The Fake Supergravity Method

To find sols of a bosonic gravity-matter system,

i. formulate fake Killing spinor eqtns.

$$\delta\psi_\mu = (D_\mu + \gamma_\mu W(\varphi))\epsilon = 0$$

$$\delta\lambda = (\gamma\varphi + W')\epsilon = 0$$

ii) Consistency conds $\longrightarrow$ 1st order diff eqtns whose sols automatically satisfy 2nd order Lagrangian EOM's of a gravity matter system (direct attack at 2nd order level difficult).

iii) Inspired by BPS-K Spinor analysis of true SG but it applies to

a) non BPS sols. of SG theories
b) sols. of gravity matter theories only vaguely related to SG
c) it even works for arbitrary D! (no $D \leq 11$ bound)

iv) Given fake BPS eqtns, one can

a) find new sols
b) use Witten-Nester method to prove stability of sols (usually)
c) fake BPS $\longrightarrow$ true BPS bound.

Review of ideas, techniques + applics.
emphasis on pedagogy!

Based on papers

1. Fake SG for flat sliced domain walls (asymp. AdS) — Skenderis + Townsend ..., De Wolfe, DZF, Gubser, Karch 1999

   applics. to AdS/CFT + brane worlds

2. AdS_d sliced domain walls in $AAdS_{d+1}$ — DZF, Nunez, Schnabl, 2008; Skenderis

   Stability of Janus sol. of II B SG — Bak, Gutperle, Hirano 2003

3. $AAdS_5$ sols with $R \times S_3$ symmetry
   new 5D charged black holes → AdS/CFT
   for $\mathcal{N}=4$ SYM on $R \times S_3$. — Elvang, DZF, Liu 2007

No time for

a. fake BPS eqtns are Hamilton-Jacobi eqtns for domain wall dynamics. — de Boer, Verlinde[2] 1999

b. fake BPS eqtns for cosmologies — Skenderis, Townsend 2006 ....

I. Fake SG for flat-sliced domain walls

1. Action for gravity-matter system $g_{\mu\nu}$ φ

$$S = \int d^D x \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right]$$

2. Write sym. ansatz for sols of interest

flat sliced dom. walls Poincaré$_d$ sym $D=d+1$

$$ds^2 = e^{2A(r)}\underbrace{\eta_{\mu\nu}\,dx^\mu dx^\nu}_{\text{Mink}_d} + dr^2 \qquad \varphi = \varphi(r)$$

3. Euler Lag. EOM's
$$R_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi + \frac{2}{d-1}g_{\mu\nu}V(\varphi)$$
$$\Box\varphi = \frac{\partial V}{\partial\varphi}$$

Insert ansatz $\longrightarrow$
$$A'^2 = \frac{1}{d(d-1)}\left[\phi'^2 - 2V(\phi)\right]$$
$$\phi'' + dA'\phi' = \frac{\partial V}{\partial\phi}$$

nonlinear 2nd order – difficult for general $V(\varphi)$.

4. Fake BPS – fake K. spin eqtns, suggested by trf rules of true SG.

$$\delta\psi_\mu = \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab} + \Gamma_\mu W(\varphi)\right)\varepsilon = 0$$
$$\delta\lambda = \left(\gamma^\mu\partial_\mu\varphi - 2(d-1)W'(\varphi)\right)\varepsilon = 0$$

Superpot. $W(\varphi)$ initially arbitrary

In domain wall ansatz, integrability eqtns are

$$\phi'(r) = -2(d-1)\,W'(\phi)$$

$$A'(r) = 2\,W(\phi(r))$$

$*$

a. easily solved by sequential integration

b. any sol of $*$ is also a sol of Euler-Lag. EOMs if $V(\varphi) = 2(d-1)^2\left(W'^2 - \frac{d}{d-1}W^2\right)$

c. fake K. spinors. $\varepsilon(r) = e^{A(r)/2}\,\varepsilon_0 \qquad \gamma^d \varepsilon_0 = \varepsilon_0$ radial γ

existence shows that analysis is correct.

II. Why does fake SG work (for all D)?

1. Consider true SG thy with base $B(x)$ and ferm. $\Psi(x)$ fields. Action $S[B,\Psi]$ and local SUSY trf rules δB, $\delta\Psi$ arbitrary $\varepsilon(x)$

Invariance under local SUSY means:

$$\delta S = \int \left[\frac{\delta S}{\delta B}\,\delta B + \frac{\delta S}{\delta \Psi}\,\delta\Psi \right] \equiv 0$$

vanishes identically for all configs of $B(x)$, $\Psi(x)$, $\varepsilon(x)$.

