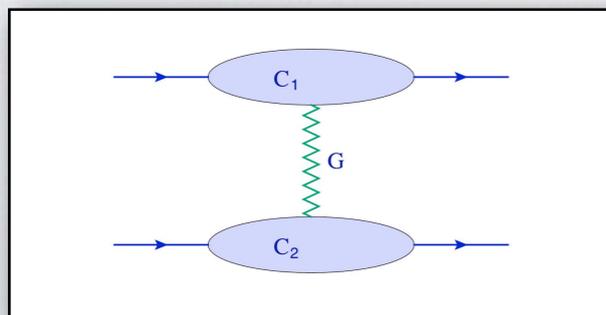
At **one loop**, for example, **soft-collinear** factorization yields

$$M^{(1),0} = \left[ Z_{1,\mathbf{R}}^{(1)} + i\pi K_1 \left( \mathbf{T}_s^2 - \frac{1}{2} \mathcal{C}_{\text{tot}} \right) \right] H^{(0)} + H^{(1),0},$$

$$M^{(1),1} = K_1 \mathbf{T}_t^2 H^{(0)} + H^{(1),1},$$



For the **octet component** of the same matrix elements, **Regge** factorization yields



$$M_{ab}^{(1),0,[8]} = \left[ C_a^{(1)} + C_b^{(1)} - i\frac{\pi}{2} (1 + \kappa) \alpha^{(1)} \right] H_{ab}^{(0),[8]},$$

$$M_{ab}^{(1),1,[8]} = \alpha^{(1)} H_{ab}^{(0),[8]},$$

# Two loops: universality breaking

📌 At **two loops** and at **NNLL** the **impact factors** display **universality breaking**

$$C_a^{(2)} = \frac{1}{2} Z_{1,\mathbf{R},aa}^{(2)} - \frac{1}{8} \left( Z_{1,\mathbf{R},aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,\mathbf{R},aa}^{(1)} \operatorname{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right] - \frac{1}{2} R_{aa}^{(2),0,[8]} - \frac{\pi^2}{4} K_1^2 \left\{ \left[ (\mathbf{T}_{s,aa}^2) \right]_{[8],[8]} - \mathcal{C}_{\text{tot},aa} \left[ \mathbf{T}_{s,aa}^2 \right]_{[8],[8]} + \frac{1}{4} \mathcal{C}_{\text{tot},aa}^2 - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}(\epsilon^0)$$

# Two loops: universality breaking

📌 At **two loops** and at **NNLL** the **impact factors** display **universality breaking**

$$C_a^{(2)} = \boxed{\frac{1}{2} Z_{1,\mathbf{R},aa}^{(2)} - \frac{1}{8} \left( Z_{1,\mathbf{R},aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,\mathbf{R},aa}^{(1)} \operatorname{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right]} - \frac{1}{2} R_{aa}^{(2),0,[8]} - \frac{\pi^2}{4} K_1^2 \left\{ \left[ \left( \mathbf{T}_{s,aa}^2 \right)^2 \right]_{[8],[8]} - \mathcal{C}_{\text{tot},aa} \left[ \mathbf{T}_{s,aa}^2 \right]_{[8],[8]} + \frac{1}{4} \mathcal{C}_{\text{tot},aa}^2 - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}(\epsilon^0)$$

Universal, real, color singlet, from jets

Define as proper impact factor

# Two loops: universality breaking

At **two loops** and at **NNLL** the **impact factors** display **universality breaking**

$$C_a^{(2)} = \left[ \frac{1}{2} Z_{1,\mathbf{R},aa}^{(2)} - \frac{1}{8} \left( Z_{1,\mathbf{R},aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,\mathbf{R},aa}^{(1)} \operatorname{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right] - \frac{1}{2} R_{aa}^{(2),0,[8]} \right] - \frac{\pi^2}{4} K_1^2 \left\{ \left[ \left( \mathbf{T}_{s,aa}^2 \right)^2 \right]_{[8],[8]} - \mathcal{C}_{\text{tot},aa} \left[ \mathbf{T}_{s,aa}^2 \right]_{[8],[8]} + \frac{1}{4} \mathcal{C}_{\text{tot},aa}^2 - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}(\epsilon^0)$$

Universal, real, color singlet, from jets

Define as proper impact factor

Non universal, from phases, color-mixed

Assign to non-factorizing remainder

# Two loops: universality breaking

At **two loops** and at **NNLL** the **impact factors** display **universality breaking**

$$C_a^{(2)} = \left[ \frac{1}{2} Z_{1,\mathbf{R},aa}^{(2)} - \frac{1}{8} \left( Z_{1,\mathbf{R},aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,\mathbf{R},aa}^{(1)} \operatorname{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right] \right] - \frac{1}{2} R_{aa}^{(2),0,[8]} - \frac{\pi^2}{4} K_1^2 \left\{ \left[ \left( \mathbf{T}_{s,aa}^2 \right)^2 \right]_{[8],[8]} - C_{\text{tot},aa} \left[ \mathbf{T}_{s,aa}^2 \right]_{[8],[8]} + \frac{1}{4} C_{\text{tot},aa}^2 - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}(\epsilon^0)$$

Universal, real, color singlet, from jets

Define as proper impact factor

Non universal, from phases, color-mixed

Assign to non-factorizing remainder

Quark and gluon impact factors derived from **qq** and **gg** amplitudes must **properly form** the **qg** amplitude. **Assuming** factorization this **fails** (Del Duca and Glover 2001). Indeed **defining**

$$\begin{aligned} \Delta_{(2),0,[8]} &= M_{qg}^{(2),0,[8]} - \left[ C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4} (1+\kappa) (\alpha^{(1)})^2 \right] H_{qg}^{(0),[8]} \\ &= \tilde{R}_{qg}^{(2),0,[8]} - \frac{1}{2} \left( \tilde{R}_{qq}^{(2),0,[8]} + \tilde{R}_{gg}^{(2),0,[8]} \right) \end{aligned}$$

# Two loops: universality breaking

At **two loops** and at **NNLL** the **impact factors** display **universality breaking**

$$C_a^{(2)} = \boxed{\frac{1}{2} Z_{1,\mathbf{R},aa}^{(2)} - \frac{1}{8} \left( Z_{1,\mathbf{R},aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,\mathbf{R},aa}^{(1)} \operatorname{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right]} - \frac{1}{2} R_{aa}^{(2),0,[8]} - \frac{\pi^2}{4} K_1^2 \left\{ \left[ \left( \mathbf{T}_{s,aa}^2 \right)^2 \right]_{[8],[8]} - C_{\text{tot},aa} \left[ \mathbf{T}_{s,aa}^2 \right]_{[8],[8]} + \frac{1}{4} C_{\text{tot},aa}^2 - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}(\epsilon^0)$$

Universal, real, color singlet, from jets

Define as proper impact factor

Non universal, from phases, color-mixed

Assign to non-factorizing remainder

Quark and gluon impact factors derived from **qq** and **gg** amplitudes must **properly form** the **qg** amplitude. **Assuming** factorization this **fails** (Del Duca and Glover 2001). Indeed **defining**

$$\begin{aligned} \Delta_{(2),0,[8]} &= M_{qg}^{(2),0,[8]} - \left[ C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4} (1+\kappa) (\alpha^{(1)})^2 \right] H_{qg}^{(0),[8]} \\ &= \tilde{R}_{qg}^{(2),0,[8]} - \frac{1}{2} \left( \tilde{R}_{qq}^{(2),0,[8]} + \tilde{R}_{gg}^{(2),0,[8]} \right) \end{aligned}$$

we **find** that

$$\Delta_{(2),0,[8]} = \frac{\pi^2 K_1^2}{2} \left[ \frac{3}{2} \left( \frac{N_c^2 + 1}{N_c^2} \right) \right] = \frac{\pi^2}{\epsilon^2} \frac{3}{16} \left( \frac{N_c^2 + 1}{N_c^2} \right) \quad \checkmark$$



