Vectorial AdS/CFT and quantum higher spins

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"Partition functions and Casimir energies in higher spin AdS_{d+1}/CFT_d" arXiv:1402.5396 with S. Giombi and I. Klebanov
"Higher spins in AdS_5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT" arXiv:1410.3273
"Vectorial AdS_5/CFT_4 duality for spin-one boundary theory" arXiv:1410.4457
and in progress with M. Beccaria

Motivation: learn about (i) structure of HS theories; (ii) limits of AdS/CFT

Free Higher Spin theory:

Flat space background: collection of free massless spin $s = 0, 1, 2, ...$ fields with gauge-invariant $\delta\varphi_{m_1 \ldots m_s} = \partial_{(m_1} \epsilon_{m_2 \ldots m_s)}$ Fronsdal actions

e.g. viewed as a formal flat limit of Vasiliev HS theory with no interactions

massless vector, massless graviton, etc.: for $s > 0$ 2 d.o.f. in $\mathrm{d} = 4$

curious fact: total number of d.o.f. is zero

$$1 + \sum_{s=1}^{\infty} 2 = 1 + 2\zeta_R(0) = 0$$

free massless spin s partition function

$$Z_{\mathrm{MHS},s} = \left[\frac{\det\Delta_{s-1\,\perp}}{\det\Delta_{s\,\perp}}\right]^{1/2} = \left[\frac{(\det\Delta_{s-1})^2}{\det\Delta_s \det\Delta_{s-2}}\right]^{1/2}$$

$\Delta_s = -\partial^2$ on symmetric rank s traceless tensor

Then total partition function is trivial:

$$(Z_{\rm MHS})_{\rm tot} = \prod_{s=0}^{\infty} Z_{{\rm MHS},s}$$
$$= \Big[\frac{1}{{\rm det}\Delta_0}\Big]^{1/2}\Big[\frac{{\rm det}\Delta_0}{{\rm det}\Delta_{1\perp}}\Big]^{1/2}\Big[\frac{{\rm det}\Delta_{1\perp}}{{\rm det}\Delta_{2\perp}}\Big]^{1/2}\Big[\frac{{\rm det}\Delta_{2\perp}}{{\rm det}\Delta_{3\perp}}\Big]^{1/2}\ldots = 1$$

cf. supersymmetric theory: B/F =1 (e.g. vanishing of vacuum energy)
here cancellation of physical spin s det and ghost det for spin $s+1$ field
should be reflecting large gauge symmetry of the theory
(cf. topological theory like antisymm tensor of rank d in $d+1$ dimensions
or Chern-Simons or 3d gravity)
Cancellation of an infinite number of factors is formal (like 1-1+1-1+...=0):
depends on grouping terms together – ∞ product requires regularization
and its value may depend on choice
choice of regularization should be consistent with underlying symmetry:
here with higher spin gauge symmetry

case of $\mathrm{d} = 4$:

$$Z_0 = \Big[\frac{1}{\det\Delta_0}\Big]^{1/2} , \qquad Z_{\mathrm{MHS},s} = (Z_0)^{\nu_s} , \qquad \nu_s = 2$$

$$\nu_s = (s+1)^2 + (s-1)^2 - 2s^2 = 2$$

$$Z_{\mathrm{tot}} = (Z_0)^{\nu_{\mathrm{tot}}} \qquad \nu_{\mathrm{tot}} = 1 + \sum_{s=1}^{\infty} \nu_s = 1 + \sum_{s=1}^{\infty} 2 = 0$$

in $\mathrm{d} = 4$: zeta-function reg. is equivalent to formal cancellation of factors Z
(cf. use of zeta-function regularization in vac energy in bosonic string:
consistent with massless vector in $D = 26$ – symmetries of critical string)
in d flat dimensions:

$$\det\Delta_s = (\det\Delta_0)^{N_s}, \quad \det\Delta_{\perp\, s} = (\det\Delta_0)^{N_s^{\perp}} , \quad N_s = \tbinom{s+\mathrm{d}-1}{s} - \tbinom{s+\mathrm{d}-3}{s-2}$$

$$N_s^{\perp} = N_s - N_{s-1}, \quad \nu_s = N_s^{\perp} - N_{s-1}^{\perp} = 2[s + \tfrac{1}{2}(\mathrm{d}-4)]\,\tfrac{(s+\mathrm{d}-5)!}{s!(\mathrm{d}-4)!}$$

in even d one may use regularization ($\epsilon \to 0$, dropping singular terms)

$$\nu_{\rm tot} = 1 + \sum_{s=1}^{\infty} \nu_s \; e^{-\epsilon[s+\frac{1}{2}(\mathrm{d}-4)]}\Big|_{\rm fin.} = 0$$

alternative reg. in any d: cutoff function $f(s,\epsilon)$ with $f(s,0)=1$ for $\Delta_{\perp,s}$

$$\nu_{\rm tot} = 1 + \sum_{s=1}^{\infty} \Big[f(s,\epsilon)\; N_s^{\perp} - f(s-1,\epsilon)\; N_{s-1}^{\perp} \Big] = 0\;.$$

it is direct analog of formal cancellation of the determinant factors $Z_{\rm tot}$

Conformally-flat case: $AdS_{\rm d}$

$Z_{\rm tot} = 1$ holds also in proper vacuum of Vasiliev theory – $AdS_{\rm d}$

Fronsdal action in AdS_d leads to similar partition function

Introduce operator in AdS_d ($k = 0, 1, ..., s-1$)

$$\Delta_s(M^2_{s,k}) \equiv -\nabla^2_s + M^2_{s,k}\varepsilon \qquad\qquad M^2_{s,k} = s - (k-1)(k+{\rm d}-2)$$

$\varepsilon = \pm 1$ for unit-radius $S^{\rm d}$ or euclidean $AdS_{\rm d}$; $\varepsilon = 0$ in flat space

Partition function of "partially-massless" field (rank k gauge parameter)

$$Z_{s,k} = \left[\frac{{\rm det}\Delta_{k\,\perp}(M^2_{k,s})}{{\rm det}\Delta_{s\,\perp}(M^2_{s,k})}\right]^{1/2}$$

For massless (maximal gauge invariance with rank $s-1$ parameter) spin s field on homogeneous conformally flat space

[Gaberdiel et al 2010; Gupta, Lal 2012; Metsaev 2014]

$$Z_{{\rm MHS},s} = Z_{s,s-1} = \left[\frac{{\rm det}\Delta_{s-1\,\perp}(M^2_{s-1,s})}{{\rm det}\Delta_{s\,\perp}(M^2_{s,s-1})}\right]^{1/2}$$

$$Z_{\mathrm{MHS},s} = \Big[\frac{\big(\det\Delta_{s-1}(M^2_{s-1,s})\big)^2}{\det\Delta_s(M^2_{s,s-1})\,\det\Delta_{s-2}(M^2_{s+2,s+1})}\Big]^{1/2}$$

$Z_{\mathrm{MHS},0} = [\det(-\nabla^2 + M_0^2)]^{-1/2}, \;\; M_0^2 = 2(\mathrm{d}-3)\varepsilon$

$Z_{\mathrm{tot}} = \prod_{s=0}^{\infty} Z_{\mathrm{MHS},s}$: here no immediate cancellation of factors
operators in numerator and denominator different for $\varepsilon \neq 0$
Using spectral zeta-function (Λ is UV cutoff, r is curvature radius)

$$\ln\det\Delta_s = -\zeta_{\Delta_s}(0)\ln(\Lambda^2 r^2) - \zeta'_{\Delta_s}(0)$$

Computing $\zeta_{\mathrm{tot}}(z) = \sum_{s=0}^{\infty}\zeta_{\Delta_s}(z)$ and then taking $z \to 0$:
$\zeta_{\mathrm{tot}}(z) = 0 + 0\times z + \mathcal{O}(z^2)$ [Giombi, Klebanov, Safdi: 2014]

$$\Big(Z_{\mathrm{MHS}}(AdS_{\mathrm{d}})\Big)_{\mathrm{tot}} = 1$$

equivalent regularization:

$$\ln\Big(Z_{\mathrm{MHS}}(AdS_{\mathrm{d}})\Big)_{\mathrm{tot}} = \sum_{s=0}^{\infty}\ln Z_{\mathrm{MHS},s}\; e^{-\epsilon[s+\frac{1}{2}(\mathrm{d}-4)]}\Big|_{\epsilon\to 0,\ \mathrm{fin.}} = 0$$

Remarks:

• proper-time cutoff for each s: power divergences Λ^n sum up to 0 too
(cf. supersymmetric theories)

