Brane adjustments

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based on work with
Fredenhagen & Keller, hep-th/0609034
Brunner & Baumgartl, 0704.2666 [hep-th]

Moduli spaces

Many phenomenologically interesting string backgrounds involve D-branes.

Stabilising their moduli then involves two kinds of moduli:

- closed string moduli (closed string background)
- D-brane moduli (position etc. of D-brane in given closed string background)

Dependencies

Obviously, these two moduli spaces are not independent of one another:

- The closed string background determines what kinds of branes are allowed, i.e. the D-brane moduli space.
- The D-branes back-react on the closed string background, and thereby may also modify the closed string moduli space.

Tree level

The back-reaction of the D-brane only arises at higher order in string perturbation theory (annulus), but the dependence of the D-brane moduli space on the bulk moduli is already visible at tree level.

This second effect is what we want to discuss in the following.

Conformal field theory

More precisely, we want to understand how a brane adjusts itself to changes of the closed string background.

This question can be analysed in conformal field theory by studying the RG equations for combined bulk and boundary perturbations.

A simple example

To illustrate the problem consider the closed string background that describes a free boson compactified on a circle of radius R, for which all conformal D-branes are known.

For all values of R we have the usual Dirichlet & Neumann branes.

But the remainder of the moduli space of conformal D-branes depends in a very sensitive manner on the value of R:

The D-brane moduli space

- if $R = \frac{M}{N} R_{sd}$ then the additional part of the moduli space of conformal D-branes is

$$SU(2)/\mathbb{Z}_M \times \mathbb{Z}_N \ .$$

[Friedan]
[MRG, Recknagel]

- if R is an irrational multiple of the self-dual radius, then the additional part of the moduli space is just the interval

$$(-1, 1) \ .$$

[Friedan], [Janik]

Bulk modulus

On the other hand, the radius R is a closed string modulus, so in this example the moduli space of D-branes depends strongly on where we are in the closed string moduli space!

So what happens to a brane associated to a generic element in $\mathrm{SU}(2)/\mathbb{Z}_{\mathrm{M}} \times \mathbb{Z}_{\mathrm{N}}$ (that exists when the radius is rational) if we change the radius of the circle?

The WZW case

For simplicity we consider in the following the theory at the self-dual radius (M=N=1), where it is equivalent to the SU(2) WZW model at level k=1.

The moduli space of conformal branes is then simply SU(2), where we write an arbitrary group element as

$$g = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}$$

$$[|a|^2 + |b|^2 = 1]$$

b=0: Dirichlet brane

a=0: Neumann brane

Conformal branes

Here the brane corresponding to g is characterised by the gluing condition

$$\left(g\, J^a_m\, g^{-1} + \bar{J}^a_{-m}\right)\, \| g \rangle\!\rangle = 0 \ ,$$

where a=1,2,3 labels a basis of su(2).

The exactly marginal bulk operator that corresponds to changing the radius is then the operator of conformal dimension (1,1)

$$\Phi = J^3 \bar{J}^3 \ .$$

Exact marginality

Exact marginality requires, in particular, that the perturbing field continues to have conformal dimension (1,1), even after the perturbation.

For closed string correlators this requires (to first order in perturbation theory) that the 3-point self-coupling vanishes:

$$\Phi(x,\bar{x})\Phi(y,\bar{y}) = \cdots + \frac{0}{|x-y|^2}\Phi(y,\bar{y}) + \cdots \quad .$$

Obviously, this is the case in the above example.

Exact marginality on disc

To check for exact marginality on the disc, we calculate the perturbed 1-point function on the upper half plane, i.e.

$$\langle \Phi(w) \rangle_\lambda = \langle \Phi(w) \rangle + \lambda \int_{\mathbb{H}+} d^2 z \, \langle \Phi(z) \, \Phi(w) \rangle + \cdots$$

A necessary condition for exact marginality is then that

$$\langle \Phi(w) \rangle_\lambda = \frac{\mathsf{C}}{|\mathrm{Im} w|^2} \; .$$

SU(2) level 1

For the case of the D-brane described by the group element g, the first order perturbation equals (here ϵ is a UV cutoff)

$$\lambda\,\mathrm{Tr}\left([t^3, g t^3 g^{-1}]^2\right)\frac{\pi}{4|\mathrm{Im}w|^2}\left(\log\epsilon - \frac{1}{2}\log|\mathrm{Im}w|^2 + \mathcal{O}(\epsilon)\right)$$

if prefactor is non-zero

modifies functional dependence!