# Three loops: factorization breaking

- The **real single-logarithmic** contribution at **NNLL** should give the **Regge trajectory** at **three loops**. As expected, **fitting** the coefficient we find a **non-universal** result. Using

$$\left[ \left( \mathbf{T}_t^2 (\mathbf{T}_s^2)^2 + \mathbf{T}_s^2 \mathbf{T}_t^2 \mathbf{T}_s^2 + (\mathbf{T}_s)^2 \mathbf{T}_t^2 \right) H^{(0)} \right]^{[8]} = \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_s^2)_{[8],[n]} \right|^2 H^{(0),[8]}$$

we find

$$\begin{aligned} \alpha_{\text{fit}}^{(3)} = & C_A K_3 + \frac{\pi^2 K_1^3}{2} \left[ \mathcal{C}_{\text{tot},ij} N_c (\mathbf{T}_{s,ij}^2)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4} + \frac{1 + \kappa}{2} N_c^3 \right. \\ & \left. - \frac{1}{3} \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_{s,ij}^2)_{[8],n} \right|^2 \right] - R^{(3),1,[8]} + \mathcal{O}(\epsilon^{-2}) \end{aligned}$$

# Three loops: factorization breaking

- The **real single-logarithmic** contribution at **NNLL** should give the **Regge trajectory** at **three loops**. As expected, **fitting** the coefficient we find a **non-universal** result. Using

$$\left[ \left( \mathbf{T}_t^2 (\mathbf{T}_s^2)^2 + \mathbf{T}_s^2 \mathbf{T}_t^2 \mathbf{T}_s^2 + (\mathbf{T}_s)^2 \mathbf{T}_t^2 \right) H^{(0)} \right]^{[8]} = \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_s^2)_{[8],[n]} \right|^2 H^{(0),[8]}$$

we find

$$\alpha_{\text{fit}}^{(3)} = \underbrace{C_A K_3}_{\text{universal}} + \frac{\pi^2 K_1^3}{2} \left[ \mathcal{C}_{\text{tot},ij} N_c (\mathbf{T}_{s,ij}^2)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4} + \frac{1 + \kappa}{2} N_c^3 \right. \\ \left. - \frac{1}{3} \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_{s,ij}^2)_{[8],n} \right|^2 \right] - R^{(3),1,[8]} + \mathcal{O}(\epsilon^{-2})$$

← non universal

# Three loops: factorization breaking

- The **real single-logarithmic** contribution at **NNLL** should give the **Regge trajectory** at **three loops**. As expected, **fitting** the coefficient we find a **non-universal** result. Using

$$\left[ \left( \mathbf{T}_t^2 (\mathbf{T}_s^2)^2 + \mathbf{T}_s^2 \mathbf{T}_t^2 \mathbf{T}_s^2 + (\mathbf{T}_s)^2 \mathbf{T}_t^2 \right) H^{(0)} \right]^{[8]} = \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_s^2)_{[8],[n]} \right|^2 H^{(0),[8]}$$

we find

$$\alpha_{\text{fit}}^{(3)} = \underbrace{C_A K_3}_{\text{universal}} + \frac{\pi^2 K_1^3}{2} \left[ \mathcal{C}_{\text{tot},ij} N_c (\mathbf{T}_{s,ij}^2)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4} + \frac{1 + \kappa}{2} N_c^3 \right. \\ \left. - \frac{1}{3} \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_{s,ij}^2)_{[8],n} \right|^2 \right] - R^{(3),1,[8]} + \mathcal{O}(\epsilon^{-2})$$

non universal

- We **define** then a single-logarithmic three-loop non-factorizing **remainder** as

$$\tilde{R}_{ij}^{(3),1,[8]} = \pi^2 K_1^3 \left[ \mathcal{C}_{\text{tot},ij} N_c (\mathbf{T}_{s,ij}^2)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4} \right. \\ \left. + \frac{1 + \kappa}{2} N_c^3 - \frac{1}{3} \sum_n (2N_c + \mathcal{C}_{[n]}) \left| (\mathbf{T}_{s,ij}^2)_{[8],n} \right|^2 \right] + \mathcal{O}(\epsilon^{-2})$$

- In the context of **Regge theory** the contributions could **source** a **new resummation** associated with **Regge cuts** in the **J** plane.

# Three loops and beyond

Explicitly, the non-factorizing remainders at three loops for qq, gg and qg amplitudes are

$$\begin{aligned}\tilde{R}_{qq}^{(3),1,[8]} &= \frac{2}{3}\pi^2 K_1^3 \frac{2N_c^2 - 5}{N_c} = \frac{\pi^2}{\epsilon^3} \frac{2N_c^2 - 5}{12N_c}, \\ \tilde{R}_{gg}^{(3),1,[8]} &= -\frac{16}{3}\pi^2 K_1^3 N_c = -\frac{\pi^2}{\epsilon^3} \frac{2}{3} N_c \\ \tilde{R}_{qg}^{(3),1,[8]} &= -\frac{1}{3}\pi^2 K_1^3 N_c = -\frac{\pi^2}{\epsilon^3} \frac{N_c}{24}.\end{aligned}$$

Note that all remainders are subleading in  $N_c$  as they must.

Furthermore, one can prove a sequence of all-order identities for the hard parts

$$\begin{aligned}\text{Im}(\hat{H}^{(n),n,[8]}) &= 0 \\ \text{Re}(\hat{H}^{(n),n,[8]}) &= \frac{1}{n!} \left(\hat{H}^{(1),1,[8]}\right)^n = O(\epsilon^n) \\ \text{Im}(\hat{H}^{(n),n-1,[8]}) &= -\pi \frac{1+\kappa}{2} \left(n\hat{H}^{(n),n,[8]}\right) = O(\epsilon^n) \\ \text{Re}(\hat{H}^{(n),n-1,[8]}) &= \text{Re}(\hat{H}^{(2),1})\hat{H}^{(n-2),n-2} + (2-n)\text{Re}(\hat{H}^{(1),0,[8]})\hat{H}^{(n-1),n-1} \\ &= O(\epsilon^{n-2})\end{aligned}$$

Up to NLL, hard parts vanish in  $d = 4$ : Reggeization is driven by infrared divergences.

# WEAVING MULTI-PARTICLE WEBS



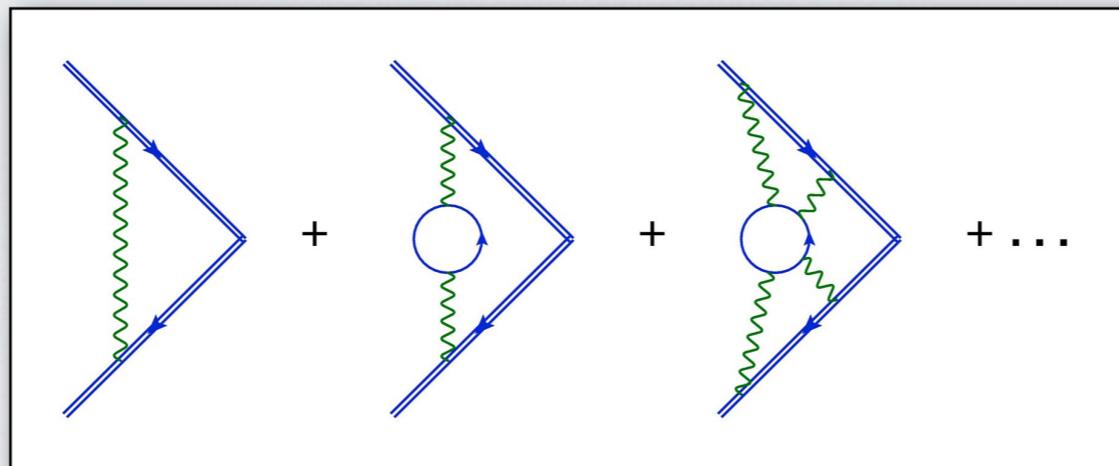
# Eikonal exponentiation

All correlators of Wilson lines, regardless of shape, resum in **exponential form**.

$$S_n \equiv \langle 0 | \Phi_1 \otimes \dots \otimes \Phi_n | 0 \rangle = \exp(\omega_n)$$

