# *Aspects of entanglement of/between disjoint regions in CFT & Holography*

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**C. De Nobili, A. Coser and E.T.;** [1501.04311]

**P. Fonda, L. Giomi, A. Salvio and E.T.;** [1411.3608] (JHEP)

**P. Calabrese, J. Cardy and E.T.;** [1408.3043] (JPA)

*Holographic Methods for Strongly Coupled Systems*

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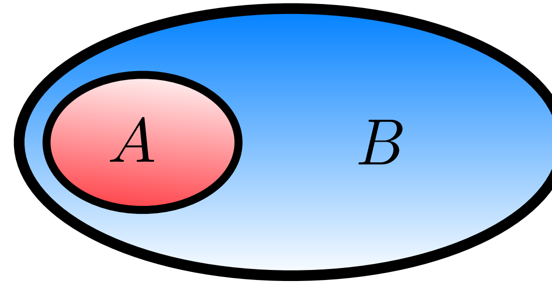
# Outline

- Introduction & some motivations
- Holographic entanglement entropy in $AdS_4$:
  - Interpolating between the disk and the infinite strip
  - Polygons
  - Holographic mutual information: disks & other shapes
- Entanglement in 2D CFT:
  - Entanglement negativity: definitions and replica limit
  - Entanglement entropies for disjoint intervals
  - Numerical extrapolations
  - Entanglement negativity at finite temperature
- Conclusions & open issues

# *Mutual Information & Entanglement Negativity*




- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$

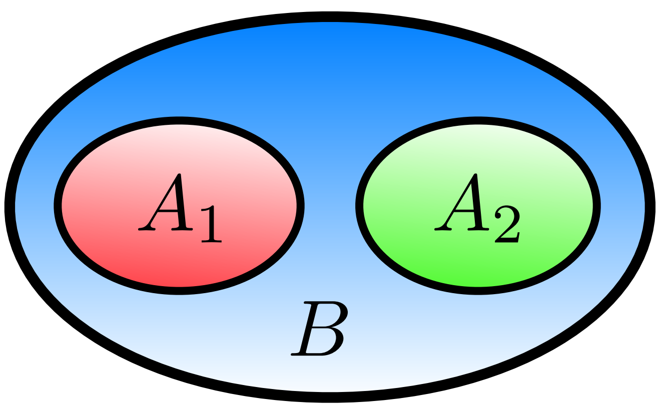
Rényi entropies

*Entanglement entropy* $S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{n\to 1} \frac{\log(\mathrm{Tr}\rho_A^n)}{1-n} = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\,\rho_A^n$

- $S_A = S_B$ for pure states

- *Tripartite* system $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$

$\rho_{A_1 \cup A_2}$ is mixed




- $S_{A_1\cup A_2}$: entanglement between $A_1 \cup A_2$ and $B$

*Entanglement between $A_1$ and $A_2$?*

- The mutual information $S_{A_1} + S_{A_2} - S_{A_1\cup A_2}$ gives an upper bound
- A *computable* measure of the entanglement is the *logarithmic negativity*

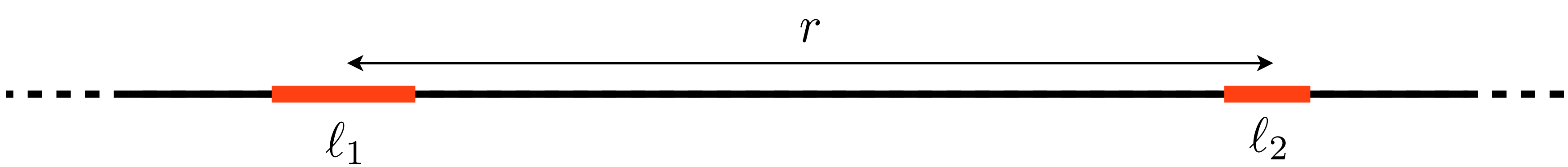
# *Short intervals expansion in 2D CFT*

- One interval on the infinite line at $T = 0$
  [Holzhey, Larsen, Wilczek, (1994)]
  [Calabrese, Cardy, (2004)]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + \text{const}$$

- Two intervals $A_1$ and $A_2$: $\text{Tr}\rho^n_{A_1 \cup A_2}$ for small intervals w.r.t. to other characteristc lengths of the system

[Headrick, (2010)]
[Calabrese, Cardy, E.T., (2011)]




$$\text{Tr}\rho^n_A = c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left( \frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j} \left( e^{2\pi i j/n} \right) \rangle^2_{\mathbf{C}}$$

$\text{Tr}\rho^n_A$ for disjoint intervals contains *all* the data of the CFT
(conformal dimensions and OPE coefficients)

The vacuum is not empty

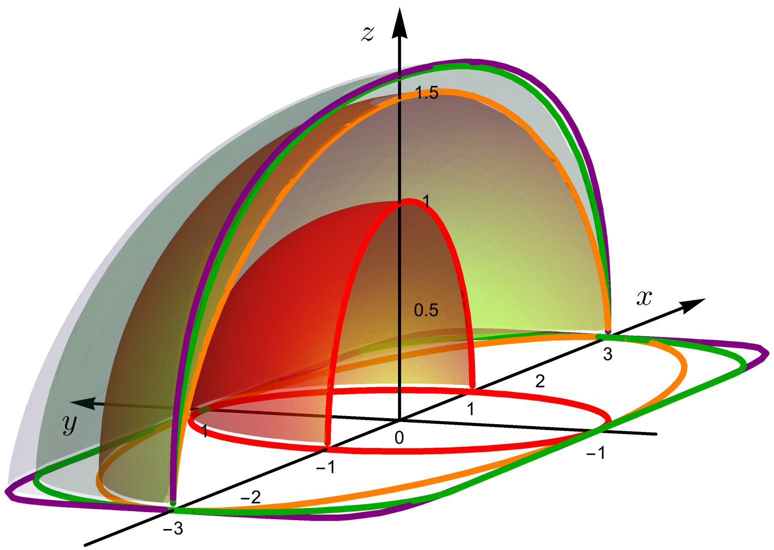
- Generalization to $2 + 1$ dimensions [Cardy, (2013)]

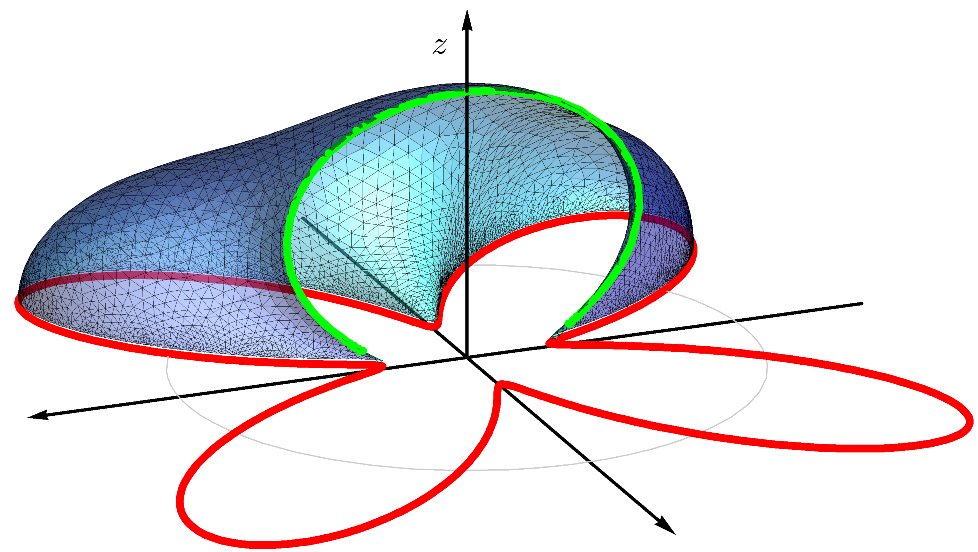
# *Holographic entanglement entropy in AdS(4)*