• $\left(Z_{\rm MHS}\right)_{\rm tot} = 1$ need not apply to quotients of flat or $AdS_{\rm d}$ space
e.g. $Z_{\rm MHS}$ on thermal quotient of $AdS_{\rm d}$ is non-trivial

• Conjecture: $\left(Z_{\rm MHS}(AdS_{\rm d})\right)_{\rm tot} = 1$ to all orders in coupling:
exact vacuum partition function of Vasiliev theory =1
(analogy with supersymmetric or topological QFT)

• This is the requirement of the vectorial AdS/CFT duality:
logarithm of partition function of dual free $U(N)$ scalar theory
has only $\mathcal{O}(N)$ term that should match classical action of Vasiliev theory
while all $g_{\rm HS} = 1/N$ corrections should be absent

AdS$_{d+1}$/CFT$_d$ "light":

free boundary CFT$_d$

(i) "vectorial": e.g. free scalar in fundamental of $U(N)$ or $O(N)$

(ii) "adjoint": e.g. free vector in adjoint of $U(N)$ or $O(N)$

no anomalous dimensions of composite operators

but correlation functions are non-trivial in N

vectorial: bilinear "single-trace" operators $\Phi_i^* \partial ... \partial \Phi_i$

adjoint: multilinear single-trace operators $\mathrm{tr}(\Phi \partial ... \partial \Phi \partial ... \partial \Phi .... \Phi)$

in general, in any $d = 3, 4, ...$ any free conformal field is ok

but restrictons of unitarity, etc.:

$d = 3$: scalars or spinor [Maldacena, Zhiboedov 11]

$d = 4$: scalar, spinor or vector [Stanev 12; Alba, Diab 13]

$d = 6$: scalar,..., tensor – e.g. (2,0) tensor multiplet in susy case

• existence of higher-spin symmetries:
[Vasiliev 04; Boulanger, Ponomarev, Skvortsov, Taronna 13]

• "vectorial" AdS/CFT:
originally in $d = 3$
free or interacting $O(N)$ fixed point theory [Klebanov, Polyakov 02]
• "adjoint" AdS/CFT:
e.g. in $d = 4$:
$g_{\rm YM} = 0$, large N limit of $\mathcal{N} = 4$ SYM – $AdS_5 \times S^5$ string duality:
i.e. $\lambda = g^2_{\rm YM} N = 0$, large N limit of standard $\text{AdS}_5/\text{CFT}_4$
• Dual higher spin theory in AdS:
contains infinite set of (massless and massive) HS fields in AdS
dual to primary operators in boundary CFT
adjoint case: related to tensionless limit of string theory

vectorial duality:

• spectrum: Flato-Fronsdal type relation:

$\Phi^*(x)\Phi(x') \to \sum \Phi^* \partial ... \partial \Phi$, e.g., in $d = 4$

$$\{0,0\} \times \{0,0\} = (2;0,0) + \bigoplus_{s=1}^{\infty} (2+s; \tfrac{s}{2}, \tfrac{s}{2})$$

corresponding relation for characters same as

AdS/CFT relation for one-particle partition functions

• correlation functions summarised by interaction vertices in AdS_{d+1}

HS theory: Vasiliev-type theory with AdS vacuum

Aim: learn about HS theory in AdS

• match quantum partition functions on both sides of duality

boundary: $S^1 \times S^{d-1}$, S^d, or Einstein space M^d

bulk: (quotient of) AdS_{d+1}, or asymptotically AdS_{d+1} space

• match Casimir energy on $R \times S^{d-1}$ to vacuum energy in AdS_{d+1}

• match a, c_r conformal anomaly coefficients to AdS_{d+1} counterparts

Some background

• consistent interacting massless higher spin gauge theories:
exist in AdS (or dS) background [Fradkin, Vasiliev 88; Vasiliev 92]
e.g. in bosonic 4d case:
infinite set $s = 1, 2, ..., \infty$ plus $s = 0$ with $m^2 = -2$
action $\sim$ quadratic Fronsdal action plus higher interactions

• vectorial AdS$_4$/CFT$_3$: [Klebanov, Polyakov 02]
free 3d complex scalar in fundamental representation of $U(N)$
$L = \partial_m \Phi_i^* \partial_m \Phi_i, \quad i = 1, ..., N$
has tower of conserved higher spin currents
$J_{m_1 ... m_s} = \Phi_i^* \partial_{(m_1} ... \partial_{m_s)} \Phi_i + ...$
singlet sector – $U(N)$ inv "single-trace" CFT primaries:
$J_s, \quad s = 1, 2, ..., \infty$ with $\Delta = s + 1$ – dual to spin s field in AdS_4
$J_0 = \Phi_i^* \Phi_i$ with $\Delta = 1$ – dual to massive scalar $\Delta(\Delta - 3) = m^2 = -2$
same spectrum of states as in HS theory in AdS_4

HS theory dual to free CFT is non-trivial:

free-theory correlators of J_s should be reproduced by
HS interactions in AdS_4 with coupling $\sim 1/N$
checked for tree 3-point functions [Giombi, Yin; Maldacena, Zhiboedov]

$$S = N \int d^{d+1}x \Big[\sum_s \phi_s(-\nabla^2 + m_s^2)\phi_s + \sum C_{s_1 s_2 s_3}(\nabla)\, \phi_{s_1}\phi_{s_2}\phi_{s_3} + ... \Big]$$

full classical action $S = N\bar{S}$ of HS theory for Vasiliev equations not known

quantum corrections: $\Gamma = N\bar{S} + \Gamma_1 + N^{-1}\Gamma_2 + ...$

one-loop $\Gamma_1(0)$ can be found as quadratic action for ϕ_s is known
[Fronsdal 78; Metsaev 94]

- HS theory "summarizes" correlators of bilinear primaries in free theory
- summing up infinite sets of correlators:

partition functions on non-trivial backgrounds should also match

Other similar $d = 3$ models:

• $O(N)$ model : N real scalars

singlet sector – higher spin conserved currents $\Phi_i \partial_{m_1} ... \partial_{m_s} \Phi_i + ...$

non-trivial for even $s = 2, 4, 6, ...$ plus scalar $\Phi_i \Phi_i$ with $\Delta = 1$

dual to "minimal" HS theory in AdS_4 containing even spins only

• "critical vector model": $L = (\partial \Phi_i)^2 + \lambda (\Phi_i \Phi_i)^2$

IR fixed point seen at large N:

scalar $\Delta = 2 + O(\frac{1}{N})$, J_s bilinears $\Delta = s + 1 + O(\frac{1}{N})$

dual to (non)minimal HS theory with $m^2 = -2$ bulk scalar

with alternative b.c.: $\Delta = 2$

• free or critical $U(N)$ or $O(N)$ fermionic 3d models: [Sezgin, Sundell 02]

dual to "type B" ($s = 1/2$) HS theories:

scalar of "type A" ($s = 0$) theory $\rightarrow$ pseudo-scalar

• higher dimensions: vectorial AdS/CFT duality should apply for $d \geqslant 3$

• singlet sector of $U(N)$ or $O(N)$ free scalar CFT$_d$

dual to *non-minimal* ($s = 1, 2, ...$) or *minimal* ($s = 2, 4, ...$)

HS theory in AdS_{d+1} + scalar with $\Delta = d - 2$, i.e. $m^2 = -2(d-2)$

[Didenko, Skvortsov 13; Giombi, Klebanov, Safdi 14]

• "non-trivial" interacting critical theory only in $d = 3$ or also in $d = 5$?