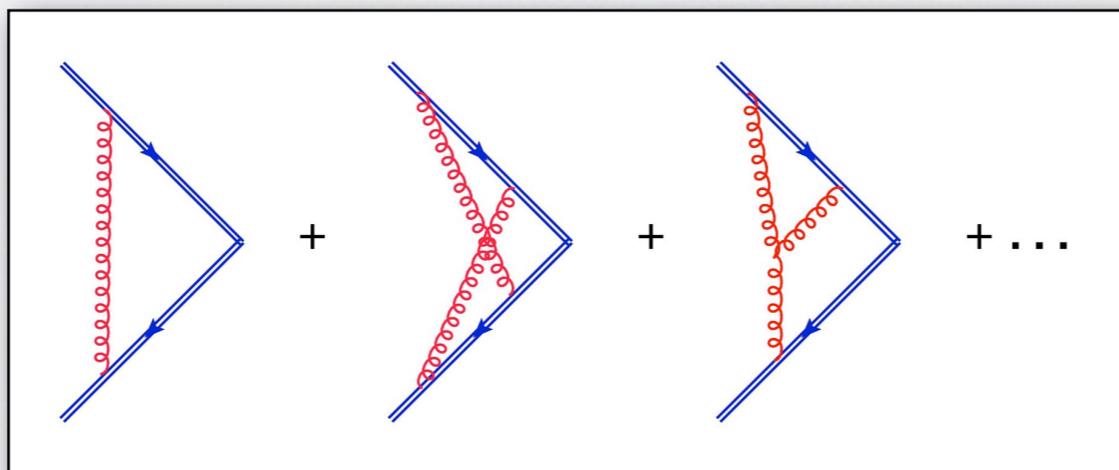
Diagrammatic rules exist to compute **directly the logarithm** of the correlators.

$$\omega_{2,\text{QED}} =$$



Only **connected** photon **subdiagrams** contribute to the logarithm.

$$\omega_{2,\text{QCD}} =$$



Only gluon **subdiagrams** which are **two-eikonal irreducible** contribute to the logarithm. They have **modified color factors**.

For **eikonal form factors**, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

# Multiparticle webs

The concept of **web** generalizes non-trivially to the case of **multiple Wilson lines**.  
(Gardi, Smillie, White, et al).

A **web** is a **set of diagrams** which **differ** only by the **order** of the **gluon attachments** on each Wilson line. They are **weighted** by **modified color factors**.

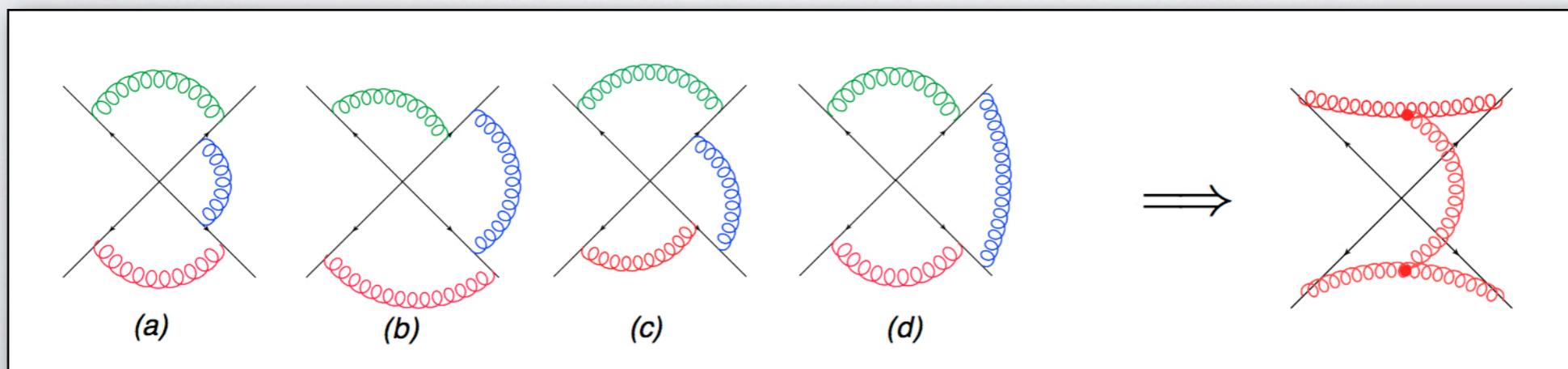
Writing each diagram as the product of its natural **color** factor and a **kinematic** factor

$$D = C(D)\mathcal{F}(D)$$

a **web**  $W$  can be expressed as a **sum of diagrams** in terms of a **web mixing matrix**  $R$

$$W = \sum_D \tilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The **non-abelian exponentiation theorem** holds: each web has the color factor of a **fully connected** gluon subdiagram (Gardi, Smillie, White).



**Bare** Wilson-line correlators **vanish** beyond tree level in **dimensional regularization**: they are given by **scale-less integrals**. We require **renormalized** correlators, which depend on the **Minkowsky angles** between the Wilson lines.

$$S_{\text{ren}}(\gamma_{ij}, \alpha_s, \epsilon) = S_{\text{bare}}(\gamma_{ij}, \alpha_s, \epsilon) Z(\gamma_{ij}, \alpha_s, \epsilon) = Z(\gamma_{ij}, \alpha_s, \epsilon), \quad \gamma_{ij} = \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}}$$

To compute the **counterterm** **Z** we make use of an **auxiliary, IR-regularized** correlator

$$\begin{aligned} \hat{S}_{\text{ren}}(\gamma_{ij}, \alpha_s, \epsilon, m) &= \hat{S}_{\text{bare}}(\gamma_{ij}, \alpha_s, \epsilon, m) Z(\gamma_{ij}, \alpha_s, \epsilon) \\ &\equiv \exp(\omega) \exp(\zeta) = \exp\left\{\omega + \zeta + \frac{1}{2}[\omega, \zeta] + \dots\right\} \end{aligned}$$

The expression of **Z** in terms of the **anomalous dimension**  $\Gamma$  follows from **RG** arguments

$$Z = \exp\left[\frac{\alpha_s}{\pi} \frac{1}{2\epsilon} \Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{1}{4\epsilon} \Gamma^{(2)} - \frac{b_0}{4\epsilon^2} \Gamma^{(1)}\right) + \left(\frac{\alpha_s}{\pi}\right)^3 \left(\frac{1}{6\epsilon} \Gamma^{(3)} + \frac{1}{48\epsilon^2} [\Gamma^{(1)}, \Gamma^{(2)}] + \dots\right)\right]$$

**Combining** informations one can **get**  $\Gamma$  directly from the **logarithm** of the **regularized S**

$$\begin{aligned} \Gamma^{(1)} &= -2\omega^{(1,-1)} \\ \Gamma^{(2)} &= -4\omega^{(2,-1)} - 2\left[\omega^{(1,-1)}, \omega^{(1,0)}\right] \end{aligned} \quad \omega = \sum_{n=1}^{\infty} \sum_{k=-n}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \epsilon^k \omega^{(n,k)}$$

Computing **regularized webs** is a game of **combinatorics** and **renormalization** theory.

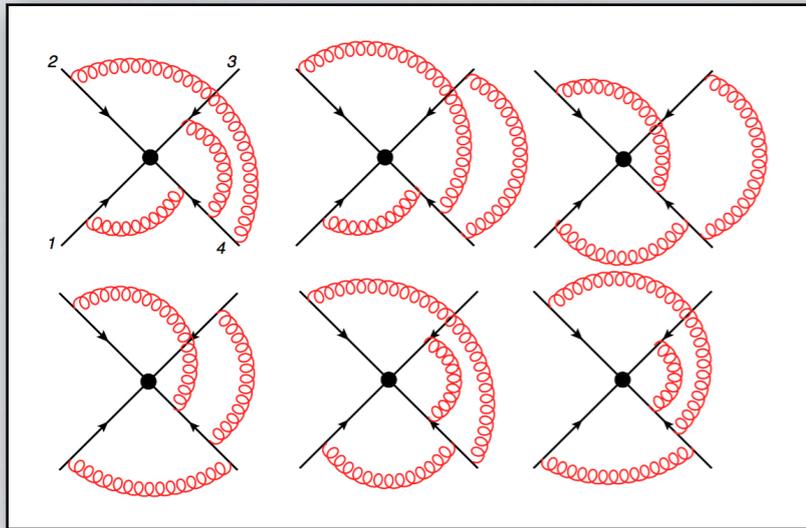


# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.

# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.

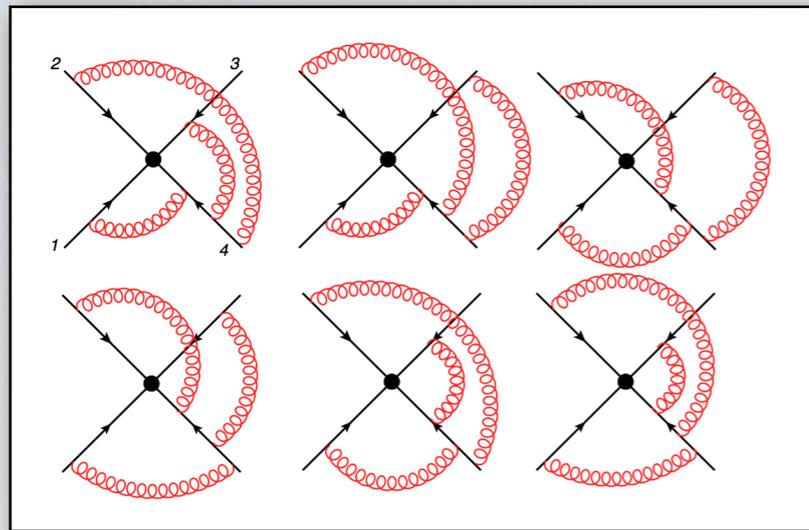


Gardi 1310.5268  
Falcioni et al. 1407.3477.

A Multiple Gluon Exchange Web

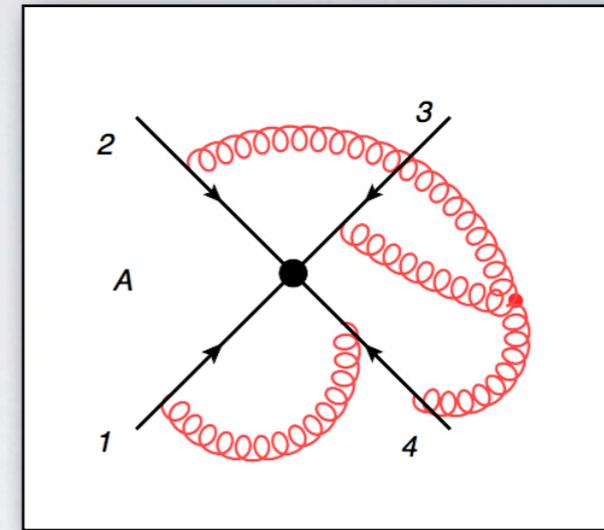
# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.



A Multiple Gluon Exchange Web

Gardi 1310.5268  
Falcioni et al. 1407.3477.

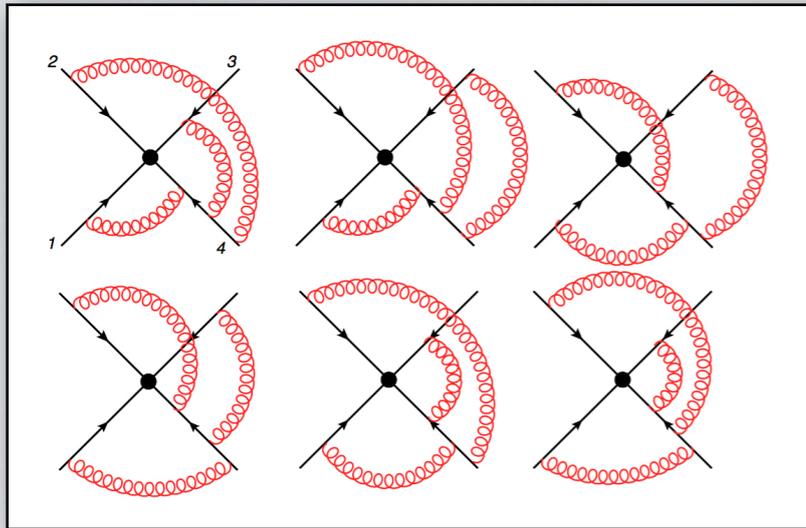


A Tripole Web

Falcioni et al.,  
in progress.

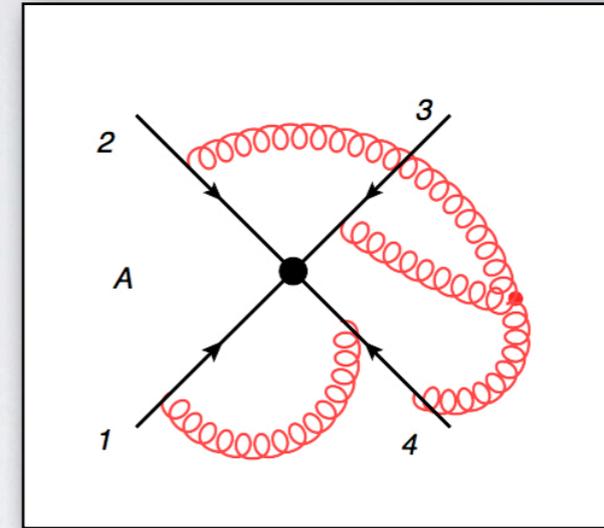
# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.



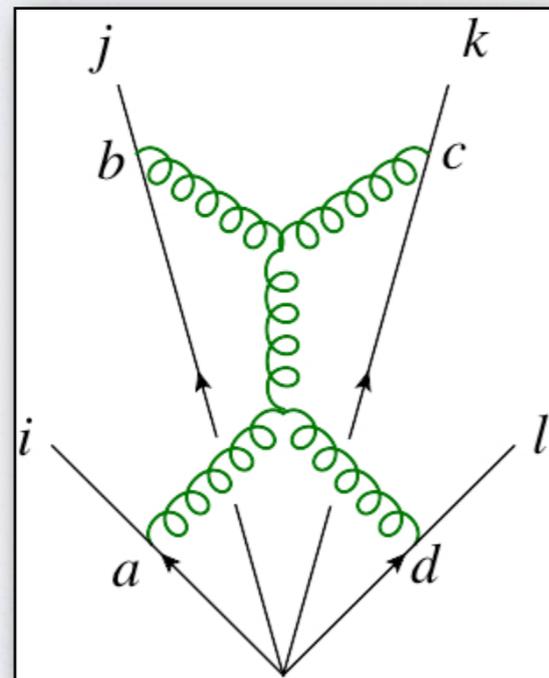
A Multiple Gluon Exchange Web

Gardi 1310.5268  
Falcioni et al. 1407.3477.



A Tripole Web

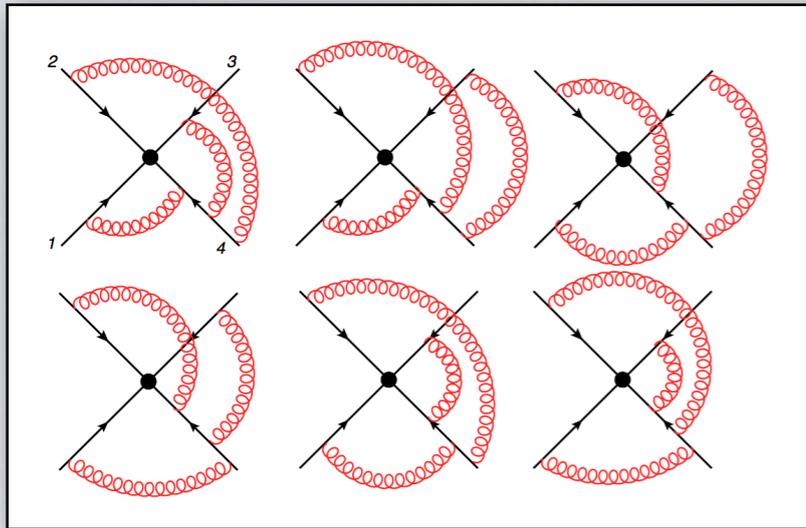
Falcioni et al.,  
in progress.