Exact marginality

The prefactor equals

$$\mathrm{Tr}([t^3, g\, t^3\, g^{-1}]^2) = -8|a|^2|b|^2 \ .$$

Thus the radius perturbation is only exactly marginal if a=0 or b=0, i.e. if the brane is a standard Neumann or Dirichlet brane!

This ties in nicely with the fact that only the standard Neumann and Dirichlet branes exist for all radii!

[Fredenhagen, MRG, Keller]

Response of the brane

But what happens if we consider a generic brane for which neither a nor b vanishes?

In order to answer this question we need to study the RG equations for combined bulk and boundary perturbations.

RG equations

Consider the perturbation

$$S = S^* + \sum_i \tilde{\lambda}_i \int \phi_i(z)\, d^2 z + \sum_j \tilde{\mu}_j \int \psi_j(x)\, dx$$

bulk perturbation

boundary perturbation

To regularise introduce length scale l, define dimensionless coupling constants

$$\tilde{\lambda}_i = \lambda_i\, l^{h_{\phi_i}-2} \ , \qquad \tilde{\mu}_j = \mu_j\, l^{h_{\psi_j}-1} \ ,$$

and introduce the UV cutoffs

$$|z_k^i - z_{k'}^{i'}| > l \ , \qquad |x_k^j - x_{k'}^{j'}| > l \ , \qquad \mathrm{Im}\, z_k^i > \frac{l}{2} \ .$$

RG equations

Now we rescale $l \mapsto (1+\delta t)l$, and ask how we have to adjust the coupling constants so as to leave the free energy unchanged.

[Cardy]

Explicit dependence:

$$\lambda_i \to (1+(2-h_{\phi_i})\delta t)\lambda_i \ , \qquad \mu_j \to (1+(1-h_{\psi_j})\delta t)\mu_j \ .$$

Implicit dependence:

$$|z_k^i - z_{k'}^{i'}| > l \ \Rightarrow \ \delta\lambda_k = \pi C_{ijk}\lambda_i\lambda_j\delta t \qquad C_{ijk}\text{: bulk OPE coefficient}$$

$$|x_k^j - x_{k'}^{j'}| > l \ \Rightarrow \ \delta\mu_k = D_{ijk}\mu_i\mu_j\delta t \qquad D_{ijk}\text{: boundary OPE coefficient}$$

$$\mathrm{Im} z > \frac{l}{2} \ \Rightarrow \ \delta\mu_k = \frac{\delta t}{2} B_{ik}\lambda_i \qquad B_{ik}\text{: bulk-boundary OPE coefficient}$$

RG equations

Altogether we thus find the first order RG equations:

$$\begin{aligned}\dot{\lambda}_k &= (2 - h_{\phi_k})\lambda_k + \pi C_{ijk}\,\lambda_i\lambda_j + \mathcal{O}(\lambda^3)\\ \dot{\mu}_k &= (1 - h_{\psi_k})\mu_k + \frac{1}{2}B_{ik}\,\lambda_i + D_{ijk}\,\mu_i\mu_j + \mathcal{O}(\mu\lambda, \mu^3, \lambda^2)\end{aligned}$$

bulk induced
boundary flow

[Fredenhagen,
MRG, Keller]

Exact marginality on disc

In general an exactly marginal bulk perturbation thus need not be exactly marginal on the disc any more.

In fact, the condition that exact marginality of ϕ_i is preserved on the disc, is that the bulk-boundary OPE coefficients vanish

$$B_{ik} = 0$$

for all marginal or relevant boundary fields ψ_k (except the identity).