[Fei, Giombi, Klebanov 14]

• singlet sector may be "dynamically" selected by

gauging $U(N)$ or $O(N)$ symmetry and taking gauge coupling to 0

(e.g. coupling to $k = \infty$ CS in $d = 3$; only pure-gauge field)

• test: compare, e.g., quantum partition functions

of large N CFT on $M^d = S^d,\ S^1 \times S^{d-1}, ...$

and of massless HS theory in AdS_{d+1} with boundary M^d

Example: $M^3 = S^3$ $\qquad Z_{\rm CFT}(S^3) = Z_{\rm HS}(AdS_4)$

free complex $U(N)$ scalar CFT: $\quad \int d^3x\sqrt{g}\,\Phi_i^*(-\nabla^2 + \frac{1}{8}R)\Phi_i$

$$\Gamma_{\rm free} = -\ln Z = N \ln \det(-\nabla^2 + \tfrac{3}{4})$$

$$= N\sum_{n=0}^{\infty}(n+1)^2 \ln[(n+\tfrac{1}{2})(n+\tfrac{3}{2})] = N\left[\tfrac{1}{4}\ln 2 - \tfrac{3}{8\pi^2}\zeta(3)\right]$$

Bulk HS theory: expand near AdS_4 vacuum: $\quad ds^2 = d\rho^2 + \sinh^2\rho\, d\Omega_3$

• vacuum value of (unknown) classical action $S = N\bar{S}$ should match (one-loop) CFT value: remains open problem

• AdS/CFT: all quantum corrections in $\quad \Gamma = N\bar{S} + \Gamma_1 + N^{-1}\Gamma_2 + ...$ should then vanish

• check directly that $\Gamma_1 = 0$

Free action of massless totally symmetric HS fields in AdS_{d+1} is known;
gauge fixing ($\delta\phi_s = \nabla\epsilon_{s-1}$) leads to 1-loop HS partition function:

$$Z_s(AdS_{d+1}) = \Big[\frac{\det\big(-\nabla^2 + \mathrm{m}'^2_{s-1}\big)_{s-1,\perp}}{\det\big(-\nabla^2 + \mathrm{m}^2_s\big)_{s,\perp}}\Big]^{1/2}$$

$$\mathrm{m}^2_s = (s-2)(s+d-2) - s \qquad \mathrm{m}'^2_{s-1} = (s-1)(s+d-2)$$

∇^2 on symmetric transverse traceless tensors (curvature radius $r = 1$)
$d = 2, s \geqslant 2$: [Gaberdiel, Gopakumar, Saha 10]; $d \geqslant 3$: [Gupta, Lal 12]

physical and ghost "mass" terms $\mathrm{m}^2_s = \Delta(\Delta - d) - s$
$\Delta = s + d - 2$ and $\Delta' = s + d - 1$ – dimensions of J_s and ∂J_s
scalar $s = 0$: $-\nabla^2 - 2(d-2)$ and no ghost numerator

Compute determinants using AdS heat kernel [Camporesi, Higuchi 92]
spectral ζ-function in non-compact case
$\zeta(z) = \sum_n d_n \lambda_n^{-z} \to \int du\, \mu(u)\, \lambda_u^{-z}$

$$\Gamma_1(AdS_{d+1}) = -\tfrac{1}{2}\zeta(0)\ln(r^2\Lambda^2) - \tfrac{1}{2}\zeta'(0) \qquad\qquad \Lambda = (\varepsilon_{\rm UV})^{-1} \to \infty$$

• even $d+1$: log UV divergence $\to$ IR divergence in CFT on S^d
must be absent – UV finiteness: $\sum_s \zeta_s(0) = 0$

• odd $d+1$: $\zeta_s(0) = 0$ but need to show that $\sum_s \zeta_s'(0) = 0$

For $(-\nabla^2 + \mathrm{m}^2)_{s\perp}$, $\quad \mathrm{m}^2 = \Delta(\Delta - d) - s$

$$\zeta_{\Delta,s}(z) = c_d\, g_s \int_0^\infty du\, \mu_s(u) \left[u^2 + (\Delta - \tfrac{1}{2}d)^2\right]^{-z}$$

$d = 3$:
$c_d = \frac{2^{d-1}}{\pi} \frac{\mathrm{Vol}(AdS_{d+1})}{\mathrm{Vol}(S^d)} \to \frac{8}{3\pi} \qquad g_s = 2s+1$
$\mu_s = \frac{\pi u}{16}\left[u^2 + (s+\frac{1}{2})^2\right] \tanh \pi u$

UV finiteness of HS theory in AdS_4 vacuum [Giombi, Klebanov 13]

$$\sum_s \zeta_s(0) = \zeta_{1,0}(0) + \sum_{s=1}^{\infty} \left[\zeta_{s+1,s}(0) - \zeta_{s+2,s-1}(0)\right]$$

$$= \tfrac{1}{360} + \tfrac{1}{24} \sum_{s=1}^{\infty} \left(\tfrac{2}{15} - s^2 + 5s^4\right) = 0$$

if regularized with Riemann ζ-function: $\zeta(0) = -\frac{1}{2},\ \zeta(-2n) = 0$
(same as adding cutoff $e^{-\epsilon s}$, $\epsilon \to 0$ and dropping singular terms)

- this regularization should be required by symmetries of theory
- finiteness is automatic if $\sum_s$ done for fixed UV cutoff Λ and then $\Lambda \to \infty$:

can be demonstrated by first summing spectral $\zeta_s(z)$ for arbitrary z

one-loop UV finiteness applies to all bosonic massless HS theories in AdS_{d+1}

Vanishing of finite part of $\Gamma_1(AdS_4)$ [Giombi, Klebanov 13]

$$\Gamma_1 = -\tfrac{1}{2}\zeta'_{1,0}(0) - \tfrac{1}{2}\sum_{s=1}^{\infty}\left[\zeta'_{s+1,s}(0) - \zeta'_{s+2,s-1}(0)\right]$$

$$\zeta'_{\Delta,s}(0) = -\tfrac{1}{3}(2s+1)\int_0^{\Delta-\frac{3}{2}} dv\, v\left[v^2 - (s+\tfrac{1}{2})^2\right]\psi(v+\tfrac{1}{2})$$

HS tower part contribution exactly cancels against scalar part

$$\zeta'_{1,0}(0) = -\tfrac{1}{1152} - \tfrac{11}{2880}\ln 2 - \tfrac{1}{8\pi^2}\zeta(3) + \tfrac{1}{8}\zeta'(-1) + \tfrac{5}{8}\zeta'(-3)$$

1-loop partition function in non-minimal HS theory in AdS_4 vanishes: consistent with no N^0 term in Γ of free $U(N)$ CFT on S^3

In minimal (even spin) HS theory – non-zero one-loop result:

$$\Gamma_{1\ \mathrm{min}} = \tfrac{1}{8}\ln 2 - \tfrac{3}{16\pi^2}\zeta(3)$$

dual to $O(N)$ real scalar CFT where no N^0 correction ?!

$$\Gamma_{\text{free O(N)}} = N\left[\tfrac{1}{8}\ln 2 - \tfrac{3}{16\pi^2}\zeta(3)\right]$$

Assume: minimal HS theory coupling $N-1$ not N [Giombi, Klebanov 13]:

$$\Gamma_{0\ \mathrm{min}} = (N-1)\bar{S} = (N-1)\big[\tfrac{1}{8}\ln 2 - \tfrac{3}{16\pi^2}\zeta(3)\big]$$
$$\Gamma_{0\ \mathrm{min}} + \Gamma_{1\ \mathrm{min}} = \Gamma_{\text{free O(N)}}$$

evidence for $g^{-1}_{\rm min} = N-1$ found also in $M^d = S^1 \times S^d$ case

- same $N-1$ in minimal type B theory (dual to free Majorana fermions)
- in minimal "type C theory" (dual to real N vectors)
  coupling should be $N-2$ [Beccaria, AT 14]

open questions:

- true meaning of $N \to N-1$

(quantum shift, analogy with CS theory, cf. quantization of HS coupling,...)

- why classical action $\bar{S}(AdS_4) = \frac{1}{8}\ln 2 - \frac{3}{16\pi^2}\zeta(3)$

or there is some interpretational subtlety ?

General d: free scalar CFT on $M^d = S^d \leftrightarrow$ HS theory in AdS_{d+1}

• Vasiliev theory in AdS_{d+1}: totally symm. ϕ_s plus $m^2 = -2(d-2)$ scalar
same spectrum as bilinear primaries in scalar CFT

• similar results about matching of partition functions as in $d = 3$, e.g.,
UV divergences vanish for any d: $\sum_s \zeta_s(0) = 0$

• use of spectral zeta-function
$\zeta_{\Delta,s}(z) = c_d\, g_s \int_0^\infty du\, \mu_s(u) \left[u^2 + (\Delta - \frac{1}{2}d)^2\right]^{-z}$
suggests natural regularization: [Giombi, Klebanov, Safdi 14]
first sum over spins for fixed z and then analytically continue in z;
equivalent to cutoff $e^{-\epsilon \bar{s}}$, $\bar{s} \equiv s + \frac{1}{2}(d-3)$
(same as Riemann zeta-function reg. in $d = 3$ only)

$$\Gamma_1 = -\tfrac{1}{2}\zeta'_{1,0}(0) - \tfrac{1}{2}\sum_{s=1}^\infty e^{-\epsilon\bar{s}} \left[\zeta'_{s+1,s}(0) - \zeta'_{s+2,s-1}(0)\right]\Big|_{\epsilon\to 0,\,\text{finite}}$$