$\Rightarrow$ Terms of each order in Ψ vanish independently.

Lowest order:
$$\delta\Psi_0 = (D_\mu + \Gamma_\mu B)\varepsilon \quad \text{zero order}$$
$$\delta B_1 = \bar\varepsilon\,\Gamma'\Psi \quad \text{1st order}$$

Lowest order term in δS is linear in Ψ:

$$\delta S_1 = \int \left[\frac{\delta S}{\delta B}\Big|_0 \delta B_1 + \frac{\delta S}{\delta \Psi}\Big|_1 \delta\Psi_0 \right] \equiv 0$$

vanishes for all configs of B, Ψ, ε

If $\varepsilon(x)$ is a K. spinor $\Leftrightarrow$ $\delta\Psi_0 = 0$, then

$$\delta S_1 \to \int \frac{\delta S}{\delta B_0}\Big|\,\bar\varepsilon\,\Gamma'\Psi = 0 \quad \leftarrow \text{vanishes}$$

for any config of $B(x)$ which supports K. spinors, but for all $\Psi(x)$

$\Rightarrow$ Local condition summed over B_I

$$\sum_I \frac{\delta S}{\delta B_I} \bar{\varepsilon} \Gamma' \psi_I = 0$$

Usually enough freedom in choice of ψ_I, so each indep bose EOM is satisfied

$$\frac{\delta S}{\delta B_I} = 0 .$$

Result: in a true SG thy, any bosonic field config which supports K. spinors is also a sol of bosonic EOMs.

2. Result extends beyond true SG because we only used structure of SG to linear order in ψ.

Explain how it works for gen'l D:

Consider action: $S = S_B + S_\psi$ where

$$S_B = \int d^D x \left[\tfrac{1}{2} R - \tfrac{1}{2} (\partial\varphi)^2 - V(\varphi) \right]$$

bosonic action for which we want sols.

Postulate bilinear S_F with generic structure of SG

$$S_F = \int d^D x \Big[\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho + \bar{\lambda} \Gamma^\mu D_\mu \lambda - A(\varphi) \bar{\lambda} \lambda$$
$$- B(\varphi) \bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu - \bar{\psi}_\mu \Gamma^\nu \partial_\nu \varphi \Gamma^\mu \lambda - C(\phi) \bar{\psi}_\mu \Gamma^\mu \lambda + \text{h.c} \Big]$$

Undetermined fns $A(\varphi)$, $B(\phi)$, $C(\phi)$

Postulate similar structure for linearized transf rules

$$\delta\psi_\mu = (D_\mu + \Gamma_\mu W(\varphi))\varepsilon \qquad \delta e^a{}_\mu = -\bar\varepsilon\gamma^a\psi_\mu + h.c.$$

$$\delta\lambda = (\Gamma^\mu\partial_\mu - E(\varphi))\varepsilon \qquad \delta\phi = -\bar\varepsilon\lambda + h.c.$$

Requirement: $\delta S_1 = 0$ determine

$$A = -(d-1)[2W'' - W] \qquad B = 4(d-1)W$$

$$C = E = 2(d-1)W'$$

$$V = 2(d-1)^2\left[W'^2 - \frac{d}{d-1}W^2\right]$$

Calculations req. a lot of Γ-algebra:

e.g. $\gamma^{\mu\nu\rho}\gamma_{ab}R_{\mu\nu}{}^{ab} \sim \gamma_\nu\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)$

but do not require dim. specific Fierz rearrangement, so it works for all D!

In next order $\delta S_3 = 0$ reqs. true SG + $D \le 11$

Since BPS argument reqs. only $\delta S_1 = 0$, we still have result:

Any bosonic field config which supports fake K. spinors i.e $\delta\psi_\mu = 0$ $\delta\lambda = 0$

is also a sol. of bosonic EOM's.