- Constant time slice in $AdS_{d+2}$
  Surfaces $\gamma_A$ s.t. $\partial\gamma_A = \partial A$
  Find the *minimal area* surface $\tilde{\gamma}_A$

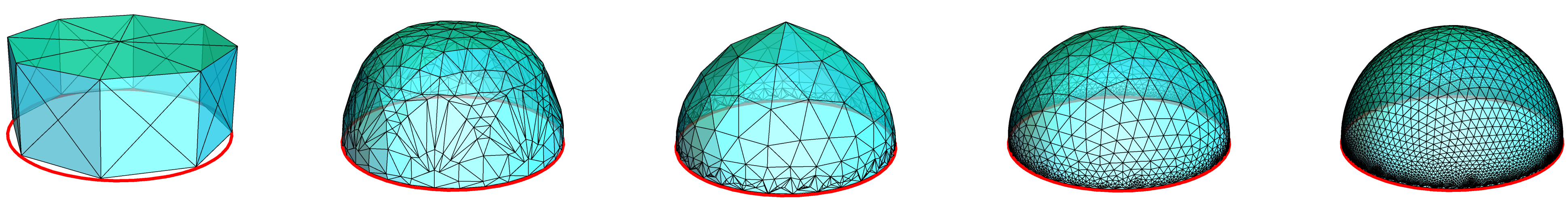
$$S_A = \frac{\text{Area}(\tilde{\gamma}_A)}{4G_N^{(d+2)}}$$

[Ryu, Takayanagi, (2006)]



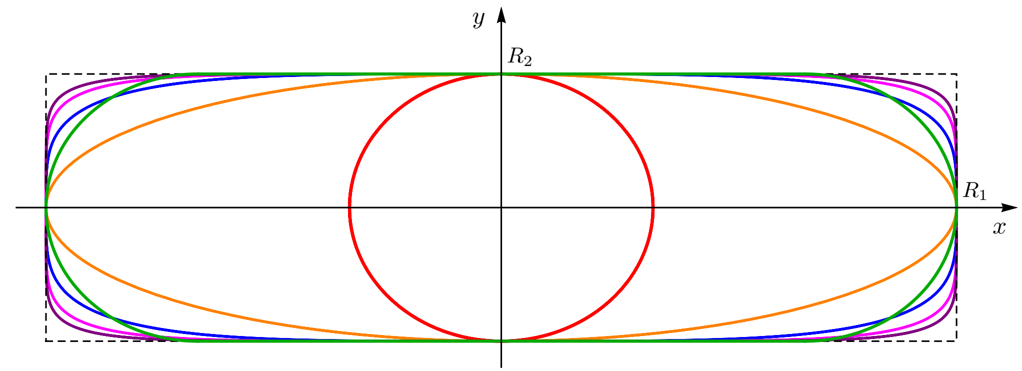



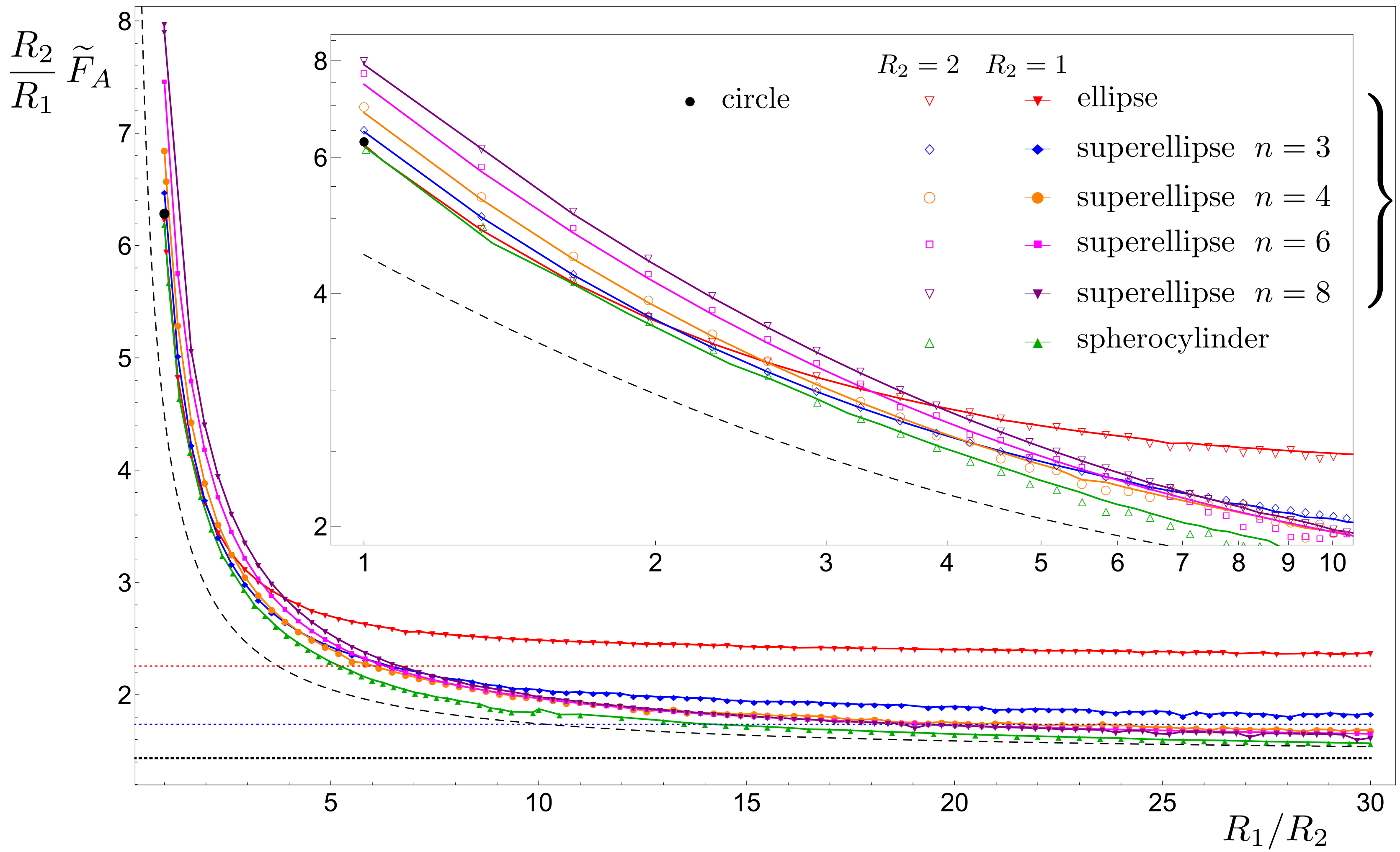

- Holographic dual of Wilson loops [Maldacena, (1998)] [Rey, Yee, (1998)]
- For arbitrary shapes of $\partial A$ and $AdS_4$ we employ a numerical method based on *Surface Evolver* (by Ken Brakke) [Fonda, Giomi, Salvio, E.T., (2014)]



# HEE in AdS(4). From the disk to the infinite strip

$$\mathcal{A}_A = \frac{P_A}{\varepsilon} - F_A + o(1) \equiv \frac{P_A}{\varepsilon} - \widetilde{F}_A$$

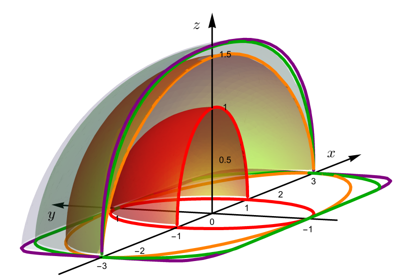



Superellipses:

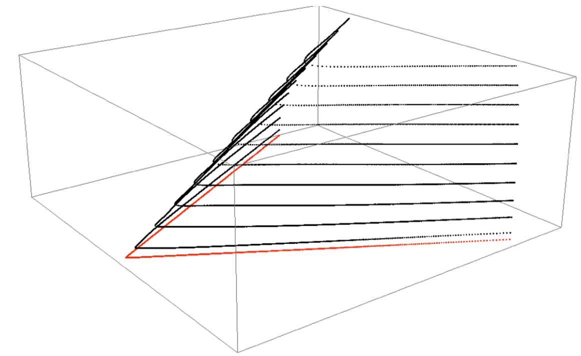
$$\frac{|x|^n}{R_1^n} + \frac{|y|^n}{R_2^n} = 1$$

squircles: $R_1 = R_2$



# HEE in AdS(4). Polygons (I)

- Infinite wedge with opening angle $\alpha$ ($|\phi| \leqslant \alpha/2$)
  [Drukker, Gross, Ooguri, (1999)] [Hirata, Takayanagi, (2006)]

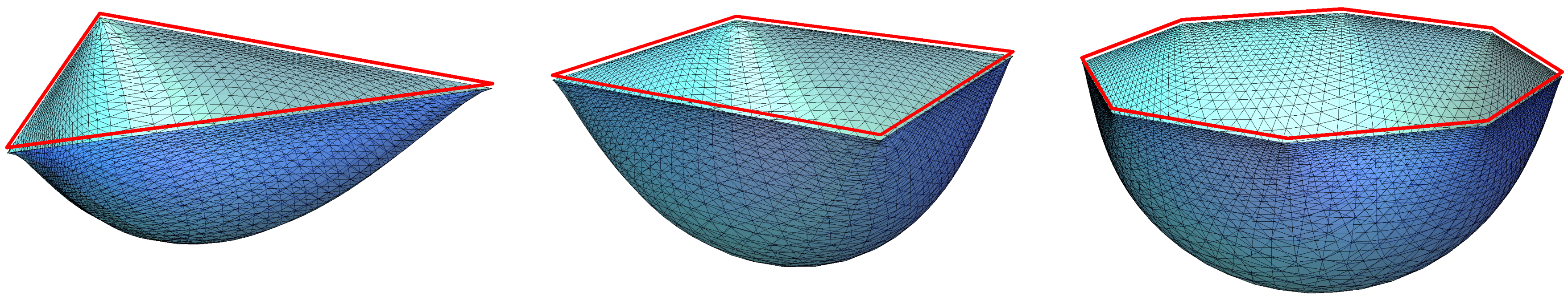


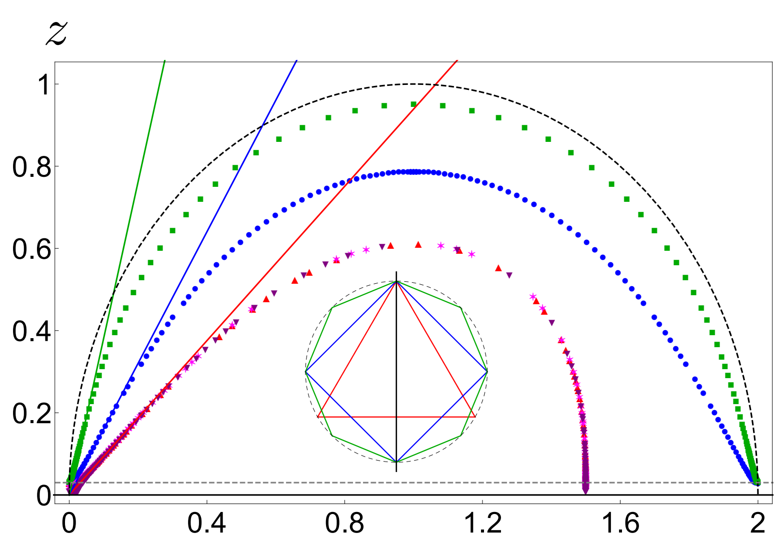
$$z = \frac{\rho}{f(\phi)}$$

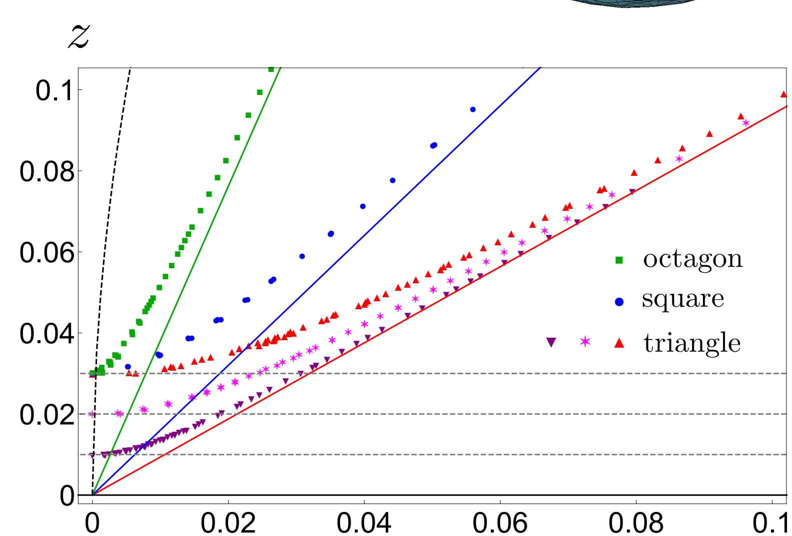
$$\phi = \int_{f_0}^{f} \frac{1}{\zeta} \left[ (\zeta^2+1) \left( \frac{\zeta^2(\zeta^2+1)}{f_0^2(f_0^2+1)} - 1 \right) \right]^{-\frac{1}{2}} d\zeta \qquad f_0 \equiv f(0)$$

$$f \to \infty \quad \text{then} \quad \phi \to \alpha/2$$

- Minimal surfaces anchored on finite polygons can be studied numerically [Fonda, Giomi, Salvio, E.T., (2014)]





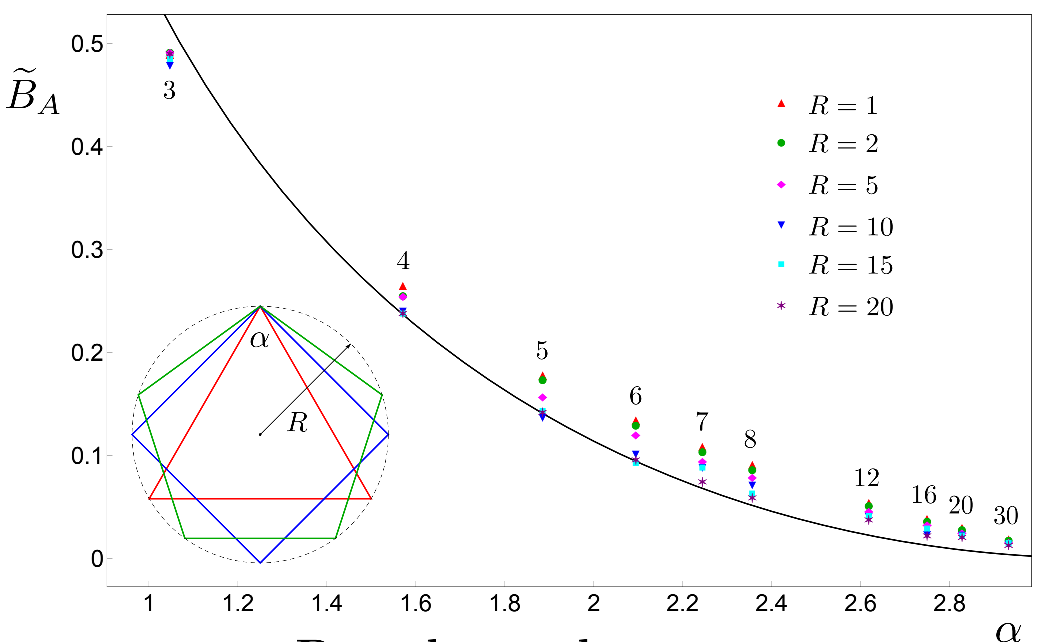
# HEE in AdS(4). Polygons (II)