Odd d: $AdS_4, AdS_6, AdS_8, ...$

$\Gamma_{\rm CFT}(S^d)$=finite $\sim N$, should be equal to $\Gamma_0(AdS_{d+1}) = N\bar{S}$

• $\Gamma_0 = N\bar{S}$ is finite:
regularized $\mathrm{Vol}(AdS_{d+1}) = \pi^{d/2}\Gamma(-\frac{1}{2}d)$ (drop power IR ∞)

• non-minimal theory ($s = 1, 2, 3, ...$): $\Gamma_1(AdS_{d+1})$ = 0

• minimal theory ($s = 2, 4, 6, ...$): find non-trivial identity (as in $d = 3$)

$$\Gamma_{1\,\rm min}(AdS_{d+1}) = \Gamma_{\rm conf.\,scalar}(S^d)$$

• consistent with AdS/CFT if minimal HS theory coupling is $N - 1$

Even d: $AdS_5, AdS_7, AdS_9, ...$

• $\Gamma_{\rm CFT}(S^d)$ has UV divergence $= -\frac{1}{2} N \zeta(0) \ln(\Lambda^2 r^2)$

$\zeta(0) = B_d(S^d) = -4{\rm a}_d$, $\quad {\rm a}_d$ = conformal anomaly of scalar in S^d
$B_d \sim \int ({\rm a}_d \mathcal{E}_d + \sum_k {\rm c}_k C_{....}....C_{....}) \;\; \to \;\; -2\,{\rm a}_d\,\chi(S^d)$
${\rm a}_4 = \frac{1}{360}, \;\; {\rm a}_6 = -\frac{1}{4\times 756}, \;\; {\rm a}_8 = \frac{23}{4\times 113400}, \cdots$

• corresponds to log IR divergence of regularized AdS_{d+1} volume:
${\rm Vol}(AdS_{d+1}) = \frac{2(-1)^{d/2}\pi^{d/2}}{\Gamma(1+\frac{1}{2}d)} \ln R \qquad R = {\varepsilon_{\rm IR}}^{-1} \;\to \infty$
• $\ln R$ term in classical HS action $\Gamma_0 = N\bar{S} \sim N{\rm Vol}(AdS_{d+1})$
should match $\ln \Lambda = \ln \varepsilon_{\rm UV}^{-1}$ term in $\Gamma_{\rm CFT}(S^d)$: $\quad \varepsilon_{\rm IR} = \varepsilon_{\rm UV} = \varepsilon$

• non-minimal theory: 1-loop correction indeed vanishes $\Gamma_1(AdS_{d+1})$=0
• minimal theory: need again $N \to N-1$ in classical HS action since

$$\Gamma_{1\,\rm min}(AdS_{d+1}) = \Gamma_{\rm conf.\,scalar}(S^d)$$

CFT in $M^d = S^1_\beta \times S^{d-1} \leftrightarrow$ HS theory in thermal AdS_{d+1}

[Giombi, Klebanov, AT 14]

• CFT$_d$ in radial quantization: operators in $R^d \to$ states in $R_t \times S^{d-1}$
spectrum of dimensions / energies – in finite $T = \beta^{-1}$ partition function

• dual theory on thermal quotient of $(AdS_{d+1})_\beta$ with boundary $S^1_\beta \times S^{d-1}$

• check matching of thermal partition functions = free energies
also: Casimir energy in $R_t \times S^{d-1} \to$ vacuum energy in AdS_{d+1}

• matching implied by equivalence of the spectra but non-trivial:
(i) singlet constraint in CFT; (ii) summation over spins in AdS
• singlet constraint: $O(N^0)$ term in CFT free energy no longer =0;
one-loop correction in HS theory in $(AdS_{d+1})_\beta$ no longer =0
• HS vacuum energy in AdS_{d+1}: vanishes after sum over spins

Standard relations: CFT_d in $R_t \times S^{d-1}$

one-particle or canonical partition function

$$\mathcal{Z}(\beta) = \mathrm{tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\omega_n}$$

"energy" zeta-function

$$\zeta_E(z) = \sum_n \mathrm{d}_n\, \omega_n^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta\, \beta^{z-1}\, \mathcal{Z}(\beta)$$

Casimir or vacuum energy

$$E_c = \tfrac{1}{2} \sum_n \mathrm{d}_n\, \omega_n = \tfrac{1}{2}\zeta_E(-1)$$

multi-particle or grand canonical partition function Z and free energy

$$\ln Z(\beta) = \mathrm{tr} \ln \left(1 - e^{-\beta H}\right)^{-1} = -\textstyle\sum_n d_n \ln(1 - e^{-\beta\omega_n})$$

$$\widehat{F}_\beta = -\ln Z(\beta) = -\sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}(m\beta)$$

Higher spin partition function in thermal AdS_{d+1} with $S^1 \times S^{d-1}$ bndry

$$Z = \prod_s Z_s = e^{-\widehat{F}(\beta)} \qquad \widehat{F} = \sum_s \widehat{F}^{(s)} \qquad \widehat{F}^{(s)} = -\ln Z_s$$

$$Z_s = \Big(\frac{\det \big[-\nabla^2 + (s-1)(s+d-2) \big]_{s-1,\perp}}{\det \big[-\nabla^2 + (s-2)(s+d-2) - s \big]_{s,\perp}} \Big)^{1/2}$$

$\widehat{F}$ is UV finite as in S^4 bndry case: $\mathrm{a}_{d+1} = 0$ (local property of AdS_{d+1})

$$\widehat{F} = \widehat{F}_c + \widehat{F}_\beta, \qquad \widehat{F}_c = \beta E_c \qquad \widehat{F}_\beta = \beta F(\beta)$$

To compute non-trivial part $\widehat{F}_\beta$:

• Hamiltonian approach [Allen, Davis 83; Gibbons, Perry, Pope 06]
and group theory to determine energy spectrum of spin s in global AdS_{d+1}
with reflective boundary conditions [Avis et al; Breitenlohner, Freedman 82]

• path integral approach – heat kernel for H^{d+1} [Camporesi, Higuchi 92]
and method of images – thermal AdS_{d+1} as quotient H^{d+1}/Z
[Gopakumar, Gupta, Lal 11]

Temperature-dependent part of AdS free energy

$$F_\beta^{(s)} = -\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}_s(m\beta) \qquad \mathcal{Z}_s(\beta) = \frac{\mathrm{d}_s\, q^{s+d-2} - \mathrm{d}_{s-1}\, q^{s+d-1}}{(1-q)^d}$$

$\mathrm{d}_s = 2[s + \frac{1}{2}(d-2)] \frac{(s+d-3)!}{(d-2)!\, s!}$ – STT tensors in d dimensions

$\mathrm{d}_s\big|_{d=3} = 2s+1, \;\; \mathrm{d}_s\big|_{d=4} = (s+1)^2, ...$

From CFT$_d$ side: $\mathcal{Z}_s$ is character of $SO(d,2)$ rep. containing spin s primary of dim $\Delta = s + d - 2$ and its descendants

[Dolan 05; Gibbons, Perry, Pope 06]

- for HS theory with $\Delta = d-2$ scalar with $\mathcal{Z}_0^{(\Delta)} = \frac{q^\Delta}{(1-q)^d}$:

$$F_\beta = \sum_{s=0}^{\infty} F_\beta^{(s)} = -\sum_{m=1}^{\infty} \frac{1}{m}\, \mathcal{Z}(m\beta)$$

$$\mathcal{Z}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=1}^{\infty} \mathcal{Z}_s(\beta) = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}$$

matches N^0 term in singlet-sector free energy of complex $U(N)$ scalar

• Non-trivial consistency check: bulk and boundary have same spectrum

• Interpretation: one-particle partition function as character $\mathcal{Z}_s(q)$ of $SO(d,2)$: matching implied by group-theoretic Flato-Fronsdal relation (e.g. in $d=4$)

$$\{0,0\}\times\{0,0\} = (d-2;0,0) + \bigoplus_{s=1}^{\infty}(d-2+s;\tfrac{s}{2},\tfrac{s}{2})$$

$$\big[\mathcal{Z}_0(\beta)\big]^2 = \mathcal{Z}_0^{(d-2)}(\beta) + \sum_{s=1}^{\infty}\mathcal{Z}_s(\beta)$$