III. Fake SG for AdS_d sliced walls in $D=d+1$ dim

Metric ansatz:

$$ds^2_{d+1} = e^{2A(r)} \underbrace{g_{ij}(x)\,dx^i dx^j}_{AdS_d \text{ metric} \quad \text{scale } L_d} + dr^2 \qquad \varphi = \varphi(r)$$

For $d+1=5$, isometry gp is $SO(3,2) \subset SO(4,2)$

5D part of Janus sol of IIB SG has this structure. Scalar $\varphi(r)$ is dilaton.

i) not truly BPS   Is it stable

ii) our program, establish fake BPS prove stability (very stg. arguments)

EOM's from Euler-Lag. + ~~sym.~~ field ansatz $D=d+1=5$

$$\phi'' + 4A'\phi' = \frac{\partial V}{\partial \phi}$$

new term.

$$A'^2 = \frac{1}{12}\left[\phi'^2 - 2V(\phi)\right] - \frac{1}{L_4^2}e^{-2A}$$

Previous fake BPS eqtns fail, succeed by including more of structure of true $D=5$ SG

i) $\mathcal{N}=2$ K. spinors: $\begin{pmatrix} \varepsilon^1_\alpha(x) \\ \varepsilon^2_\alpha(x) \end{pmatrix}$ ← pair of Dirac spinors

ii) matrix superpot. $\mathbb{W}(\phi) = W_a \tau^a$

$a = 1,2,3$

Pauli τ^a

New fake K. spinor eqtns:

$$\left[\nabla_c^{AdS_4} + \Gamma_c \left(\tfrac{1}{2} A' \Gamma^{\hat{r}} + \mathbb{W}(\phi) \right) \right] \varepsilon = 0$$

$$\left[\partial_r + \Gamma^{\hat{r}} \mathbb{W} \right] \varepsilon = 0$$

$$\left[\Gamma^{\hat{r}} \phi' - 6 \mathbb{W}' \right] \varepsilon = 0$$

Analysis of integ. conds. more complic. One finds

i) $V(\phi) = 9 \, \mathrm{Tr} \left(\mathbb{W}'^2 - \tfrac{4}{3} \mathbb{W}^2 \right)$

$$\left[\mathbb{W}', 3\mathbb{W}'' + \mathbb{W} \right] = 0$$ ← consistency cond. on $\mathbb{W}(\phi)$.

ii) Given $\mathbb{W}(\phi)$ which satisfies i) the first order fake BPS eqtns are

$$\phi'(r) = 6 \sqrt{\mathrm{Tr}\, \mathbb{W}'^2 / 2}$$ ← easily integrated

$$e^{-2A(r)} / L_d^2 = \frac{4 \, \mathrm{Tr} \left(\mathbb{W}^2 \mathbb{W}'^2 \right) - \mathrm{Tr} \{ \mathbb{W}, \mathbb{W}' \}^2}{\mathrm{Tr}\, \mathbb{W}'^2}$$

algebraic eqtn for scale factor!

Summary: flow eqtns simpler than flat wall case but finding $\mathbb{W}(\phi)$ is more difficult.

IV. Fake SG for $R \times S_3$ sliced walls in $D=5$

$\leftarrow$ works for all D.

$$ds_5^2 = -e^{2A(r)}dt^2 + dr^2 + e^{2B(r)}d\Omega_3^2$$

C unit S_3 metric

$$\phi = \phi(r)$$

Motivation from AdS/CFT: bdy gauge thy on $R \times S_3$ as in several recent studies of $\mathcal{N}=4$ SYM.

holog. RG flow: IR cutoff at radius of S_3?

1. With only $g_{\mu\nu}$, φ prev. BPS eqtns allow only undeformed AdS_5 as sol.

$$ds_5^2 = -L^2 ch^2(r/L)\, dt^2 + dr^2 + L^2 sh^2(r/L)\, d\Omega_3^2$$

Otherwise eqtns are inconsistent $1=0$.

2. Literature on 5D black holes suggests adding abelian gauge field: $F_{\mu\nu}$ electric

with $F_{rt} = \partial_r A_t(r) = a'(r)$

We need BPS eqtns for 4 fns.