WZW example

In the case of the above su(2) example we find that the exactly marginal bulk perturbation by $\Phi = J^3 \bar{J}^3$ has a non-vanishing bulk-boundary OPE coefficient

$$B_{\Phi c} = -2\sqrt{2}\,|a|\,|b|$$

with the marginal boundary current corresponding to

$$t^c = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\,, \quad \text{where} \quad a\,b^* = |ab|e^{i\varphi}\,.$$

Boundary flow

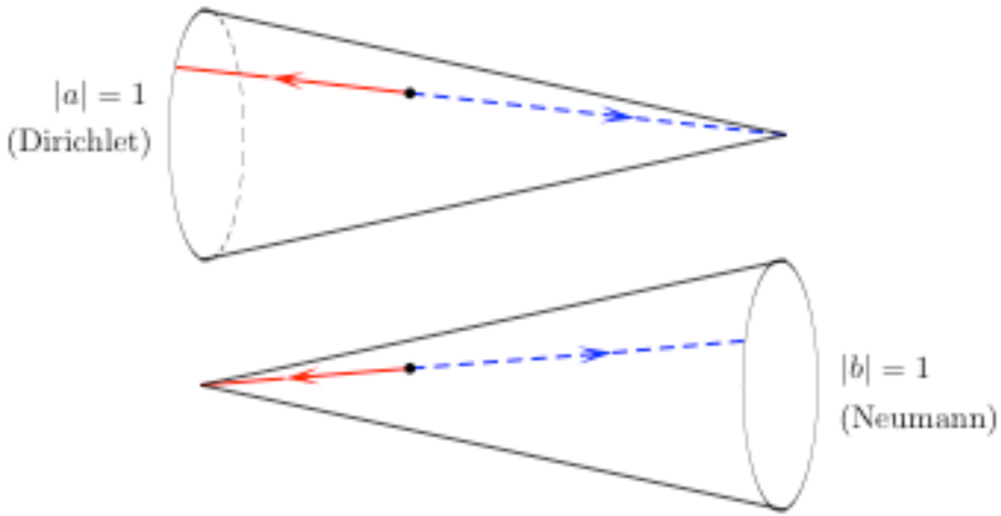
This boundary current modifies the boundary condition g by

$$\delta g = i\, t^c\, g = \frac{1}{\sqrt{2}} \begin{pmatrix} -a\frac{|b|}{|a|} & b^*\frac{|a|}{|b|} \\ -b\frac{|a|}{|b|} & -a^*\frac{|b|}{|a|} \end{pmatrix} .$$

This leaves the phases of a and b unmodified, but decreases the modulus of a, while increasing that of b.

The flow on SU(2)

In fact, one can integrate the RG equations exactly in the boundary coupling (at first order in the bulk perturbation), and one finds that the RG flow is along a geodesic on SU(2).



increase radius

decrease radius

A supersymmetric example

This analysis was performed for the simplest bosonic example, a free boson compactified on a circle.

Is it possible to do a similar analysis also for more interesting/realistic examples?

In the following I want to explain how this can be done by combining these conformal field theory arguments with matrix factorisation techniques.

[Baumgartl, Brunner, MRG]

The quintic

To illustrate this method we want to consider the Fermat quintic, i.e. the Calabi-Yau manifold described by the equation

$$W_0 = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$$

in complex projective space $\mathbb{CP}^4$.

At this point in the closed string moduli space, its conformal field theory description is known: it is the Gepner model corresponding to the tensor product of five N=2 models with k=3.

D-branes in Gepner models

For such a Gepner model two classes of branes are known: these are the Recknagel-Schomerus (RS) branes that are characterised by the property that they preserve the 5 N=2 superconformal algebras separately:

$$\begin{aligned} \left(L_n^{(i)} - \tilde{L}_{-n}^{(i)} \right) \| B \rangle\rangle &= 0 \\ \left(G_r^{\pm(i)} - \tilde{G}_{-r}^{\pm(i)} \right) \| B \rangle\rangle &= 0 \end{aligned}$$

[Here I have described B-type branes.]

Permutation branes

In addition there are the permutation branes that are characterised by

$$\begin{aligned} \left(L_n^{(i)} - \tilde{L}_{-n}^{(\pi(i))} \right) \| B \rangle\!\rangle &= 0 \\ \left(G_r^{\pm(i)} - \tilde{G}_{-r}^{\pm(\pi(i))} \right) \| B \rangle\!\rangle &= 0 \, , \end{aligned}$$

where $\pi \in S_5$ is a permutation of the five N=2 algebras.