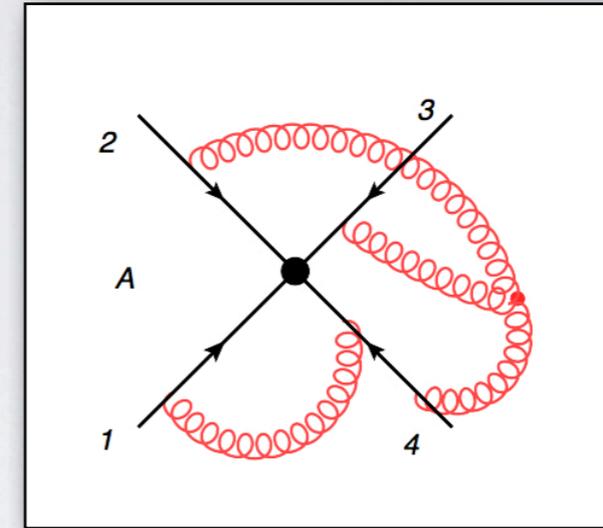
A Quadrupole Web

# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.



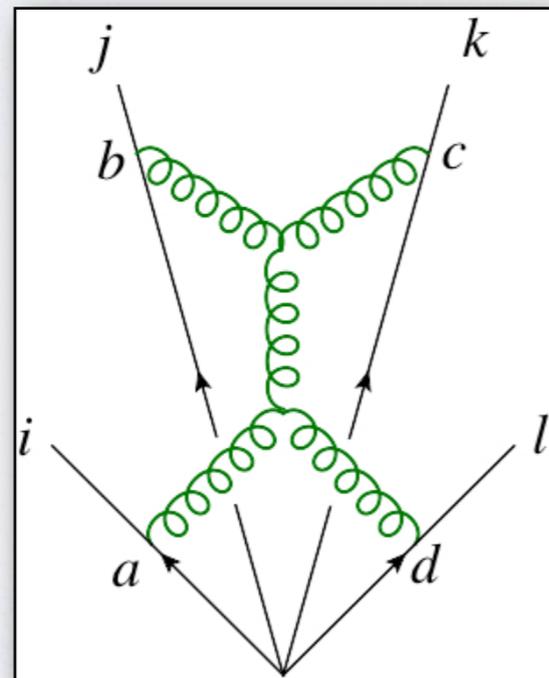
Gardi 1310.5268  
Falcioni et al. 1407.3477.



Falcioni et al.,  
in progress.

A Multiple Gluon Exchange Web

A Tripole Web

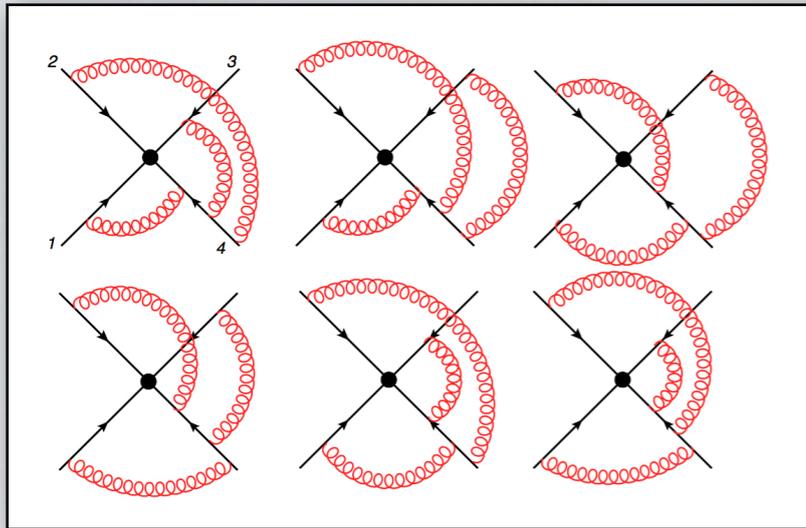


A Quadrupole Web



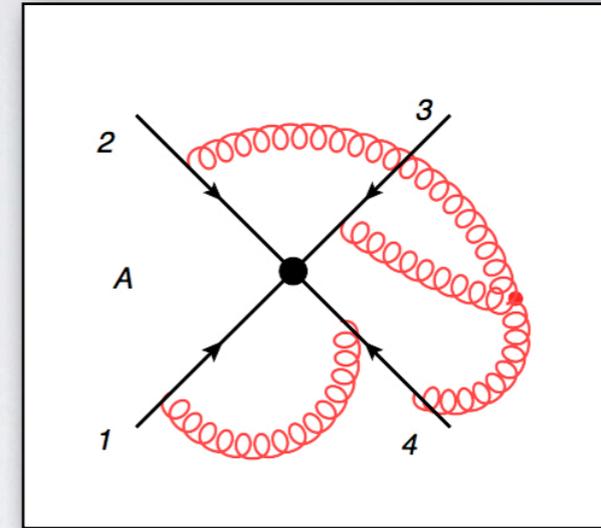
# Three-loop progress

The computation of the **three-loop** multi-particle **soft anomalous dimension** is **under way**.



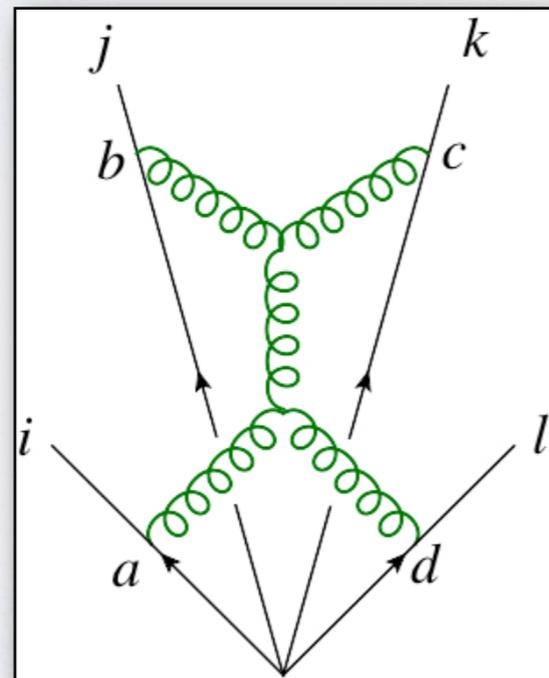
Gardi 1310.5268  
Falcioni et al. 1407.3477.

A Multiple Gluon Exchange Web



Falcioni et al.,  
in progress.

A Tripole Web



A Quadrupole Web



Almelid, Duhr and Gardi,  
in progress, in the  
**massless** limit.

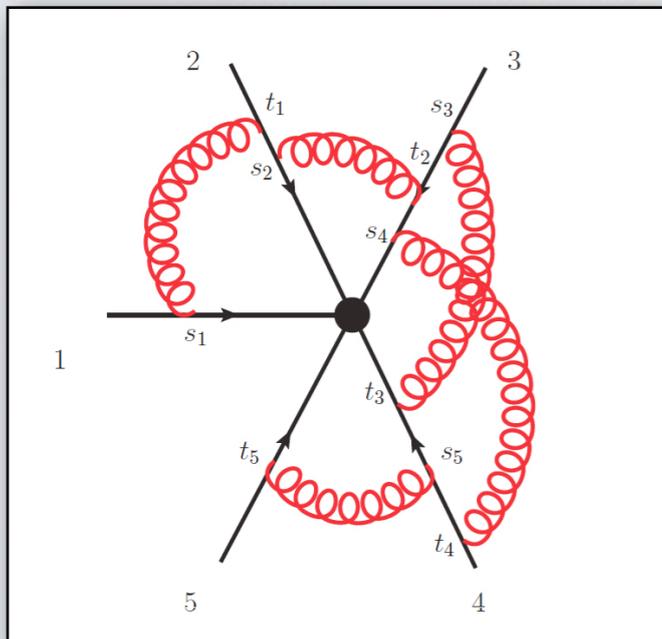
# Multiple Gluon Exchange Webs

**Multiple Gluon Exchange Webs (MGEWs)** arise from a path integral weighted with the **free part** of the quantum YM action

$$\mathcal{S} \left( \gamma_{ij}, \alpha_s(\mu), \epsilon, \frac{m}{\mu} \right) \Big|_{\text{MGEW}} \equiv \int [DA] \Phi_{\beta_1}^{(m)} \otimes \Phi_{\beta_2}^{(m)} \otimes \dots \otimes \Phi_{\beta_L}^{(m)} \exp \left\{ iS_0[A] \right\},$$