$A(r)$ $B(r)$ $\varphi(r)$ $a(r)$

$\leftarrow$ More complic. than previous. Succeed with two-stage procedure

A. linearized fake SG action + trf rules

B. conditions of consistency for

$$\delta\psi_\mu = 0 \qquad \delta\lambda = 0$$

A. Linearized SG thy starting from bosonic action

$$S_B = \int d^5x \left[\tfrac{1}{2} R - \tfrac{1}{2} (\partial\varphi)^2 - \tfrac{1}{4} Q(\varphi) F_{\mu\nu} F^{\mu\nu} - V(\varphi) \right]$$

Fermion trf rules (inspired by true SG)

$$\delta\psi_\mu = \left[D_\mu + i X(\varphi) \left(\Gamma_\mu{}^{\nu\rho} - 4 \delta_\mu^\nu \Gamma^\rho \right) F_{\nu\rho} + \Gamma_\mu W + i c A_\mu \right] \varepsilon$$

$$\delta\lambda = \left[\Gamma^\mu \partial_\mu \varphi + i Y(\varphi) \Gamma^{\nu\rho} F_{\nu\rho} - 6 W' \right] \varepsilon$$

gravitino is charged

(2x2 matrix structure unnecessy, omitted)

Postulate appropriate S_F and δB. Long analysis determines $W(\phi)$ $X(\phi)$ $Y(\phi)$ $Q(\phi)$ ....

System has linearized local SUSY if

$$W = \frac{1}{(2+3k^2)2L} \left[2 e^{-k\varphi} + 3k^2 e^{2\varphi/3k} \right] \xrightarrow[\varphi \to 0]{} \frac{1}{2L}$$

(std. normalization

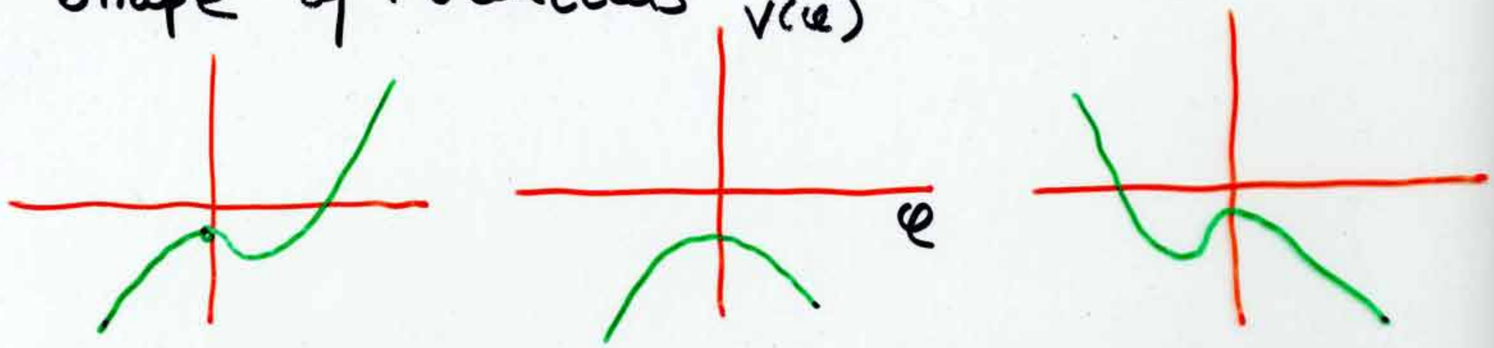
$$V = 18 \left(W'^2 - \tfrac{4}{3} W^2 \right)$$

$$X = a e^{k\varphi} \qquad Y = 6kX \qquad Q = e^{2k\phi}$$

$$a^2 = \frac{1}{48(2+3k^2)} \qquad c = -\sqrt{\frac{3}{2+3k^2}} \frac{1}{L}$$

<u>Summary</u>: Previous freedom in $W(\varphi)$ is reduced to sum of exponentials with <u>one</u> free param. k

D. Shape of Potentials



$0 < k < 1/\sqrt{3}$ $\qquad$ $1/\sqrt{3} < k < 2/\sqrt{3}$ $\qquad$ $2/\sqrt{3} < k$

All sols approach local max at $\varphi = 0$ as $r \longrightarrow \infty$ AdS bdy.

$$V(\varphi) \approx -\frac{6}{L^2} - \frac{2}{L^2}\varphi^2 + \cdots = -\frac{6}{L^2} + \frac{1}{2}m^2\varphi^2$$

Cos. const. for AdS$_5$ of scale L

Identify mass $m^2 = -4/L^2$

Indep. of k, saturates BF bound.

ϕ is dual to $\Delta = 2$ operator.