[Recknagel]
cf. also [MRG, Schafer-Nameki]

Rational constructions

Unfortunately, these constructions only describe very special D-branes at isolated points in the closed string moduli space.

This is therefore not sufficient to study the questions about the moduli space we are interested in....

To make progress we use that the topological aspects of B-type D-branes can be described in a different manner.

Matrix factorisations

Kontsevich has suggested that the B-type D-branes of the Landau-Ginzburg model with superpotential W (that flows in the IR to the conformal field theory in question) can be characterised in terms of matrix factorisations of W as

$$E(x_i) \cdot J(x_i) = W(x_i) \; \cdot \mathbf{1}$$

Here E and J are polynomial (r x r)-matrices in the variables x_i.

Matrix factorisations

Equivalently, we can describe this in terms of the (2r x 2r) matrix

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \ ,$$

that satisfies then the condition

$$\{Q, Q\} = (2W) \cdot \mathbf{1} \ .$$

Matrix factorisations

Either condition can be understood from a physics point of view by analysing the supersymmetry variation of the Landau-Ginzburg model on a world-sheet with boundary (Warner problem).

[Brunner, et.al.]
[Kapustin, Li]

The matrices describe (world-sheet) fermionic degrees of freedom at the boundary. They compensate the above variation terms.

A single minimal model

The simplest example is the one with superpotential $W = x^d$. It flows in the IR to a single N=2 minimal model at level k (d=k+2).

The matrix factorisations of this superpotential are all equivalent to direct sums of the fundamental factorisations (m=1,.., d-1)

[Herbst et al]

$$E_m = x^m \qquad J_m = x^{d-m} \ .$$

[The corresponding branes are the standard B-type branes of this minimal model.]

Tensoring factorisations

Matrix factorisations can be tensored. For example, for the superpotential $W = x_1^{d_1} + x_2^{d_2}$ the simple factorisations of each monomial can be tensored to give a (tensor) factorisation of W given by

[Ashok et al]

$$E = \begin{pmatrix} x_1^{m_1} & x_2^{m_2} \\ x_2^{d_2-m_2} & -x_1^{d_1-m_1} \end{pmatrix} \qquad J = \begin{pmatrix} x_1^{d_1-m_1} & x_2^{m_2} \\ x_2^{d_2-m_2} & -x_1^{m_1} \end{pmatrix}$$

Tensor branes

In particular, by tensoring five such one-dimensional factorisations together one describes precisely the RS (tensor) branes.

This identification can be checked by by comparing the topological open string spectrum of these branes.

[Brunner, et.al.]
[Kapustin, Li]

- In conformal field theory: consider the chiral primaries in open string spectrum.
- From matrix factorisation point of view: the topological spectrum is the cohomology of an operator that is associated to the factorisations.

Permutation factorisations

The factorisations that correspond to the permutation branes are also known.

[Brunner, MRG]
[Enger, et. al.]

In particular, the `transposition' branes involving two factors of the same central charge arise from writing

$$W = x_1^d + x_2^d = \prod_{l=1}^{d} (x_1 - \eta_l x_2) = E\, J \ ,$$

where the product runs over the d'th roots of -1.

[Ashok et al]

A family of factorisations

Using matrix factorisation techniques we can
now however also discuss whole families
of branes (not just isolated points).

Linear ansatz

To see how this goes we make the ansatz

$$J_1 = x_1 - \eta x_2 \ , \qquad J_3 = a x_3 - b x_4 \ , \qquad J_5 = a x_5 - c x_4 \ ,$$

and look for common solutions of

$$J_1 = J_3 = J_5 = 0 \quad \text{and} \quad W_0 = 0 \ ,$$

where as before

$$W_0 = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \ .$$

Linear ansatz

To see how this goes we make the ansatz

$$J_1 = x_1 - \eta x_2 \,, \qquad J_3 = a x_3 - b x_4 \,, \qquad J_5 = a x_5 - c x_4 \,.$$