- Area of the minimal surfaces anchored on polygons

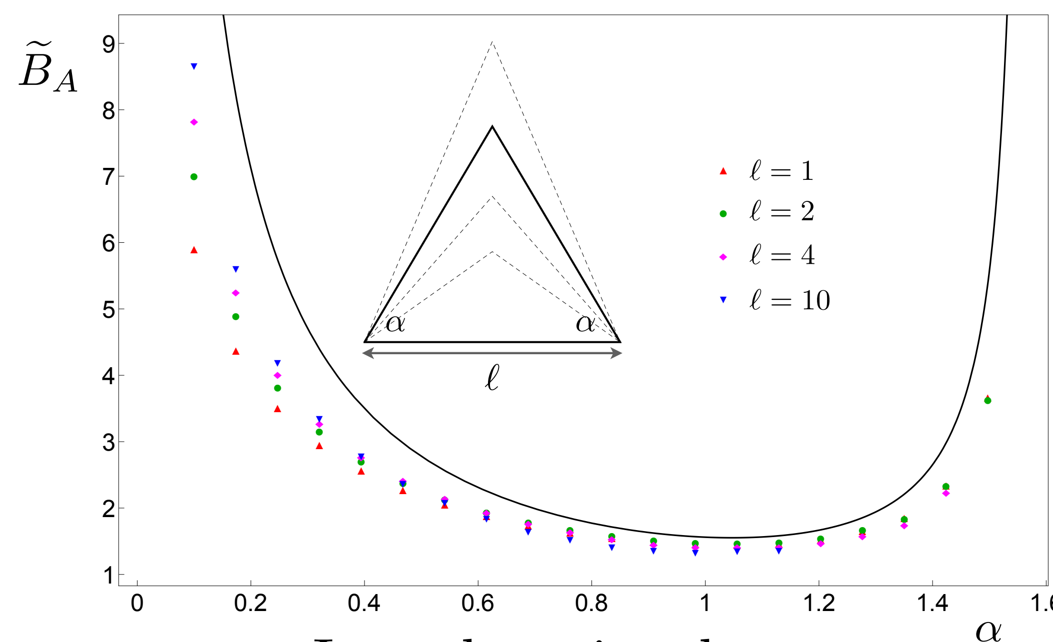
$$\mathcal{A}_A = \frac{P_A}{\varepsilon} - B_A \log(P_A/\varepsilon) - W_A + o(1) \equiv \frac{P_A}{\varepsilon} - \widetilde{B}_A \log(P_A/\varepsilon)$$

$$B_A \equiv 2\sum_{i=1}^{N} b(\alpha_i) \qquad b(\alpha) \equiv \int_0^\infty \left(1 - \sqrt{\frac{\zeta^2 + f_0^2 + 1}{\zeta^2 + 2f_0^2 + 1}}\right) d\zeta$$

- Numerical checks with *Surface Evolver*



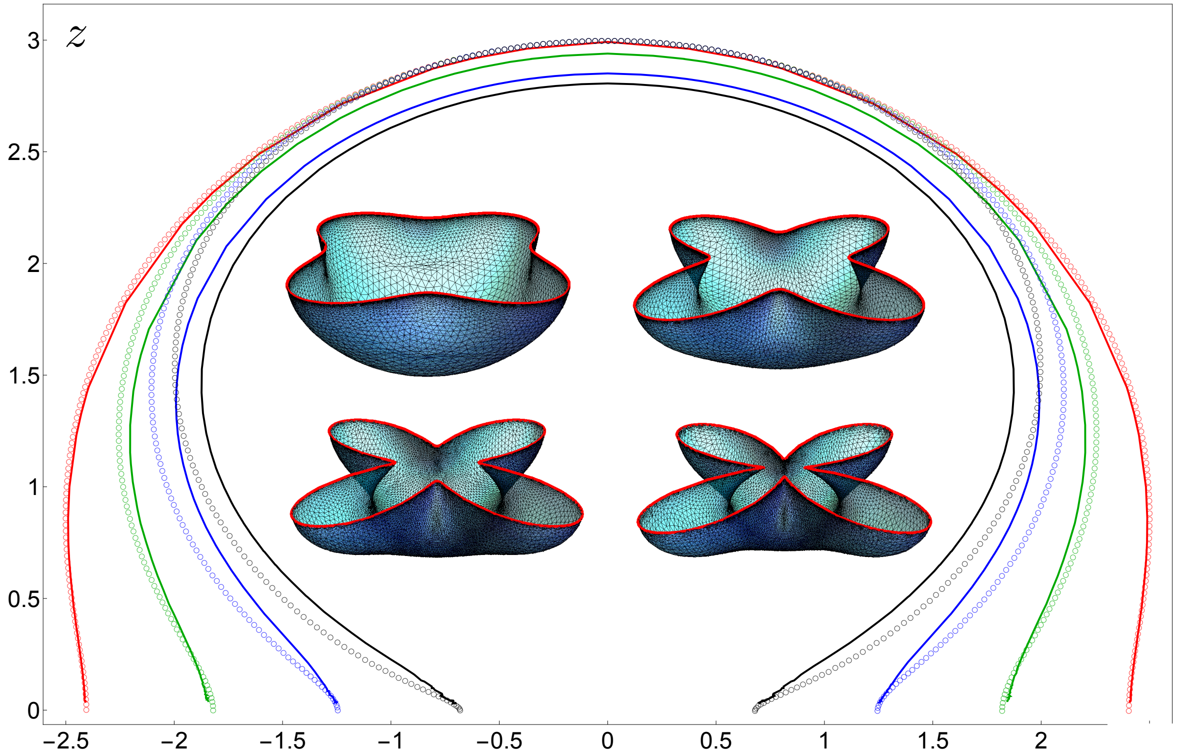

Regular polygons



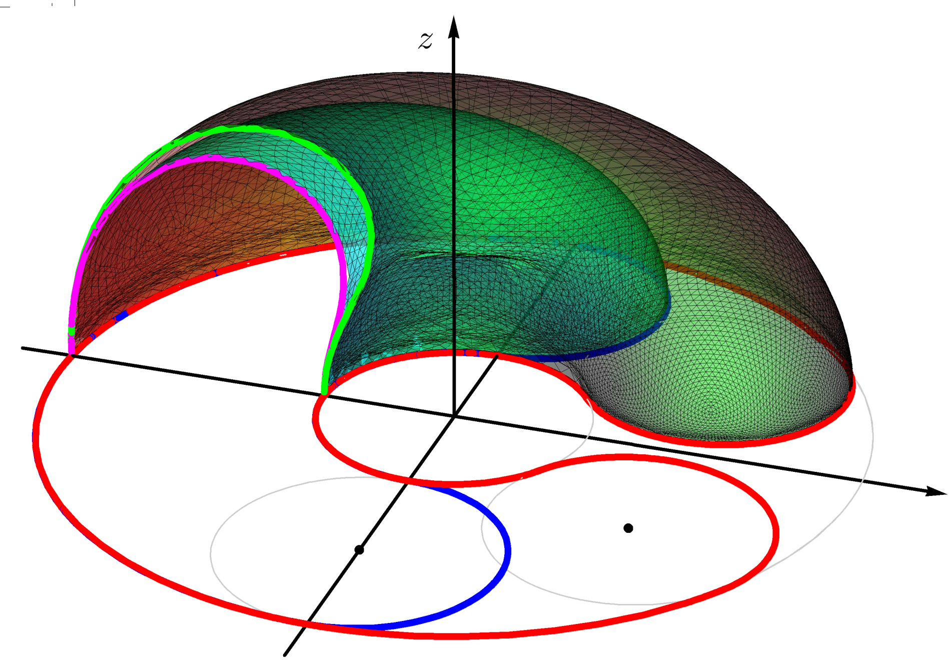

Isosceles triangles

- $W_A$ influenced by the regularization [Drukker, Gross, Ooguri, (1999)]