• For minimal Vasiliev theory in AdS_{d+1}:

$$F_{\beta\,\min} = \sum_{s=0,2,4,..}^{\infty} F_\beta^{(s)} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\min}(m\beta)$$

$$\mathcal{Z}_{\min}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=2,4,...}^{\infty}\mathcal{Z}_s(\beta) = \tfrac{1}{2}\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} + \tfrac{1}{2}\frac{q^{d-2}(1+q^2)}{(1-q^2)^{d-1}}$$

matches order N^0 term in free energy of $O(N)$ singlet-sector CFT

Casimir energy

similar pattern: order N in CFT to match classical HS part

no 1-loop correction in non-minimal case: HS AdS vacuum energy vanishes

$$\zeta_E(z) = \frac{1}{\Gamma(z)} \int_0^\infty d\beta\, \beta^{z-1}\, \mathcal{Z}(\beta)\,, \qquad \mathcal{Z}(\beta) = \frac{e^{-(d-2)\beta}(1+e^{-\beta})^2}{(1-e^{-\beta})^{2d-2}}$$

$$E_c = \tfrac{1}{2}\zeta_E(-1) = \textstyle\sum_{s=0}^\infty E_{c,s} = 0$$

$\mathcal{Z}(\beta) = \mathcal{Z}(-\beta)$ property: implies vanishing of $\zeta_E(-1)$ for all d

individual spin contributions:

$$E_{c,s} = \tfrac{1}{2} \sum_{n=1}^{\infty} \tbinom{n+d-2}{d-1} \Big[\mathrm{d}_s(n+s+d-3) - \mathrm{d}_{s-1}(n+s+d-2) \Big]$$

$$d=3: \qquad E_{c,s} = \tfrac{1}{8}s^4 - \tfrac{1}{12}s^2 + \tfrac{1}{240}$$

AdS_4: $E_{c,s}$ computed using standard ζ-function in n [Allen, Davis 83]

• $E_{vac}=0$ in $\mathcal{N}>4$ extended gauged supergravities from susy sum rules

$\sum_s (-1)^{2s} d(s)\, s^p = 0, \qquad p<\mathcal{N}=1,...,8, \qquad s=0,\frac{1}{2},1,\frac{3}{2},2$

$E_c=0$ in $\mathcal{N}>4$ extended gauged supergravities [Allen, Davis 83]

and also at each KK level of spectrum of 11-d supergravity on S^7

[Gibbons, Nicolai 84; Inami, Yamagishi 84]

• cancellation in purely bosonic HS theory:

$$E_c(AdS_4)=\tfrac{1}{480}+\sum_{s=1}^{\infty}\left(\tfrac{1}{8}s^4-\tfrac{1}{12}s^2+\tfrac{1}{240}\right)=0$$

since $\zeta(0)=-\frac{1}{2},\ \zeta(-2)=\zeta(-4)=0$

$$E_c(AdS_5)=-\tfrac{1}{1440}\sum_{s=0}^{\infty} s(s+1)\Big[18s^2(s+1)^2-14s(s+1)-11\Big]=0$$

• instead of susy here ζ-function reg. (consistent with symmetries)

• no need to use special prescription to sum over s in each d:

automatically get zero if sum over spins is done first for finite z in $\zeta_E(z)$

Non-minimal vs minimal HS theory:

odd d: in CFT $E_c = 0$ (no conf. anomaly: as in flat space)

and in AdS_{d+1} sum overs spins gives $E_c = 0$

in both non-minimal (all s) and minimal (even s) HS theory

even d: in CFT $E_c \sim N$ and should match classical HS action

1-loop $E_c = 0$ in non-minimal case but $E_c \neq 0$ in minimal HS case:

using $\zeta_E(z)$ find that

$$E_c^{\rm min} = \sum_{s=0,2,4,...} E_{c,s} = \sum_{n=0}^{\infty} \frac{(n+d-3)!}{(d-2)!n!} \big[n + \tfrac{1}{2}(d-2)\big]^2$$

i.e. same as Casimir energy of single real conformal scalar in $R \times S^{d-1}$

• again consistent with $N \to N - 1$ shift of coupling constant

in minimal HS theory dual to $O(N)$ real scalar CFT

• equivalence of scalar Casimir energy in $R \times S^{d-1}$ and minimal HS energy in AdS_{d+1} requires use of same (zeta-function) regularization of sum over radial quantum number n on both sides of AdS/CFT duality

Conclusions

• quantum tests of vectorial – higher spin AdS/CFT for scalar CFT

• massless HS theories in AdS_{d+1} at one loop:
UV finite partition function; vanishing vac energy; matching free energies

• importance of definition / regularization of sum over infinite set of spins

Questions:
• leading large N term – classical action of Vasiliev theory?
• meaning of $N \to N-1$ shift in minimal HS theory?
• correlation functions:
sum over spin prescription in intermediate channel;
consistency with $N \to N-1$?

AdS_5/CFT_4: mixed $SO(2,4)$ representations

• type A HS theory dual to $U(N)$ or $O(N)$ scalars:
bilinear currents are totally symmetric traceless tensors
• $d \geqslant 4$: conformal fields and dual HS in AdS not only totally symmetric
• $d = 4$: mixed-symmetry reps – $SO(4)$ Young tableau with two rows
lengths $h_1 = j_1 + j_2 = s$, $h_2 = j_1 - j_2$, $SU(2) \times SU(2)$ weights (j_1, j_2)
conformal fields in $SO(2,4)$ reps. $(\Delta; j_1, j_2)$
$j_1 = j_2$: totally symmetric case
• such mixed-symmetry fields appear in e.g. $d = 4$ free fermion
or free Maxwell vector theory and dual type B and C HS theories in AdS_5
and thus also in $\mathcal{N} = 4$ Maxwell multiplet (superdoubleton) theory
• important for understanding (limits of) adjoint AdS/CFT
Aim:
• compute boundary conformal anomalies a and c;
partition function and Casimir energy for generic $(\Delta; j_1, j_2)$ field
• check AdS/CFT in type B and type C theories in AdS_5

AdS_5	CFT_4 (singlet sector)
non-minimal type A theory $(2;0,0)+\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N complex scalars : $U(N)$
minimal type A theory $(2;0,0)+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N real scalars : $O(N)$
non-minimal type B theory $2\,(3;0,0)+$ $2\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=1}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Dirac fermions : $U(N)$
minimal type B theory $2\,(3;0,0)+$ $\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Majorana fermions : $O(N)$
non-minimal type C theory $2\,(4;0,0)+(4;1,0)_c+$ $2\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N complex Maxwell vectors : $U(N)$
minimal type C theory $2\,(4;0,0)+$ $\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N real Maxwell vectors : $O(N)$

4d conformal anomaly

$$\mathcal{A} = -\mathrm{a}\,\mathcal{E} + \mathrm{c}\,C^2 + \mathrm{g}\,D^2 R$$

Casimir energy on S^3 [Cappelli, Coste 89]

$$E_c = \tfrac{3}{4}\left(\mathrm{a} + \tfrac{1}{2}\mathrm{g}\right)$$

g and E_c both depend on regularization (natural: ζ-function or heat kernel)
$\mathcal{N} \geqslant 3$ supersymmetric case (e.g. $\mathcal{N} = 4$ SYM)

$$\mathcal{N} \geqslant 3 \text{ susy}: \qquad E_c = \tfrac{3}{4}\mathrm{a} \qquad \mathrm{a} = \mathrm{c} \qquad \mathrm{g} = 0$$

• extract 4d conformal anomaly from bulk description:
(cf. "tree-level" 5d derivation of conf. anom. [Henningson, Skenderis 98])
1-loop correction:
$\mathcal{O}$ for 5d field ϕ dual to 4d field $(\Delta; j_1, j_2)$ [Metsaev]

$$\mathcal{O} = -D^2 + X\,, \qquad X = \Delta\,(\Delta - 4) - h_1 - |h_2| = (\Delta - 2)^2 - 2\,j_1$$

on asymptotically AdS_5 space $ds^2 = z^{-2}\,[dz^2 + g_{\mu\nu}(x, z)\,dx^\mu\,dx^\nu]$

1-loop partition function with Dirichlet-type "+" or Neumann-type "−" b.c.

$$Z^{\pm} = (\det \mathcal{O})^{-1/2}_{\pm}$$

boundary conformal anomaly $\mathcal{A}^{\pm}$ as variation of $Z^{\pm}$:
$\delta \log Z^{\pm} = -\frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, \delta\sigma\, \mathcal{A}^{\pm}, \quad \delta g_{\mu\nu} = 2\,\delta\sigma\, g_{\mu\nu}$
early attempt [Mansfield, Nolland, Ueno 03]: $\mathcal{A}^{+} = (\Delta - 2)\,\bar{A}$
in general $\mathcal{A} = \mathcal{A}^{-} - \mathcal{A}^{+} = -2\mathcal{A}^{+}$ and $\bar{A}$ is function of (Δ, j_1, j_2)
now found explicitly in case of S^4 boundary; conjectured for $R_{\mu\nu} = 0$