A **general integral representation** for diagrams  $D$  contributing to MGEWs can be written down to **all orders**, starting from a coordinate space representation of the Wilson lines

$$\mathcal{F}^{(n)}(D) = \kappa^n \Gamma(2n\epsilon) \int_0^1 \prod_{k=1}^n \left[ dx_k \gamma_k P_\epsilon(x_k, \gamma_k) \right] \phi_D^{(n)}(x_i; \epsilon)$$



A five-loop MGEW diagram

The variables  $x_k$  measure **collinearity** to Wilson lines, the **overall UV pole** is **extracted**, the coordinate-space gluon **propagators** give

$$P_\epsilon(x, \gamma) \equiv \left[ x^2 + (1-x)^2 - \gamma x(1-x) \right]^{-1+\epsilon}$$

**Non-abelian** information is encoded in the **order of gluon attachments** on each Wilson line, through the **kernel**

$$\phi_D^{(n)}(x_i; \epsilon) = \int_0^1 \prod_{k=1}^{n-1} dy_k \left[ (1-y_k)^{-1+2\epsilon} y_k^{-1+2k\epsilon} \right] \Theta_D[\{x_k, y_k\}]$$

**Replacing** the set of  $\mathcal{G}$  functions by **unity** one recovers **abelian** exponentiation.

Individual diagrams contain multiple UV poles and give uniform weight results. One must: combine them into webs (where leading poles cancel); subtract subdivergences via commutator counterterms; organize the result in a color basis. In the “right” variables

$$\gamma \rightarrow -\alpha - \frac{1}{\alpha}, \quad P_\epsilon(x, \gamma) \rightarrow p_\epsilon(x, \alpha)$$

$$p_\epsilon(x, \alpha) = -\left(\alpha + \frac{1}{\alpha}\right) \left[q(x, \alpha)\right]^{-1+\epsilon}, \quad q(x, \alpha) = x^2 + (1-x)^2 + \left(\alpha + \frac{1}{\alpha}\right) x(1-x)$$

Expansion in powers of  $\epsilon$  will generate logarithms of  $q(x, \alpha)$ . Assembling the results

$$\bar{\omega}_{a_1, \dots, a_L}^{(n, -1)} = \sum_A C_{a_1, \dots, a_L}^{(A)} F_A^{(n)}(\alpha_1, \dots, \alpha_n)$$

$$F_A^{(n)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \int_0^1 \left[ \prod_{k=1}^n dx_k p_0(x_k, \alpha_k) \right] \mathcal{G}_n(x_1, \dots, x_n; q(x_1, \alpha_1), \dots, q(x_n, \alpha_n))$$

A lot of experimental and conceptual evidence is accumulating favoring the following

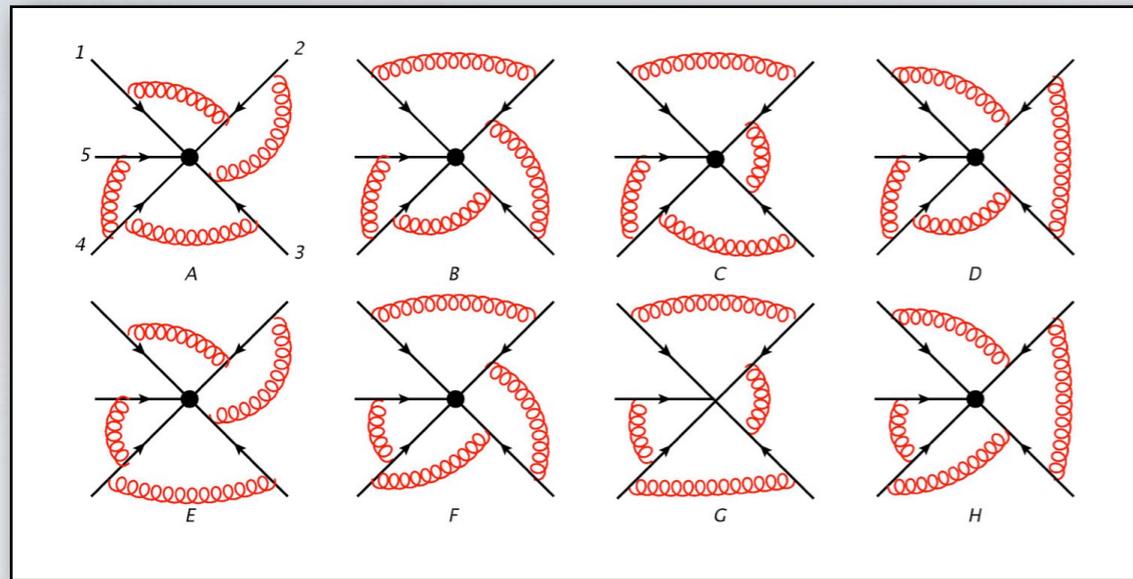
- Factorization conjecture: the subtracted MGEW kernel  $\mathcal{G}$  is a sum of products of logarithmic functions of individual cusp variables of uniform weight  $n-1$ .
- Alphabet conjecture: the Symbol of all subtracted MGEW kinematic coefficients  $F_A$  is restricted to the letters

$$\left\{ \alpha_k, \eta_k \equiv \alpha_k / (1 - \alpha_k^2) \right\}$$



# Four loops, five legs

The **explicit evaluation** of **all** three-loop MGEWs is **nearing completion**. We can however push **further** into the **L&L** space ....



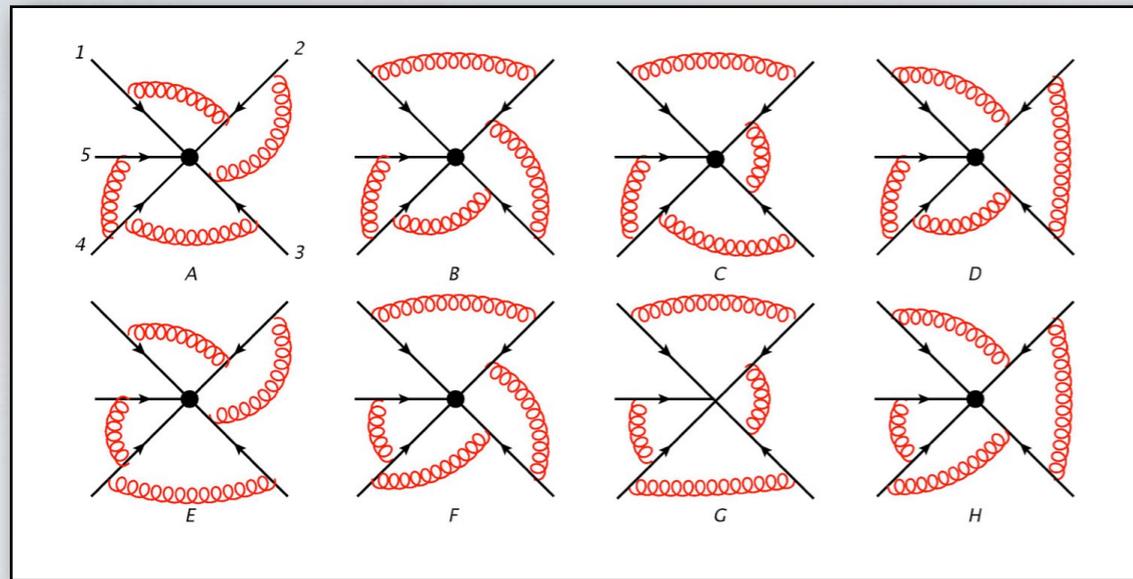
The four-loop five-leg MGEW with attachments 1-2-2-2-1

The **four-loop, five-line** MGEW 1-2-2-2-1

- Contributes to a **single color structure** of the four-loop anomalous dimension.
- It contains **eight diagrams** connected by a mirror symmetry.
- Needs an elaborate set of **nested commutator** counterterms.