# HEE in AdS(4) for other simply connected regions



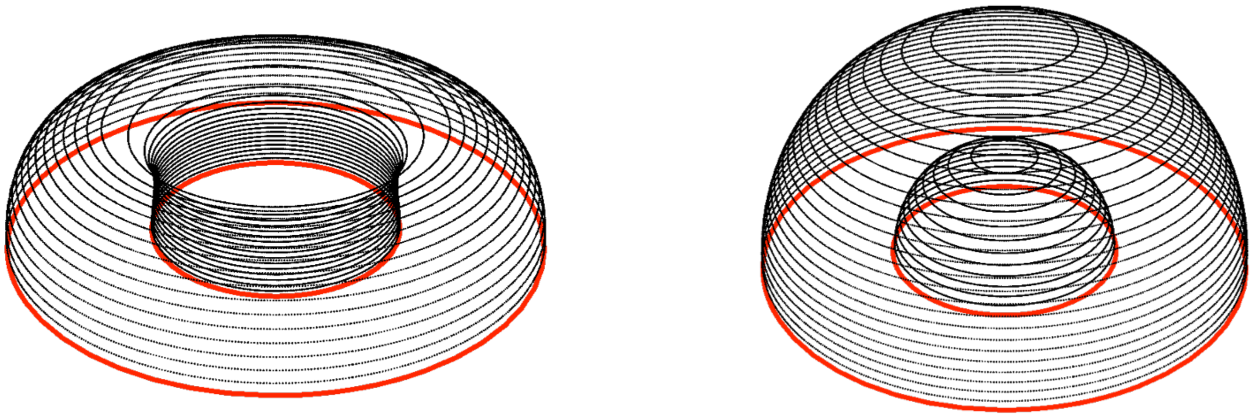

- Smooth perturbations around the hemisphere (star shaped regions)
  [Hubeny, (2012)]



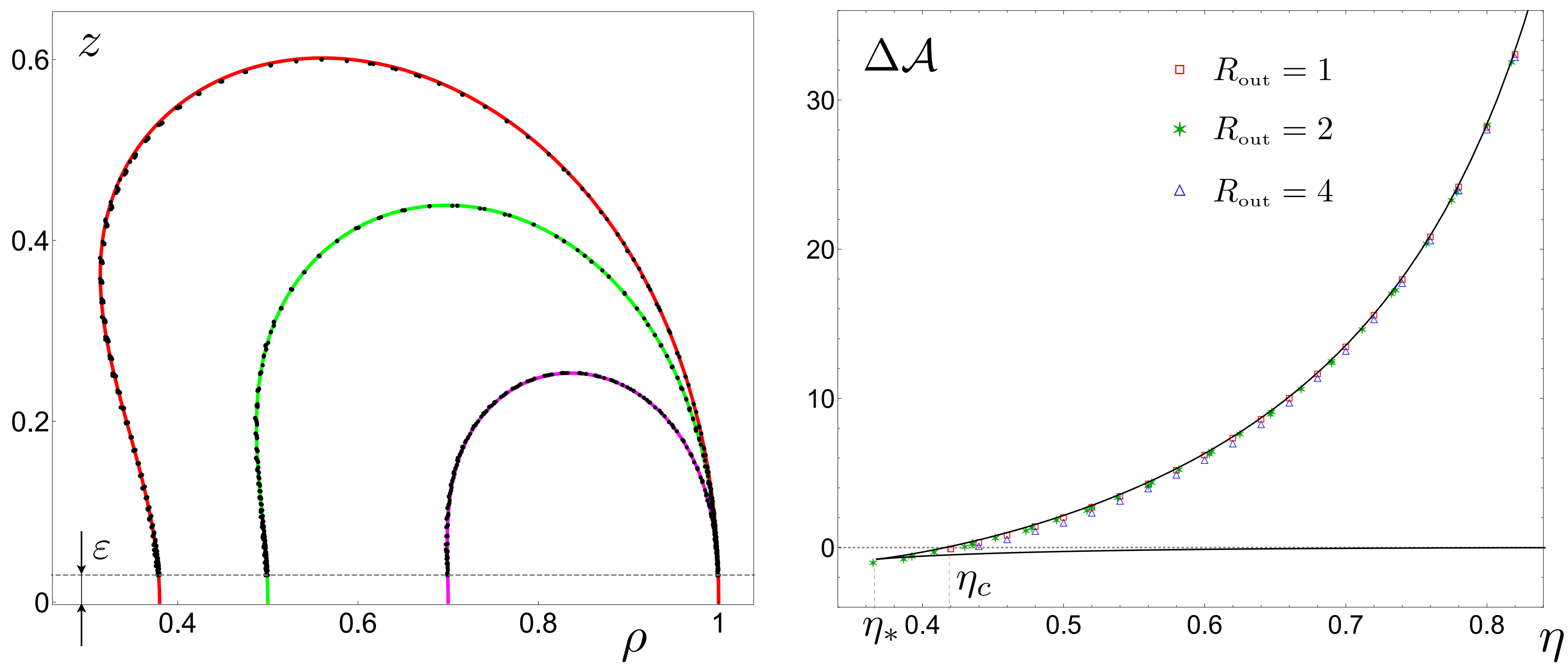

- *Surface Evolver* allows to consider surfaces which are difficult to parameterize
  [Fonda, Giomi, Salvio, E.T., (2014)]

# HEE in AdS(4). Annulus

- Competition between two configurations of minimal surfaces ($\eta \equiv R_{\rm in}/R_{\rm out}$)
  [Gross, Ooguri, (1998)] [Zarembo, (1999)] [Olesen, Zarembo, (2000)] [Drukker, Fiol, (2005)]

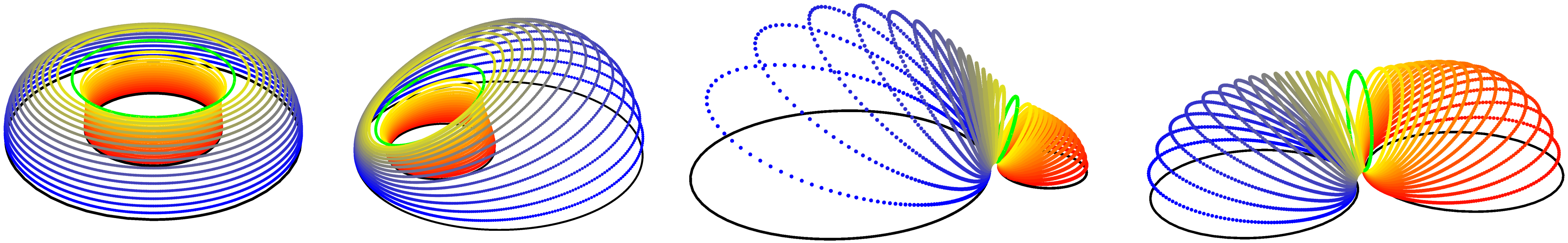


- Critical ratio $\eta_c$ when $\Delta\mathcal{A} \equiv \mathcal{A}_{\rm dis} - \mathcal{A}_{\rm con}$ vanishes