Partition function on $S^1 \times S^3$ and Casimir energy

one-particle partition functions same as conformal characters [Dolan 05]
"massive" conformal rep. $(\Delta;\, j_1, j_2)$: $\Delta > 2 + j_1 + j_2$
long representation of $SO(2,4)$ – massive AdS_5 HS field partition function

$$\widehat{\mathcal{Z}}^{+}(\Delta;\, j_1, j_2) = (2j_1 + 1)(2j_2 + 1)\, \frac{q^{\Delta}}{(1-q)^4}$$

"massless" rep: $\Delta = 2 + j_1 + j_2$ corresponds to conserved current in CFT

massless HS gauge field in AdS_5 (subtract ghost in 5d or cons. cond. in 4d)

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \widehat{\mathcal{Z}}^+(\Delta;\, j_1, j_2) - \widehat{\mathcal{Z}}^+(\Delta+1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2})$$

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \mathcal{Z}_+(\Delta;\, j_1, j_2) = \frac{q^\Delta}{(1-q)^4}\Big[(2j_1+1)(2j_2+1) - 4\,q\,j_1\,j_2\Big]$$

Casimir energy on S^3

compute from $\mathcal{Z}$:

$$E_c = \tfrac{1}{2}\,(-1)^F \sum_n \mathrm{d}_n\,\omega_n = \tfrac{1}{2}(-1)^F\,\zeta_E(-1)$$

$$\zeta_E(z) = \sum_n \frac{\mathrm{d}_n}{\omega_n^z} = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\,\beta^{z-1}\,\mathcal{Z}(e^{-\beta})$$

$E_c(\Delta;\, j_1, j_2) = \; E_c^- \; - \; E_c^+ = -2\,E_c^+$

massive rep:

$$\widehat{E}_c(\Delta;\, j_1, j_2) = -\tfrac{1}{720}(-1)^{2j_1+2j_2}\,(2j_1+1)(2j_2+1)(\Delta-2)$$
$$\times\Big[6\,(\Delta-2)^4 - 20\,(\Delta-2)^2 + 11\Big]$$

massless rep. $\Delta = 2 + j_1 + j_2$

$$E_c(\Delta;\, j_1, j_2) = \widehat{E}_c(\Delta;\, j_1, j_2) - \widehat{E}_c\left(\Delta + 1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}\right)$$

Conformal anomaly a-coefficient

euclidean AdS_5 with S^4 boundary

$$\log Z^+ = -\tfrac{1}{2} \log \det{}_+ \mathcal{O} = \tfrac{1}{2}\, \zeta'(0) = -4\mathrm{a}^+ \log \mathrm{R} + \ldots$$

$\zeta(z)$ from $\mathbb{H}^5$ heat kernel for "massive" 5d operator $\mathcal{O}$

gives for $\mathrm{a} = -2\mathrm{a}^+$ in massive case

$$\begin{aligned}\widehat{\mathrm{a}}(\Delta;\, j_1, j_2) =& \tfrac{1}{720}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)(\Delta-2) \\ &\times \Big[-3(\Delta-2)^4 + 10\big(j_1^2 + j_2^2 + j_1 + j_2 + \tfrac{1}{2}\big)(\Delta-2)^2 \\ &\qquad - 15(j_1 - j_2)^2 (j_1 + j_2 + 1)^2 \Big]\end{aligned}$$

in massless case:

$$\mathrm{a}(\Delta;\, j_1, j_2) = \widehat{\mathrm{a}}\big(\Delta;\, j_1, j_2\big) - \widehat{\mathrm{a}}\big(\Delta + 1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}\big)$$

Conformal anomaly c-coefficient

if a is known, to find c compute $(\mathrm{c}-\mathrm{a})$ on Ricci flat 4d space: $\mathcal{A}=(\mathrm{c}-\mathrm{a})\mathcal{E}$

for low ($s\leqslant 2$) spins $\mathrm{c}=-2\mathrm{c}^{+}$ [Mansfield et al 03; Ardehali et al 13]

$$\widehat{\mathrm{c}}^{+}-\widehat{\mathrm{a}}^{+}=-\tfrac{1}{360}\,(-1)^{2\,(j_1+j_2)}(\Delta-2)\,d(j_1,j_2)\,\big[1+f(j_1)+f(j_2)\big]$$
$$d(j_1,j_2)=(2j_1+1)(2j_2+1),\qquad f(j)\equiv j\,(j+1)\,\big[6j\,(j+1)-7\big]$$

proposal in general case:

$$\widehat{\mathrm{c}}(\Delta;\,j_1,j_2)=\tfrac{1}{720}\,(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)\,(\Delta-2)$$

$$\times\Big[-6\,(\Delta-2)^4+20\,(\Delta-2)^2+6\,(j_1^4+j_2^4)+20\,j_1^2j_2^2+12\,(j_1^3+j_2^3)$$

$$+20\,(j_1^2j_2+j_1j_2^2)-6\,(j_1^2+j_2^2)+20\,j_1j_2-12\,(j_1+j_2)-8\Big]$$

E_c, a and c are (5-th order) polynomials in $\Delta-2$, and in j_1,j_2

$\mathcal{N}$	ϕ	ψ	V_μ	E_c	a	c
1	–	1	1	$\frac{7}{64}$	$\frac{3}{16}$	$\frac{1}{8}$
2	2	2	1	$\frac{13}{96}$	$\frac{5}{24}$	$\frac{1}{6}$
3, 4	6	4	1	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$

$\mathcal{N} > 1$ superconformal multiplets

Maxwell supermultiplets

$$\mathcal{N} = 3, 4: \qquad E_c = \tfrac{3}{4}\mathrm{a} \qquad \mathrm{a} = \mathrm{c} \qquad \mathrm{g} = 0$$

$\mathcal{N} = 4$ Maxwell multiplet same as $\mathcal{N} = 4$ superdoubleton of $PSU(2,2|4)$

$\{\mathcal{N} = 4\} = \{1,0\}_c + 4\{\frac{1}{2},0\}_c + 6\{0,0\}$

$$K(\{\mathcal{N} = 4\}) = K(\mathcal{N} = 4 \text{ Maxwell}) \qquad K \equiv (E_c, \mathrm{a}, \mathrm{c})$$

$\mathcal{N}$	ϕ	Φ	ψ	Ψ	$T_{\mu\nu}$	V_μ	ψ_μ	$g_{\mu\nu}$	E_c	a	c
1	–	–	–	–	–	1	1	1	$\frac{47}{16}$	3	$\frac{17}{4}$
2	–	–	2	–	1	4	2	1	$\frac{145}{96}$	$\frac{41}{24}$	$\frac{13}{6}$
3	6	–	9	1	3	9	3	1	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$
4	20	2	20	4	6	15	4	1	$-\frac{3}{4}$	-1	-1

Conformal supergravity multiplets

short multiplets with highest spin 2 – 4d conformal supergravity multiplets

$$\mathcal{N} = 3, 4 : \qquad E_c = \tfrac{3}{4}\mathrm{a} \qquad \mathrm{a} = \mathrm{c}$$

Field	$(\Delta;\, j_1, j_2)$	E_c	a	c
$\phi\ (\Box)$	$(3; 0, 0)$	$\frac{1}{240}$	$\frac{1}{360}$	$\frac{1}{120}$
$\Phi\ (\Box^2)$	$(4; 0, 0)$	$-\frac{3}{40}$	$-\frac{7}{90}$	$-\frac{1}{15}$
$\psi\ (\partial)$	$(\frac{5}{2};\, \frac{1}{2}, 0) + (\frac{5}{2};\, 0, \frac{1}{2})$	$\frac{17}{960}$	$\frac{11}{720}$	$\frac{1}{40}$
$\Psi\ (\partial^3)$	$(\frac{7}{2};\, \frac{1}{2}, 0) + (\frac{7}{2};\, 0, \frac{1}{2})$	$-\frac{29}{960}$	$-\frac{3}{80}$	$-\frac{1}{120}$
$T_{\mu\nu}\ (\Box)$	$(3; 1, 0) + (3; 0, 1)$	$\frac{1}{40}$	$-\frac{19}{60}$	$\frac{1}{20}$
$V_\mu\ (\Box)$	$(3;\, \frac{1}{2}, \frac{1}{2})$	$\frac{11}{120}$	$\frac{31}{180}$	$\frac{1}{10}$
$\psi_\mu\ (\partial^3)$	$(\frac{7}{2};\, 1, \frac{1}{2}) + (\frac{7}{2};\, \frac{1}{2}, 1)$	$-\frac{141}{80}$	$-\frac{137}{90}$	$-\frac{149}{60}$
$g_{\mu\nu}\ (\Box^2)$	$(4;\, 1, 1)$	$\frac{553}{120}$	$\frac{87}{20}$	$\frac{199}{30}$