If η is a fifth root of -1 we get from $J_1 = 0$

$$x_1 = \eta x_2 \,,$$

and hence W_0 equals then

$$W_0 = x_3^5 + x_4^5 + x_5^5 \,.$$

Linear ansatz

To see how this goes we make the ansatz

$$J_1 = x_1 - \eta x_2 \,, \qquad J_3 = a x_3 - b x_4 \,, \qquad J_5 = a x_5 - c x_4 \,.$$

Next, if a is non-zero we get from $J_3 = J_5 = 0$

$$x_3 = \frac{b}{a} x_4 \,, \qquad x_5 = \frac{c}{a} x_4 \,,$$

and hence $W_0 = 0$ becomes the equation

$$W_0 = \frac{x_4^5}{a^5}\left(b^5 + a^5 + c^5\right) = 0 \,.$$

Joint solution

Thus we have a joint solution if

$$a^5 + b^5 + c^5 = 0 \ .$$

Then the Nullstellensatz implies that W_0 can be written as

$$W_0 = J_1 \cdot E_1 + J_3 \cdot E_3 + J_5 \cdot E_5 \ ,$$

where all factors are polynomials.

[Brunner, et.al.]
[Brunner, MRG, Keller]

Hence we get a corresponding matrix factorisation!

D-brane moduli space

The moduli space of these branes is thus the complex curve

$$M = a^5 + b^5 + c^5 = 0$$

in complex projective space $\mathbb{CP}^2$.

Geometrically these branes are D2-branes wrapping the 2-cycle on $W_0 = 0$ described by

$$(x_1, x_2, x_3, x_4, x_5) = (u, \eta u, av, bv, cv) \ ,$$

where $(u, v) \in \mathbb{CP}^1$.

cf [Ashok et al]

Open string spectrum

The matrix factorisation description now allows us to calculate the (topological) open string spectrum on each of these D2-branes.

At each point in the moduli space M=0 there are two `fermions' that correspond to marginal boundary fields in conformal field theory.

[In fact, all marginal boundary fields are described in this manner.]

Marginal directions

In particular, we always have one exactly marginal boundary field $\Psi_1 = \partial_b Q(a,b,c)$, where $Q(a,b,c)$ denotes the above factorisation.

This is the field that moves one along the D-brane moduli space.

In addition, there is a second marginal boundary field Ψ_2 which is however only exactly marginal at special points.

[Its three-point function does not vanish, except at special points where different branches of the moduli space meet.]

Bulk perturbations

Now we have understood as much as we need about the brane moduli space at the Gepner point in order to come back to the problem we are interested in:

What happens to these branes as we switch on a closed string modulus?

Complex structure deformation

The complex structure deformations of our Gepner model can be easily described in the LG language: they correspond to adding to the superpotential homogenous polynomials.

In the following we shall consider one such class of deformations, namely those of the form

$$W = W_0 + \lambda\Phi \ , \ \Phi = x_1^3 \cdot \left(t_3 x_3^2 + t_4 x_4^2 + t_5 x_5^2 \right) \ .$$

Geometry

It is known that at a generic point in the complex structure moduli space of the quintic there are only finitely many distinct lines, and not any family of curves. Thus one should expect that these complex structure deformations are typically obstructed.

[Albano, Katz]

We therefore want to understand what happens to these branes as the bulk perturbation is switched on.

Matrix factorisations obstruction

From a matrix factorisation point of view, try to find

$$Q = Q(a, b, c) + \lambda Q^{(1)} + \lambda^2 Q^{(2)} + \cdots$$

so that Q becomes a matrix factorisation of

$$W = W_0 + \lambda \Phi \ .$$

To first order in λ we find

$$\{Q, Q\} = 2W_0 + \lambda \Big\{ Q(a, b, c), Q^{(1)} \Big\} + \mathcal{O}(\lambda^2) \ .$$

Matrix factorisation obstruction

Thus a necessary condition is that we can write

$$2\,\Phi = \left\{ Q(a,b,c), Q^{(1)} \right\} \ ,$$

i.e. that Φ is exact with respect to $Q(a,b,c)$.