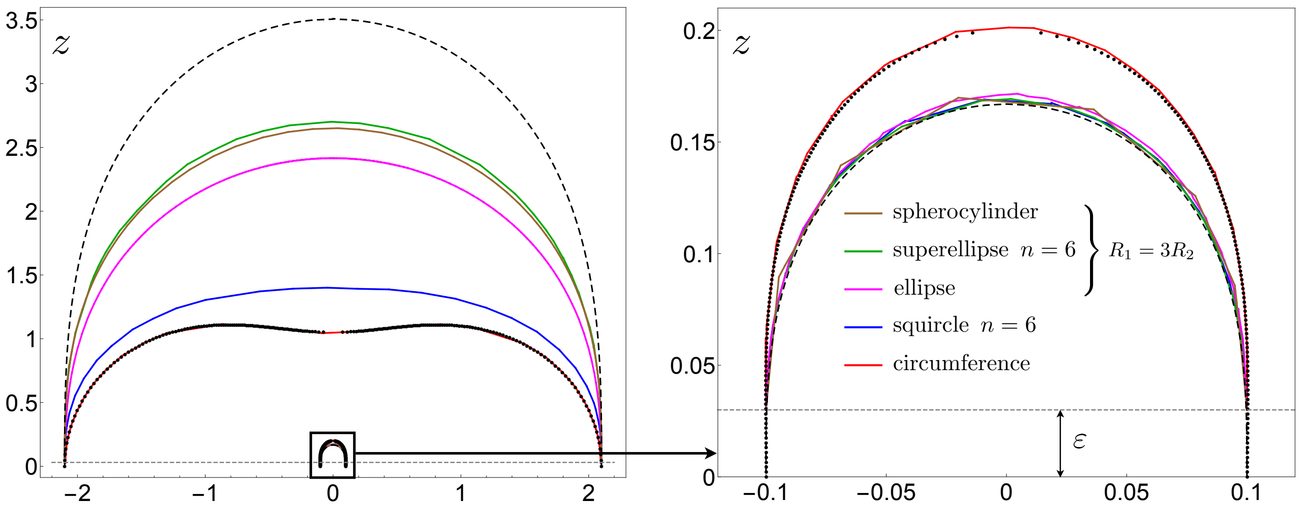


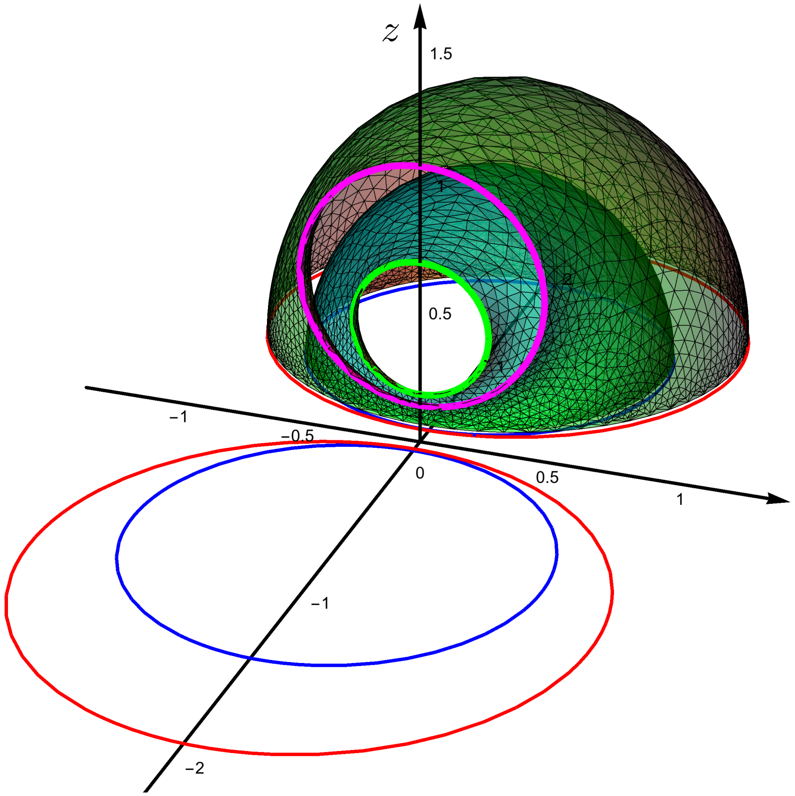
# Holographic mutual information in AdS(4). Disjoint disks

- Isometries of $\mathbb{H}_3$ can be employed to map the case of the annulus into the case of two disjoint disks