• $\mathcal{N}=4$ CSG + four $\mathcal{N}=4$ Maxwell is anomaly free [Fradkin, AT 81]

$$K(\mathcal{N}=4\ \mathrm{CSG})+4\,K(\mathcal{N}=4\ \mathrm{Maxwell})=0 \qquad\qquad K=(E_c,\mathrm{a},\mathrm{c})$$

• $\mathcal{N}=4$ CSG multiplet: isomorphic to
(i) supercurrent multiplet of $\mathcal{N}=4$ Maxwell theory
(ii) short massless multiplet of 5d $\mathcal{N}=8$ sugra with AdS_5 isometry $PSU(2,2|4)$
• 5d expressions for conf anomaly and Casimir energy for $\mathcal{N}=4$ CSG
are directly related to 1-loop contribution of $\mathcal{N}=8$ 5d supergravity

$$K(\mathcal{N}=4\ \mathrm{CSG})=\ -2\,K^{+}(\mathcal{N}=8\ 5\mathrm{d\ SG})$$

this is 1-loop generalization of tree-level relation [Liu, AT 98]
• implies that

$$K^{+}(\mathcal{N}=8\ 5\mathrm{d\ SG})=2\,K(\mathcal{N}=4\ \mathrm{Maxwell})$$

• this may be interpreted as expressing the fact that
states of $\mathcal{N}=8$ 5d supergravity are in product of
two $\mathcal{N}=4$ superdoubletons [Gunaydin, Minic, Zagerman 98]

Applications to AdS/CFT

Adjoint AdS_5/CFT_4: 1-loop correction in IIB 10d supergravity on S^5

type IIB superstring on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ SYM theory

$Z_{\rm SYM}$ on M^4 = $Z_{\rm string}$ on asymptotically AdS_5 with bndry M^4

implies matching of conformal anomalies and Casimir energies

direct comparison possible due to non-renormalization: on SYM side

$$K\big(\mathcal{N} = 4\ \mathrm{SU}(N)\ \mathrm{SYM}\big) = (N^2 - 1)\,\mathrm{k}\ , \qquad\qquad K \equiv (E_c, a, c)$$

$\mathrm{k} = (\frac{3}{16}, \frac{1}{4}, \frac{1}{4})$ for single $\mathcal{N} = 4$ Maxwell multiplet

at N^2 order (string tree level – classical type IIB supergravity)

demonstrated in [Henningson, Skenderis 98] (conformal anomalies)

and [Balasubramanian, Kraus 99] (vacuum energy)

string one-loop order: assume contributions of massive string modes vanish

(i) string modes: long $PSU(2,2|4)$ multiplets, should not contribute

(ii) masses depend on 't Hooft coupling ($m^2 \sim \alpha'^{-1} \sim \sqrt{\lambda}$)

contribution would contradict expected non-renormalization

	$(\Delta;\, j_1, j_2)$	$SU(4)$		$(\Delta;\, j_1, j_2)$	$SU(4)$
$p \geq 2$	$(p;\, 0, 0)$	$(0, p, 0)$	$p \geq 3$	$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(2, p-3, 1)_c$
	$(p+\frac{1}{2};\, \frac{1}{2}, 0)$	$(0, p-1, 1)_c$		$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(0, p-3, 1)_c$
	$(p+1;\, 1, 0)$	$(0, p-1, 0)_c$		$(p+2;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-3, 1)_c$
	$(p+1;\, 0, 0)$	$(0, p-2, 2)_c$		$(p+2;\, 1, 0)$	$(2, p-3, 0)_c$
	$(p+2;\, 0, 0)$	$(0, p-2, 0)_c$		$(p+3;\, 1, 0)$	$(0, p-3, 0)_c$
	$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(0, p-2, 1)_c$		$(p+\frac{5}{2};\, 1, \frac{1}{2})$	$(1, p-3, 0)_c$
	$(p+1;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-2, 1)$	$p \geq 4$	$(p+2;\, 0, 0)$	$(2, p-4, 2)$
	$(p+\frac{3}{2};\, 1, \frac{1}{2})$	$(1, p-2, 0)_c$		$(p+3;\, 0, 0)$	$(0, p-4, 2)_c$
	$(p+2;\, 1, 1)$	$(0, p-2, 0)$		$(p+4;\, 0, 0)$	$(0, p-4, 0)$
				$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(2, p-4, 1)_c$
				$(p+\frac{7}{2};\, \frac{1}{2}, 0)$	$(0, p-4, 1)_c$
				$(p+3;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-4, 1)$

Table 1: Field content of compactification of type IIB supergravity on S^5

$O(N^0)$ term should come from loop of massless string modes:
one-loop correction in 10d type IIB supergravity compactified on S^5
sum of contributions of massless $\mathcal{N} = 8$ $5d$ supergravity multiplet
and tower of massive KK multiplets [Kim, Romans, van Nieuwenhuizen 85]
thus should find

1-loop 10d IIB SG on S^5: $\qquad E_c^+ = -\frac{3}{16}, \quad \mathrm{a}^+ = -\frac{1}{4}, \quad \mathrm{c}^+ = -\frac{1}{4}$

[contributions of 5d fields with standard ("Dirichlet") b.c.: $K^+ = -\frac{1}{2}K$]

$$K^+(\text{10d IIB SG on } \mathrm{S}^5) \;\; = \;\; -K(\mathcal{N} = 4 \text{ Maxwell})$$

vacuum energy does not vanish in 1-loop type IIB supergravity on S^5
different from $\mathcal{N} > 4$ gauged SG in 4d [Allen 83]
and 11d SG on S^7 [Gibbons, Nicolai 84]
but similar to 11d SG on S^4 [Beccaria, AT]
use general expressions for $\mathrm{a}, \mathrm{c}, E_c$ and table of KK states to compute
massless level: states of 5d $\mathcal{N} = 8$ SG give ($p = 2$)

$$p = 2: \qquad E_c = \tfrac{3}{8}, \qquad \mathrm{a} = \tfrac{1}{2}, \qquad \mathrm{c} = \tfrac{1}{2}$$

• same up to -1/2 as of $\mathcal{N}=4$ 4d conformal supergravity multiplet

$p=3$ and $p\geqslant 4$ massive KK multiplets give

$$p\geq 3: \qquad E_c=\tfrac{3p}{16} \qquad \mathrm{a}=\tfrac{p}{4}\,, \qquad \mathrm{c}=\tfrac{p}{4}$$

• $K=(E_c,\mathrm{a},\mathrm{c})$ are thus universally described by $(p=2,3,4,...)$

$$K^+(\text{KK level } p \text{ of 10d IIB SG on } \mathrm{S}^5) \;\; = \;\; p\,K(\mathcal{N}=4 \text{ Maxwell})$$

• applies also for $p=1$:

$\mathcal{N}=4$ superdoubleton multiplet = Maxwell multiplet

linearity in p: E_c, a and c are 5th order polynomials in $\Delta-2$ (and thus in p)

• non-linearity in p cancels out after multiplying by dimensions of $SO(6)$ reps and summing over the members of each supermultiplet

cf. 5d states at level p appear in product of p $\mathcal{N}=4$ doubletons [Gunaydin]

• how to sum over p: correct prescription

$$\sum_{p=1}^{\infty} p=0 \qquad \text{i.e.} \qquad \sum_{p=2}^{\infty} p=-1$$

interpretation: $p = 1$ term – $\mathcal{N} = 4$ Maxwell multiplet = superdoubleton
should not to be included – gauged away
cf. decoupled $U(1)$ D3-brane contribution or $SU(N)$ vs $U(N)$ on SYM side

true if use sharp cutoff $\sum_{p=1}^{P} p = \frac{1}{2}P^2 + \frac{1}{2}P \to 0$
can be justified for E_c by ζ-function regularization directly in 10d
regularization consistent with symmetries of theory

should be applied directly in 10d rather than in 5d:
should be based on spectrum of original 10d operators