# Four loops, five legs

The **explicit evaluation** of **all** three-loop MGEWs is **nearing completion**. We can however push **further** into the **L&L** space ....



The **four-loop, five-line MGEW 1-2-2-2-1**

- Contributes to a **single color structure** of the four-loop anomalous dimension.
- It contains **eight diagrams** connected by a mirror symmetry.
- Needs an elaborate set of **nested commutator** counterterms.

The four-loop five-leg MGEW with attachments 1-2-2-2-1

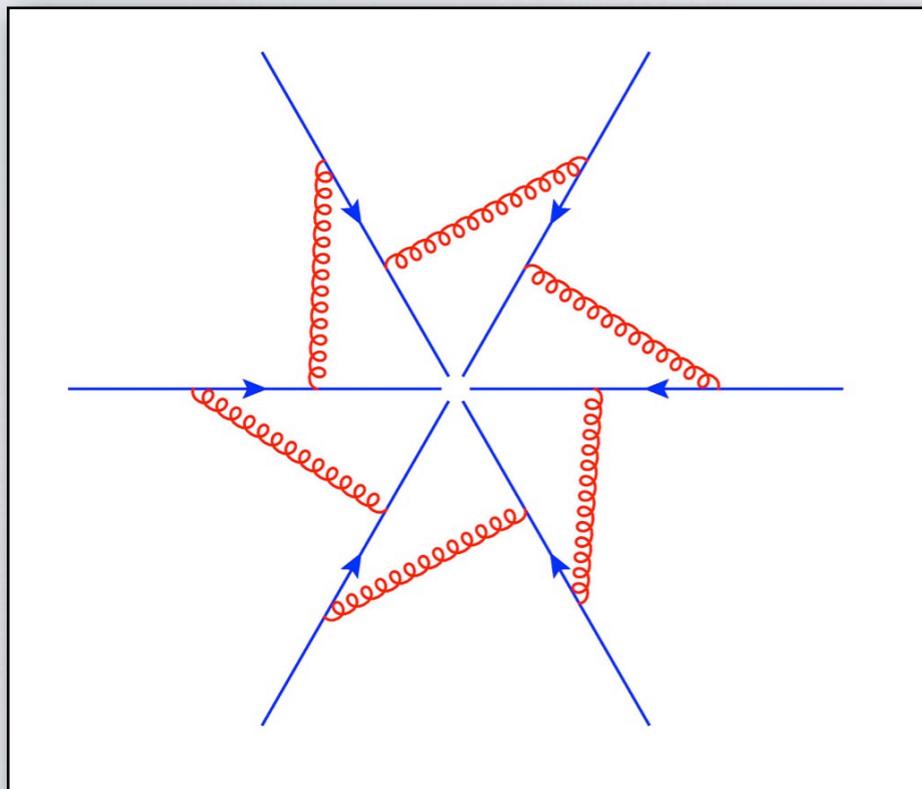
The result is a **simple function** of the logarithms  $L_{ij} = \log \left( \frac{q(x_i, \alpha_{ij})}{x_i^2} \right)$ ,  $\Sigma_i = \log \left( \frac{x_i}{1-x_i} \right)$

$$\begin{aligned}
 \mathcal{G}_{(4)}(x_1, x_2, x_3, x_4) = & -\frac{1}{144} \left\{ L_{12}^3 - 3 L_{23}^3 + 3 L_{34}^3 - L_{45}^3 + 3 L_{12}^2 \left[ L_{23} + L_{34} - 3 L_{45} \right] \right. \\
 & + 3 L_{23}^2 \left[ L_{12} - 3 L_{34} + 5 L_{45} \right] - 3 L_{34}^2 \left[ 5 L_{12} - 3 L_{23} + L_{45} \right] - 3 L_{45}^2 \left[ L_{23} + L_{34} - 3 L_{12} \right] \\
 & + 6 \left[ L_{12} L_{23} L_{34} - 3 L_{12} L_{23} L_{45} + 3 L_{12} L_{34} L_{45} - L_{23} L_{34} L_{45} \right] \\
 & \left. + 24 \left[ \Sigma_2^2 \left( L_{12} + L_{23} + L_{34} - 3 L_{45} \right) - \Sigma_3^2 \left( L_{23} + L_{34} + L_{45} - 3 L_{12} \right) \right] \right\}
 \end{aligned}$$

# The Escher Staircase

Special diagrams contributing to MGEWs have special features, notably those which **do not contain subdivergences**. The **most symmetric** example is the “**Escher Staircase**”, with kernel

$$\begin{aligned} \phi_{ES}^{(n)}(x_i; \epsilon) &= \int_0^\infty \prod_{k=1}^{n-1} d\xi_k \left[ \xi_k^{-1+2k\epsilon} (1 + \xi_k)^{-2(k+1)\epsilon} \right] \widehat{\Theta}_{ES} \left[ \{x_k, \xi_k\} \right] \\ &= \int_{A_1}^{B_1} \frac{d\xi_1}{\xi_1} \int_{A_2(\xi_1)}^{B_2(\xi_1)} \frac{d\xi_2}{\xi_2} \cdots \int_{A_{n-1}(\xi_1, \dots, \xi_{n-2})}^{B_{n-1}(\xi_1, \dots, \xi_{n-2})} \frac{d\xi_{n-1}}{\xi_{n-1}} + \mathcal{O}(\epsilon) \end{aligned}$$



Escher Staircase with six loops and six legs

where  $\xi_k = y_k / (1 - y_k)$

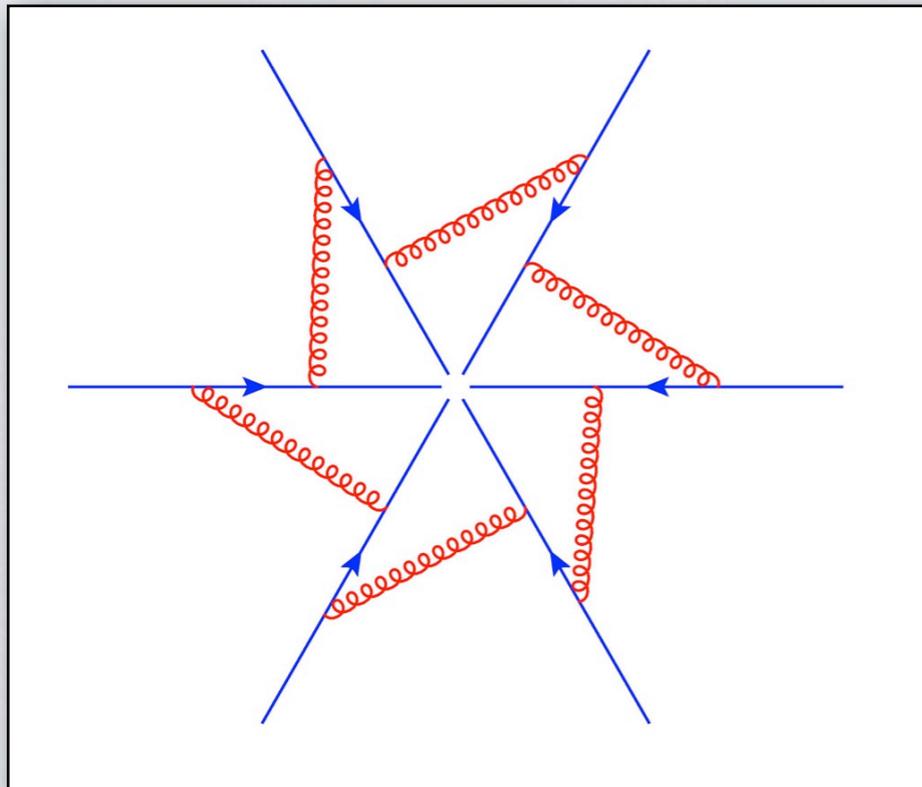
- The integral has a ***d log*** form, with **intricate limits**.
- The **nested** theta functions can be made **explicit**

$$\begin{aligned} A_k(\xi_1, \dots, \xi_{k-1}) &= \frac{x_{k+1}}{1 - x_k} (1 + \xi_{k-1}) \\ B_k(\xi_1, \dots, \xi_{k-1}) &= \frac{\prod_{j=k+1}^n (1 - x_j)}{\prod_{j=k+2}^{n+1} x_j} \prod_{j=1}^{k-1} \frac{1 + \xi_j}{\xi_j} \end{aligned}$$