B. Subst. ansatz for fields in K. spinor eqtns.

$$\delta\psi_\mu = 0 \qquad \delta\lambda = 0$$

Ansatz: $ds_5^2 = -e^{2A}dt^2 + dr^2 + e^{2B}d\Omega_3^2$

$$\varphi = \varphi(r) \qquad A_t = a(r) \qquad A_\mu = 0 \quad \mu \neq t$$

Consistency conds $\longrightarrow$ 4 1st order ODE's for A, B, φ, a

$$\phi'^2 = 36\,W'^2 + 4Y^2 e^{-2A} a'^2$$

$$B'W' = -\tfrac{1}{3} W \phi'$$

$$A'\phi' = -12\,W W' - 16 X Y a'^2 e^{-2A}$$

$$A'B' + B'^2 = 8W^2 - 16X^2 a'^2 e^{-2A} + e^{-2B}$$

We decouple and solve by regarding

$$B(\varphi(r)) \qquad \frac{dB}{dr} = \frac{dB}{d\phi}\frac{d\varphi}{dr} = \dot{B}\varphi'$$

2nd eq. $\rightarrow$ $\dot{B} = -\frac{1}{3}\frac{W}{W'}$

integrate $e^{-2B} = c\, e^{(k - \frac{2}{3k})\phi} W'(\varphi)$

Continue to find complete sol + verify ~~that~~ it satisfies Lag. EOM's.

F. Recognized sing. sols similar to 5D black hole literature. Suggests useful to write in terms of harmonic fn. $H(y) = 1 + q/y^2$ and new radial coord. y

Result for fake BPS sols – extremal

$$ds_5^2 = -H^{-2p} f\, dt^2 + H^p\left(f^{-1} dy^2 + y^2 d\Omega_3^2\right)$$

$$A_t = \pm\sqrt{\frac{3}{2+3k^2}}\,\frac{q}{y^2+q} \qquad e^{2\phi/3k} = H^p$$

$$H(y) = 1 + q/y^2 \qquad f(y) = 1 + \frac{y^2}{L^2} H^{3p}$$

$$p = \frac{2}{2+3k^2} \qquad q = \text{free param. of sol.}$$

Properties:

1) electric charge from gauge field EOM

$$\partial_\mu\left(\sqrt{-g}\, Q(\varphi) F^{\mu\nu}\right) = \sqrt{-g}\, J^\nu$$ if we add current source

$$q_{elec} = \frac{1}{2\pi^2}\int dy\, d\Omega_3 \sqrt{-g}\, J^t = \int d\Omega_3 \sqrt{-g}\, Q F^{yt}$$

$$= 2\sqrt{\frac{3}{2+3k^2}}\, q$$

Since $\nexists$ source in actual thy, we have a pt. charge at origin ($y \to 0$). $\Rightarrow$ singularity confirmed by $\phi \sim \ln H \sim \ln y$ $\quad R$ ....

2) fake BPS sols sing. charged black holes

Sing. expected, since reg. BPS black holes in true $D=5$ SG have angular momentum (not spherically symmetric).

Sols in
3 The two cases $k = 1/\sqrt{3}$, $2/\sqrt{3}$ coincide with known sols of true D=5 $\mathcal{N}=2$ SG $U(1)^3$ thy

Duff, Pope ....

Lifted to D=10 IIB by Behrndt, Cvetic, Sabra 1998

Interpreted as contin. distribs. of giant graviton "Superstars" Myers + Tafjord 2001

Sing. is admissible since it comes from distributed sources in D=10.

For general k, our sols. are qualitatively similar, although they probably do not lift to 10D IIB SG.

4. Exact fake K. spinors which shows that analysis of consistency conds is correct. (The sols are $1/2$ fake BPS)

5. Find non-extremal sols in which sing. is clothed by black hole horizon

Easy to do by following Behrndt, Cvetic Chamseddine 1999

Simple changes from BPS case:

$$f(y) \longrightarrow 1 + \frac{y^2}{L^2} H^{3p} - \frac{\mu}{y^2} \qquad \mu > 0$$

$$A_t \rightarrow \frac{\tilde{q}}{q} A_t \qquad \tilde{q}^2 = q(q+\mu)$$

$\mu \leftarrow$ non extremality param. ($\sim$ mass)

6. non-extremal sols for general k found by (properties not explored). Gao + Zhang hep-th/0411104

V. AdS/CFT interp. of sols.

All fields $g_{\mu\nu}$, A_μ, φ $\longrightarrow$ bdy at VeV rate.