Vectorial AdS_5/CFT_4

no supersymmetry, free CFT at the boundary in any d

$d = 4$ or AdS_5 : first non-trivial case where mixed-symmetry representations appear in type B and type C theories

type C theory: dual to (complex or real) N 4d Maxwell fields

can be obtained by taking the product of two spin 1 doubletons

complex Maxwell field case: $F^*_{\mu\nu}(x)F_{\kappa\rho}(x') \to F^*\partial...\partial F$

dimension 4 states $F^*_{..}F_{..}$:

(i) scalar $F^*_{\mu\nu}F^{\mu\nu}$ and pseudoscalar $F^*_{\mu\nu}\widetilde{F}^{\mu\nu}$ in rep $(4;0,0)$;

(ii) antisymmetric tensor $F^*_{\mu[\nu}F_{\kappa]\mu}$ – massive selfdual + anti-selfdual rank 2 tensors: $(4;1,0)_c = (4;1,0) + (4;0,1)$

(iii) spin 2 conserved stress tensor $(4;1,1)$ and its parity-odd counterpart with one $F_{\mu\nu}$ replaced by $\widetilde{F}_{\mu\nu}$

(iv) conserved current with symmetries of Weyl tensor, i.e. massless state $(4;2,0)_c$ described by Young tableu with 2 rows and 2 columns

AdS_5	CFT_4 (singlet sector)
non-minimal type A theory $(2;0,0)+\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N complex scalars : $U(N)$
minimal type A theory $(2;0,0)+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N real scalars : $O(N)$
non-minimal type B theory $2\,(3;0,0)+$ $2\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=1}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Dirac fermions : $U(N)$
minimal type B theory $2\,(3;0,0)+$ $\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Majorana fermions : $O(N)$
non-minimal type C theory $2\,(4;0,0)+(4;1,0)_c$ $2\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N complex Maxwell vectors : $U(N)$
minimal type C theory $2\,(4;0,0)+$ $\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\ldots}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N real Maxwell vectors : $O(N)$

sum over spins prescription: sum with fixed cutoff

implied by use of spectral ζ-function

$$\sum_s K(s) \equiv \sum_s e^{-\epsilon\,(s+\frac{1}{2})}\, K(s)\,\Big|_{\epsilon\to 0,\ \text{finite part}}\,, \qquad K=(E_c,\mathrm{a},\mathrm{c})$$

$s=j_1+j_2$ is total spin and summation over all states

non-minimal type A theory:

$$\sum_{s=1}^{\infty} K^+(2+s;\ \tfrac{s}{2},\tfrac{s}{2}) = 0$$

minimal type A theory:

$$\sum_{s=2,4,\dots}^{\infty} K^+(2+s;\ \tfrac{s}{2},\tfrac{s}{2}) = K(3;\,0,0)$$

i.e. AdS_5 HS theory 1-loop correction is exactly 1-loop contribution

of single real massless 4d scalar: $K(3;\,0,0)=(\frac{1}{240},\frac{1}{360},\frac{1}{120})$

consistent with AdS/CFT duality if minimal HS theory action $N\to N-1$

non-minimal type B theory:

$$2\,K^+(3;\,0,0) + 2\sum_{s=1}^{\infty} K^+(2+s;\,\tfrac{s+1}{2},\tfrac{s-1}{2}) = 0$$

$2\,K^+(3;\,0,0) = -K(3;0,0)$ contribution of two 5d scalars
symmetric representation term vanishes separately
contributions of $(\Delta;j_1,j_2)$ and $(\Delta;j_2,j_1)$ are equal: doubling

minimal type B theory:

$$2\,K^+(3;\,0,0) + 2\sum_{s=2,4,\ldots}^{\infty} K^+(2+s;\,\tfrac{s+1}{2},\tfrac{s-1}{2}) = K(\tfrac{5}{2};\,\tfrac{1}{2},0)_c$$

r.h.s. is same as contribution of single 4d Majorana fermion
$K(\frac{5}{2};\,\frac{1}{2},0)_c = 2K(\frac{5}{2};\,\frac{1}{2},0) = (\frac{17}{960},\frac{11}{720},\frac{1}{40})$

non-minimal type C theory:

$$2\,K^+(4;\,0,0)+K^+(4;\,1,0)_c$$
$$+\,2\sum_{s=2}^{\infty}K^+(2+s;\,\tfrac{s}{2},\tfrac{s}{2})+\sum_{s=2}^{\infty}K^+(2+s;\,\tfrac{s+2}{2},\tfrac{s-2}{2})_c$$
$$=2\,K(3;\,\tfrac{1}{2},\tfrac{1}{2})=-4\,K^+(3;\,\tfrac{1}{2},\tfrac{1}{2})$$

sum of all AdS_5 1-loop contributions is no longer zero – is twice of $K(3;\,\frac{1}{2},\frac{1}{2})=(\frac{11}{120},\frac{31}{180},\frac{1}{10})$ – same as of one complex 4d Maxwell field

already in non-minimal type C theory case one needs $N\to N-1$?!

minimal type C theory:

$$2\,K^+(4;\,0,0)+\sum_{s=2}^{\infty}K^+(2+s;\,\tfrac{s}{2},\tfrac{s}{2})+\sum_{s=2,4,\dots}^{\infty}K^+(2+s;\,\tfrac{s+2}{2},\tfrac{s-2}{2})_c$$
$$=2\,K(3;\,\tfrac{1}{2},\tfrac{1}{2})=-4\,K^+(3;\,\tfrac{1}{2},\tfrac{1}{2})$$

here boundary vector field is real:

need shift $N\to N-2$ in the coefficient of the classical HS action

Supersymmetric cases

• supersymmetry not a necessary ingredient in vectorial AdS/CFT duality

but may consider also supersymmetric AdS_5/CFT_4 dual pairs

(supersymmetric AdS_4/CFT_3 cases [Sezgin, Sundell 03,Leigh, Petkou 03])

• $\mathcal{N} = 1$ supersymmetric HS theory in AdS_5 [Alkalaev, Vasiliev 02]

boundary theory – N free spin $(0, \frac{1}{2})$ $\mathcal{N} = 1$ supermultiplets

similar susy generalizations of type A, B and C theory examples

• most supersymmetric case of free unitary boundary CFT:

N free $\mathcal{N} = 4$ Maxwell supermultiplets

• spectrum of dual AdS_5 HS theory: product of two $\mathcal{N} = 4$ superdoubletons

[Gunaydin et al 98; Sezgin, Sundell 02]

low-spin $s \leqslant 2$ part same as in type IIB supergravity compactified on S^5

• this HS theory should correspond to "leading Regge trajectory" part of

"zero tension" limit of $AdS_5 \times S^5$ superstring [Bianchi et al 03]

• particular maximally supersymmetric case of vectorial AdS/CFT duality

as a truncation of $g_{\rm YM} = 0$ limit of the adjoint AdS/CFT

when 5d fields are combined into supermultiplets many cancellations happen

• $K^+ = (E_c^+, \mathrm{a}^+, \mathrm{c}^+)$ for infinite set of HS 5d fields appearing in product of two superdoubletons $\{\mathcal{N}\}$ each representing $\mathcal{N}$-super Maxwell theory

$$K^+(\{\mathcal{N}\} \otimes \{\mathcal{N}\}) = 2\, K(\{\mathcal{N}\}) = 2\, K(\mathcal{N}\text{-Maxwell})$$

r.h.s. is twice the contribution of $\mathcal{N}$-super Maxwell theory or $\mathcal{N}$-superdoubleton

• get direct super-generalization of the relation in type C theory
"anomaly of a product is twice anomaly of a factor":
may be viewed as analog of relation for the characters or partition functions
$\mathcal{Z}(\{\mathcal{N}\} \otimes \{\mathcal{N}\}) = [\mathcal{Z}(\{\mathcal{N}\})]^2$

• admits the following interpretation:
1-loop contribution of states of $\mathcal{N} = 8$ 5d supergravity is already equal to that of two $\mathcal{N} = 4$ Maxwell multiplets; thus all other states appearing in the product $\{\mathcal{N}\} \otimes \{\mathcal{N}\}$ should give zero contribution: they should be in long massless supermultiplets of $PSU(2,2|4)$ giving 0 contributions

Conclusions

• quantum tests of vectorial – higher spin AdS/CFT:
general mixed representations in $AdS_{d+1},\ d = 2, 4, 6$

• supersymmetric examples: cancellations, simple patterns of
contrubutions of KK multiplets;
subleading terms in a-anomaly coefficients:
$a_{d=4} = N^2 - 1, \quad a_{d=6} = 4N^3 - \frac{9}{4}N - \frac{7}{4}, \quad a_{d=2} = 6(N_5 N_1 + 1)$

• applications: to adjoint AdS/CFT in "zero-tension"' limit