# The Escher Staircase

Special diagrams contributing to MGEWs have special features, notably those which **do not contain subdivergences**. The **most symmetric** example is the “**Escher Staircase**”, with kernel

$$\begin{aligned} \phi_{ES}^{(n)}(x_i; \epsilon) &= \int_0^\infty \prod_{k=1}^{n-1} d\xi_k \left[ \xi_k^{-1+2k\epsilon} (1 + \xi_k)^{-2(k+1)\epsilon} \right] \widehat{\Theta}_{ES} \left[ \{x_k, \xi_k\} \right] \\ &= \int_{A_1}^{B_1} \frac{d\xi_1}{\xi_1} \int_{A_2(\xi_1)}^{B_2(\xi_1)} \frac{d\xi_2}{\xi_2} \cdots \int_{A_{n-1}(\xi_1, \dots, \xi_{n-2})}^{B_{n-1}(\xi_1, \dots, \xi_{n-2})} \frac{d\xi_{n-1}}{\xi_{n-1}} + \mathcal{O}(\epsilon) \end{aligned}$$



Escher Staircase with six loops and six legs

where  $\xi_k = y_k / (1 - y_k)$

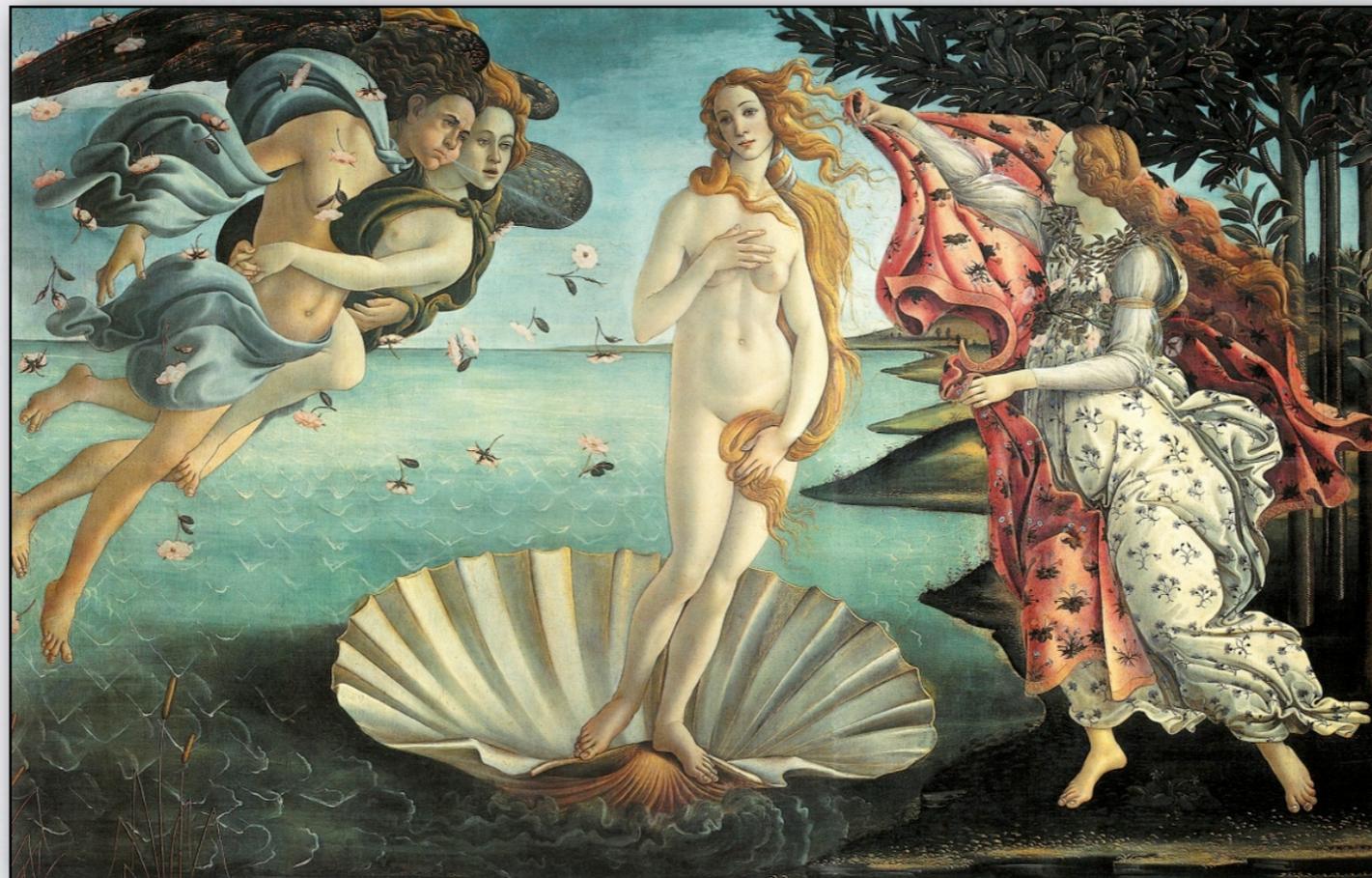
- The integral has a  **$d \log$**  form, with **intricate limits**.
- The **nested** theta functions can be made **explicit**

$$\begin{aligned} A_k(\xi_1, \dots, \xi_{k-1}) &= \frac{x_{k+1}}{1 - x_k} (1 + \xi_{k-1}) \\ B_k(\xi_1, \dots, \xi_{k-1}) &= \frac{\prod_{j=k+1}^n (1 - x_j)}{\prod_{j=k+2}^{n+1} x_j} \prod_{j=1}^{k-1} \frac{1 + \xi_j}{\xi_j} \end{aligned}$$

The result is **remarkably simple!**

$$\phi_{ES}^{(n)}(x_i; 0) = \frac{1}{n!} \left[ \log \left( \prod_{i=1}^n \frac{1 - x_i}{x_i} \right) \right]^{n-1} \Theta^{(n)}(x_i)$$

# OUTLOOK



# Summary

- The past five years have seen **rapid progress** on the structure of **infrared singularities** of **multi-particle** gauge theory amplitudes.
- In the **massless case**, the **dipole formula** is being put to the test at **three** and **four loops**.
- The **high-energy limit** of the dipole formula provides **insights** into **Reggeization** and **beyond**, at least for **divergent contributions** to the amplitude.
- **High-energy factorization** generically **breaks down** at **NNLL**, with **computable** corrections, probably related to **Regge cuts** in the angular momentum plane.
- We **recover** from **first principles** the non-factorizing non-logarithmic **two-loop remainder** of **Del Duca and Glover**.
- We **explicitly predict** the leading **non-factorizing** high-energy **logarithms** at **three loops** for **qq**, **gg** and **qg** amplitudes in QCD.
- Similar **results** can be derived for **multi-parton** amplitudes in **multi-Regge kinematics**.
- We understand **in detail** the general structure of **non-abelian eikonal exponentiation** for **multi-particle** amplitudes.
- Special **classes of diagrams** can be computed to **very high** and in some cases to **all orders**.
- The computation of the **full multi-particle soft anomalous dimension matrix** at the **three-loop** order is feasible in a **finite time**.

THANK YOU TO THE ORGANIZERS!

DIMITRI COLFERAI  
DANIEL DE FLORIAN  
SVEN MOCH  
GUIDO MONTAGNA  
FULVIO PICCININI

THANK YOU!