[Hori, Walcher]

Explicitly one finds that this is only the case provided that

$$t_3 b^2 + t_4 a^2 + t_5 c^2 = 0.$$

In fact, this is also a sufficient condition.

Discrete solutions

On the other hand, Bezout's theorem implies that there are only ten discrete solutions of the joint equations (for nontrivial (t_3, t_4, t_5))

$$a^5 + b^5 + c^5 = 0 = t_3 b^2 + t_4 a^2 + t_5 c^2 .$$

If (a, b, c) is not one of these ten points, then the matrix factorisation is obstructed under the perturbation by Φ !

Bulk induced RG flow

So what happens in conformal field theory?

We expect that the situation is similar to what happens for the free boson: the bulk perturbation will induce a non-trivial RG flow on the boundary that will drive the brane to one of the ten allowed brane embeddings!

Bulk induced RG flow

Actually, while we do not have an explicit conformal field theory realisation of this brane, we know enough to check this explicitly:

$$\begin{aligned}\dot{\lambda}_k &= (2-h_{\phi_k})\lambda_k + \pi C_{ijk}\,\lambda_i\lambda_j + \mathcal{O}(\lambda^3)\\ \dot{\mu}_k &= (1-h_{\psi_k})\mu_k + \frac{1}{2}B_{ik}\,\lambda_i + D_{ijk}\,\mu_i\mu_j + \mathcal{O}(\mu\lambda,\mu^3,\lambda^2)\end{aligned}$$

bulk induced
boundary flow

Bulk induced RG flow

Actually, while we do not have an explicit conformal field theory realisation of this brane, we know enough to check this explicitly:

The important terms in the RG analysis are the bulk-boundary OPE coefficients for all marginal (or relevant) boundary fields. In the current context where the bulk perturbation is topological, these are topological quantities that can be calculated in the matrix factorisation description.

Bulk-boundary coefficients

In our example one finds, using the Kapustin-Li formula,

$$B_{\Phi,\Psi_1} = \frac{\eta^4}{25c^4}\left(t_3 b^2 + t_4 a^2 + t_5 c^2\right)$$
$$B_{\Phi,\Psi_2} = 0\ .$$

[Here we have worked in a chart of the brane moduli space where $ac \neq 0$, and we have rescaled the moduli variables so that $a = 1$.]

Boundary RG flow

The bulk perturbation therefore only switches on the boundary moduli field Ψ_1 , and as in the bosonic WZW case, we can interpret the RG flow as a flow in the original moduli space.

In fact, the relevant RG equation is simply

$$\dot{b} = \lambda \frac{\eta^4}{25c^4} \left(t_3 b^2 + t_4 a^2 + t_5 c^2 \right) \quad .$$

correct
fixed points

[Baumgartl,
Brunner, MRG]

Gradient flow

This RG flow is actually a gradient flow

$$\dot{b} = \partial_b \mathcal{W}(a, b, c) \ ,$$

cf. [Friedan, Konechny]

where $\mathcal{W}(a, b, c)$ is globally defined (but multi-valued) on the whole D-brane moduli space.

[Explicitly, $\mathcal{W}(a, b, c)$ is the integral of a holomorphic 1-form on the brane moduli space.]

cf. [Agangic, Vafa]

Effective superpotential

The potential $\mathcal{W}(a, b, c)$ has also got a nice interpretation:

It is precisely the term of the (exact) effective superpotential that is linear in the bulk coupling constant.

[Baumgartl, Brunner, MRG]

[This follows from the fact that $\mathcal{W}(a, b, c)$ is just the generating function of the amplitudes involving one bulk insertion as well as arbitrary many boundary insertions.]

Conclusions

- Branes adjust via boundary RG flow to changes in bulk moduli space.
- RG flow is gradient flow of effective superpotential.
- Using matrix factorisation techniques this can be very explicitly calculated.

Future directions

- Use this approach to calculate effective superpotential exactly. Check mirror symmetry.
- Study backreaction of branes onto closed string background. [MRG, Keller, in progress]
- Study brane adjustment under relevant bulk perturbations.
  [For A-type brane of a single minimal N=2 model this could be done quite explicitly.]
  [MRG, Lawrence]