- Analytic expressions can be checked with *Surface Evolver*



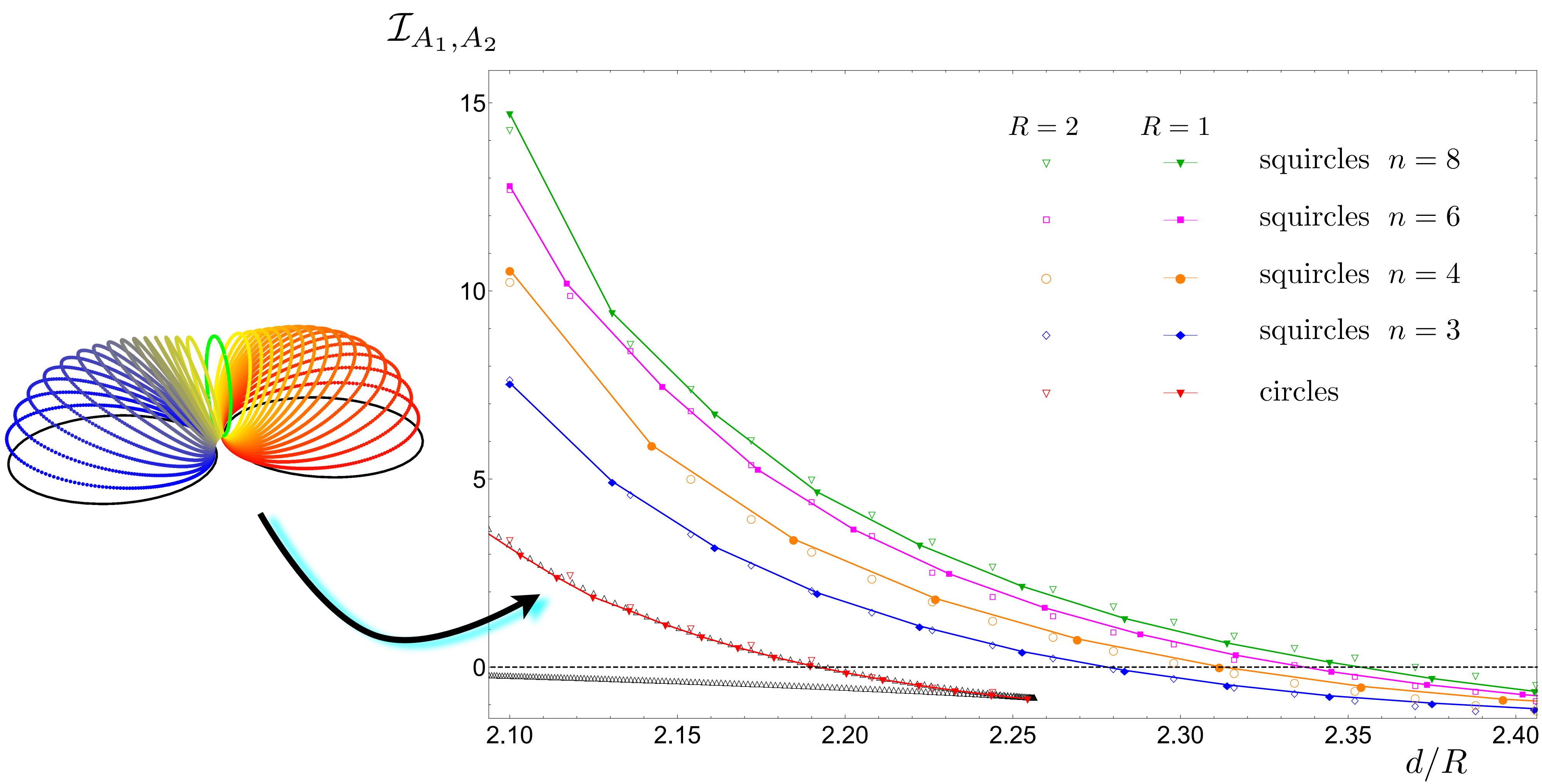



# Holographic mutual information in AdS(4). Squircles

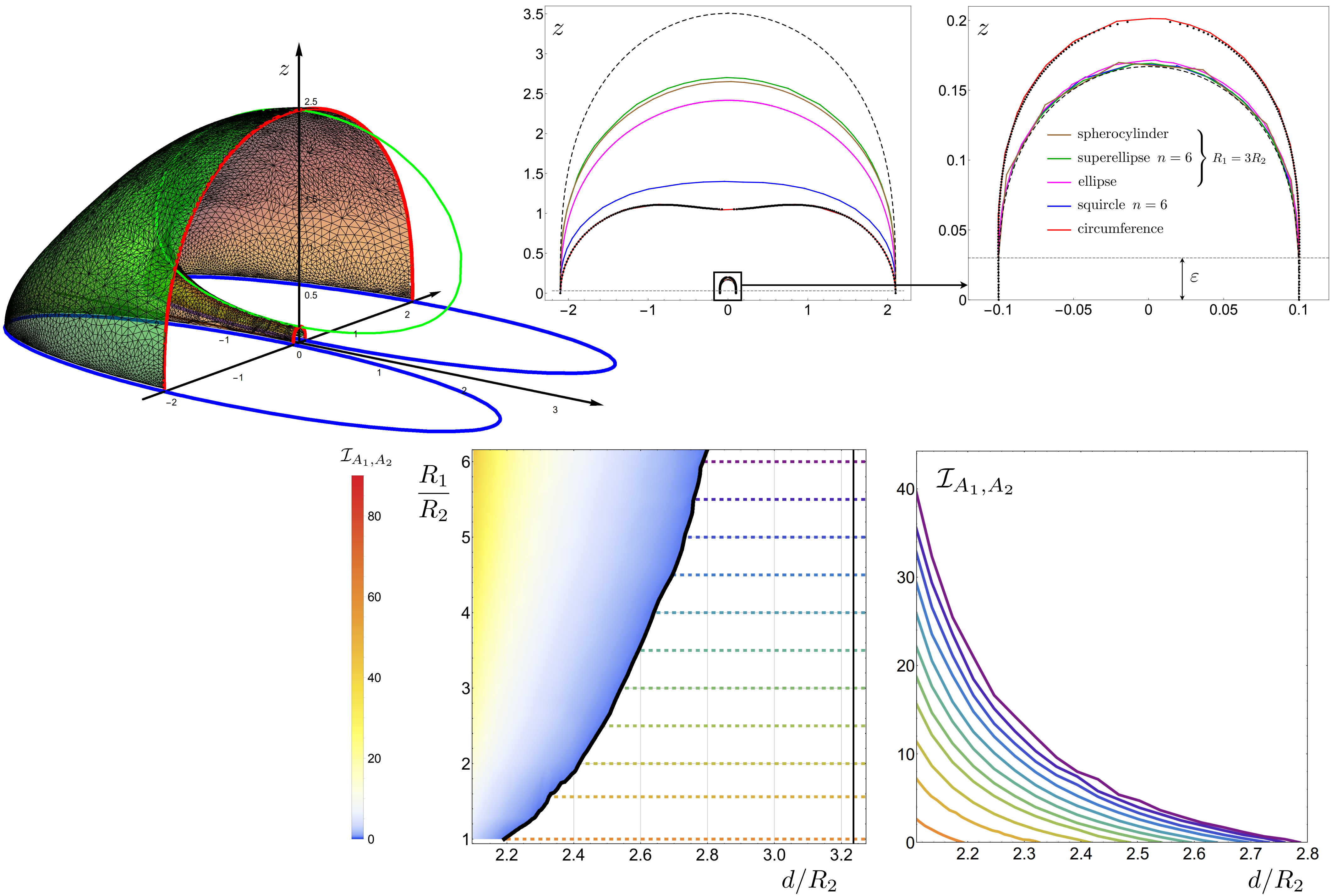
$$I_{A_1,A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1 \cup A_2} \equiv \frac{\mathcal{I}_{A_1,A_2}}{4G_N}$$

$$\mathcal{I}_{A_1,A_2} = F_{A_1 \cup A_2} - F_{A_1} - F_{A_2} + o(1)$$

- Beyond a critical distance $\mathcal{I}_{A_1,A_2} = 0$ and the disconnected configuration is the minimal one