So Sols are dual to state of bdy gauge thy. with 1 pt fns. $\langle T_{ij} \rangle$ $\langle J_i \rangle$ $\langle \mathcal{O}_{\varphi} \rangle$

stress tensor, R current, $\Delta = 2$ $\mathrm{Tr}(X^I X^J)$

Compute VeV's using Holographic Renormalization: systematic, correct, difficult. Henningson + Skenderis 1998 ⋮

Outline method:

1. AAdS metrics param. by radial coord $\rho \geq 0$, $\rho \to 0$ bdy.

   $\perp$ coords x^i $i = 0, 1, 2, 3$

$$ds_5^2 = \frac{L^2 d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(\rho, x) dx^i dx^j$$

Near bdy ρ, y related by $y = \frac{L}{\sqrt{\rho}}(1 + a_1 \rho + a_2 \rho^2 + \dots)$

2. All fields have near bdy expansions in ρ

$$g_{ij}(\rho, x^i) = g^{(0)}_{ij}(x) + g^{(2)}_{ij}(x)\,\rho + g^{(4)}_{ij}(x)\,\rho^2 + \cdots$$

$$A_i = A_i^{(0)} + A_i^{(2)}\rho$$

$$\phi = \tilde{\phi}^{(0)}(x)\,\rho\ln\rho + \phi^{(0)}(x)\,\rho + \cdots$$

source ↑   ← VeV

3. On-Shell action $S = \int d^5x \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}Q F_{\mu\nu}^2 - V\right]$

<u>suitably renormalized</u> is generating fnl. of gauge thy correlators.

1 pt fns from variation:

$$\delta S_{ren} = \int d^4x \sqrt{-g^{(0)}}\left[\frac{1}{2}\langle T_{ij}\rangle \delta g^{ij}_{(0)} + \langle J^i\rangle \delta A_i^{(0)} + \mathcal{O}(\varphi)\cdot\delta\hat{\varphi}^{(0)}\right]$$

Formalism gives $\langle\ \rangle$'s in terms of bdy. expansion:

$$\langle T_{ij}\rangle = g^{(4)}_{ij} - \frac{1}{2} g^{(2)\,k}_{i}\, g^{(2)}_{kj} + \cdots$$

$$\langle J_i\rangle = 2A_i^{(2)} = 2F^{(2)}_{\rho i}$$ ← gauge inv.

$$\langle \mathcal{O}_\varphi\rangle = 2\phi_{(0)}$$

Input data from sols. to find:

$$E = \langle T^{tt}\rangle \sim \frac{3L^2}{8} + \frac{3M}{2} + \frac{6}{2+3k^2}\,q$$

$$q_{elec} = \langle J^t\rangle \sim 2\sqrt{\frac{3}{2+3k^2}}\,\tilde{q}$$ ← agrees w. prev.

$$\langle \mathcal{O}_\varphi\rangle = \frac{3\sqrt{2}\,k\,q}{2+3k^2}$$

(Norm. factors involving L, G_5 omitted)

Interpret $E = \frac{\pi}{4G}\left[\frac{3L^2}{8} + \frac{3\mu}{2} + \frac{6}{2+3b^2}\, q\right]$

Casimir energy of $\mathcal{N}=4$ SYM on $R \times S^3$

Balasubramanian + Kraus

$$M = E - E_{AdS} = \frac{3\mu}{2} + \frac{6}{2+3b^2}\, q \geq \sqrt{\frac{3}{2+3b^2}}\, |q_{elec}| \checkmark$$

4. Witten-Nester Method: technical difficulty due to Chern-Simons term. One can show that

i. The fake BPS sols. are lowest energy among all sols of same asymptotics with

$$\varepsilon^{\mu\nu\rho\lambda\sigma} F_{\nu\rho} F_{\lambda\sigma} = 0 \quad \checkmark\checkmark$$

ii Inequality ✓ holds as a BPS bound for all sols. which satisfy ✓✓

5. Holog. c-thm (similar to flat sliced walls) follows from $B'' \leq 0$

$1/(B'(r))^3$ monotonically decreases from bdy → deep interior. Agrees with central charges of conf. alg at endpts.

Conclusion: The Fake SG method is inspired by true SG but goes far beyond in its applications. Allows construction of several new non-trivial sols of gravity-matter systems and study of their stability.