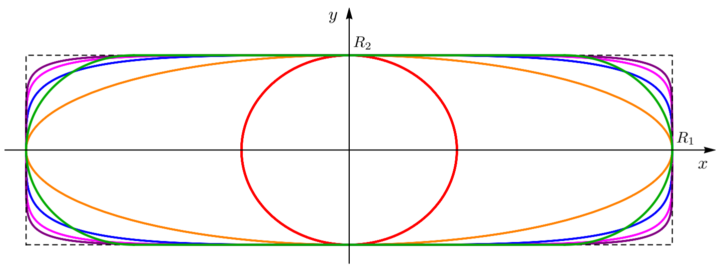


# Holographic mutual information in AdS(4). Disjoint ellipses

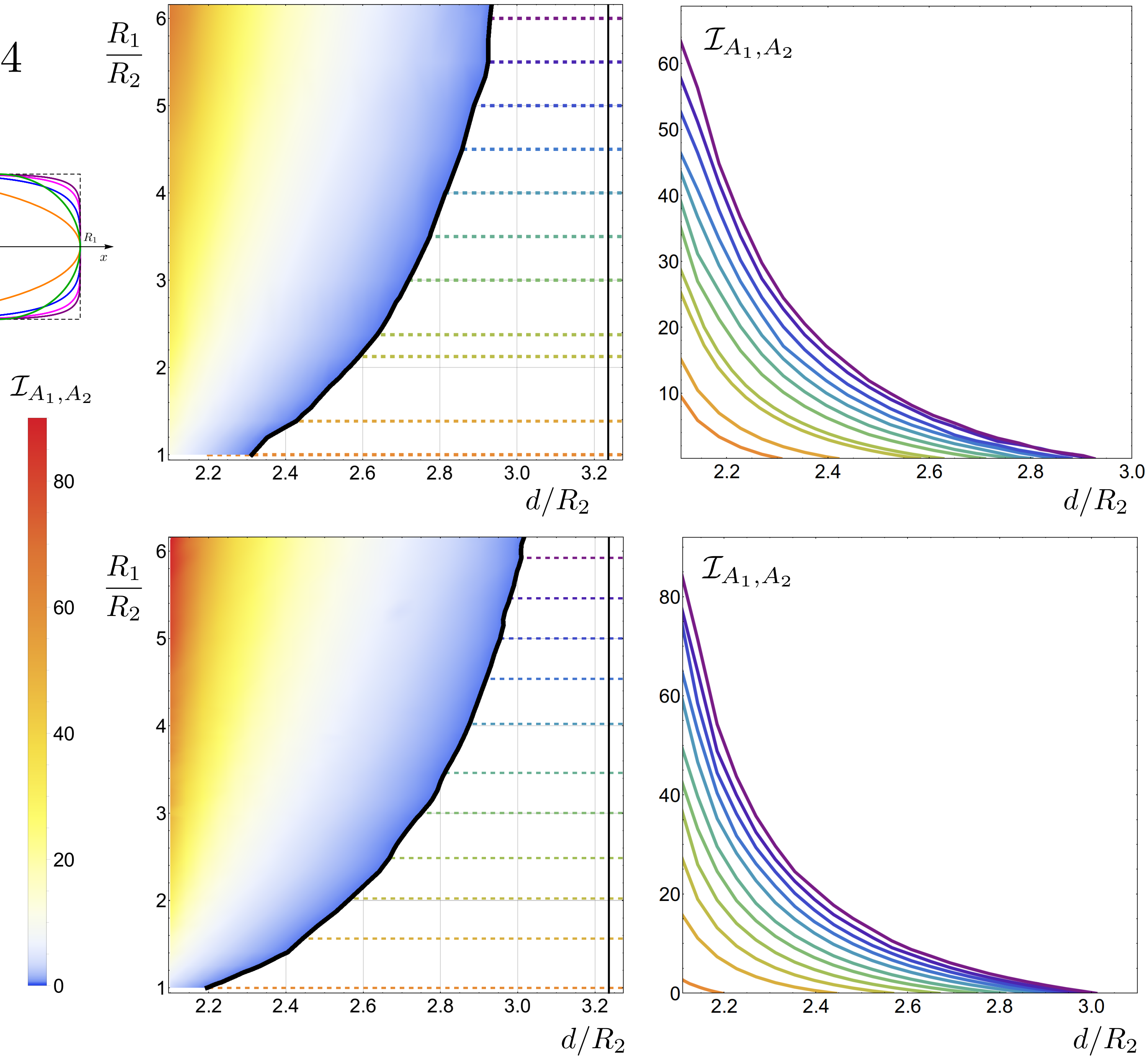


# *Holographic mutual information in AdS(4). Other shapes*

- Superellipses $n = 4$



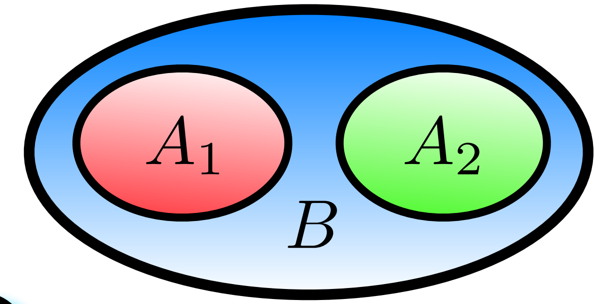

- Two dimensional spherocylinder



# *Entanglement between disjoint regions: Negativity*

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state

  $\rho^{T_2}$ is the *partial transpose* of $\rho$




$$\langle e_i^{(1)} e_j^{(2)} | \, \rho^{T_2} \, | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \, \rho \, | e_k^{(1)} e_j^{(2)} \rangle$$

$(|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k})$

[Peres, (1996)] [Zyczkowski, Horodecki, Sanpera, Lewenstein, (1998)] [Eisert, (2001)] [Vidal, Werner, (2002)]

- *Trace norm*

$$||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i$$

$\lambda_j$ eigenvalues of $\rho^{T_2}$

$\mathrm{Tr}\, \rho^{T_2} = 1$

*Logarithmic negativity*

$$\mathcal{E}_{A_2} = \ln ||\rho^{T_2}|| = \ln \mathrm{Tr}|\rho^{T_2}|$$

$\mathcal{E}$ measures "how much" the eigenvalues of $\rho^{T_2}$ are negative

- *Bipartite* system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in *any* state $\rho$ $\longrightarrow$ $\mathcal{E}_1 = \mathcal{E}_2$

# *Replica approach to Negativity*

[Calabrese, Cardy, E.T., (2012)]

- A *parity effect* for $\mathrm{Tr}(\rho^{T_2})^n$

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

- Analytic continuation on the *even* sequence $\mathrm{Tr}(\rho^{T_2})^{n_e}$ (*make 1 an even number*)

$$\mathcal{E} = \lim_{n_e\to 1} \log\left[\mathrm{Tr}(\rho^{T_2})^{n_e}\right] \qquad \lim_{n_o\to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\,\rho^{T_2} = 1$$

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- Pure states $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $(\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2)$

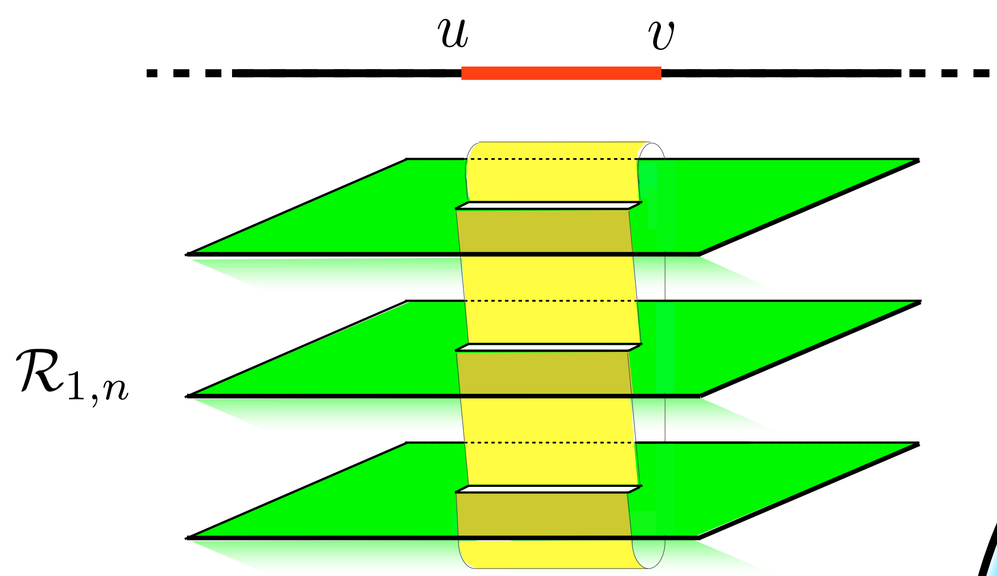
*Schmidt decomposition*

$$\mathrm{Tr}(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \quad \text{odd} \\ \left(\mathrm{Tr}\,\rho_2^{n/2}\right)^2 & n = n_e \quad \text{even} \end{cases}$$

- Taking $n_e \to 1$ we have $\mathcal{E} = 2\log \mathrm{Tr}\rho_2^{1/2}$ (Renyi entropy $1/2$)

# 2D CFT: Renyi entropies as correlation functions

- One interval ($N = 1$): the Renyi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]




[Holzhey, Larsen, Wilczek, (1994)]

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + c_1'$$

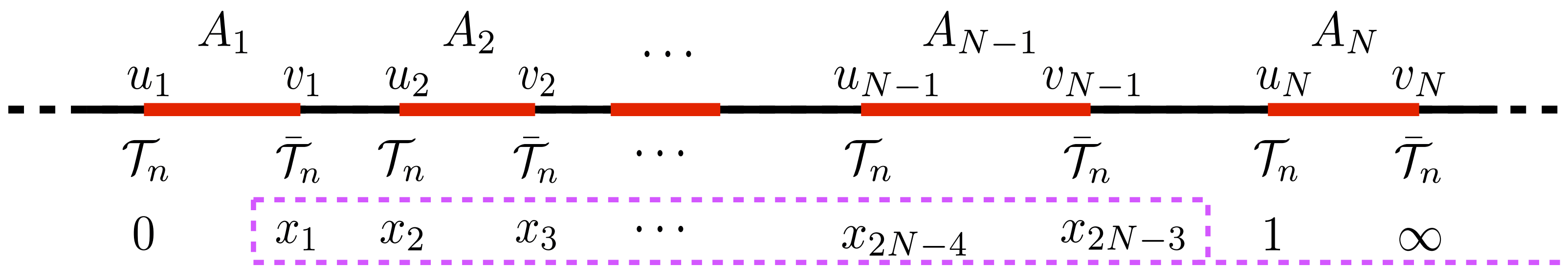
$$\text{Tr}\rho_A^n = \frac{\mathcal{Z}_{1,n}}{\mathcal{Z}^n} = \langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \rangle = \frac{c_n}{|u-v|^{2\Delta_n}}$$

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Twist fields have been largely studied in the 1980s [Zamolodchikov, (1987)] [Dixon, Friedan, Martinec, Shenker, (1987)] [Knizhnik, (1987)] [Bershadsky, Radul, (1987)]
- Integrable field theories [Cardy, Castro-Alvaredo, Doyon, (2008)] [Doyon, (2008)]

# *2D CFT: Renyi entropies for many disjoint intervals*

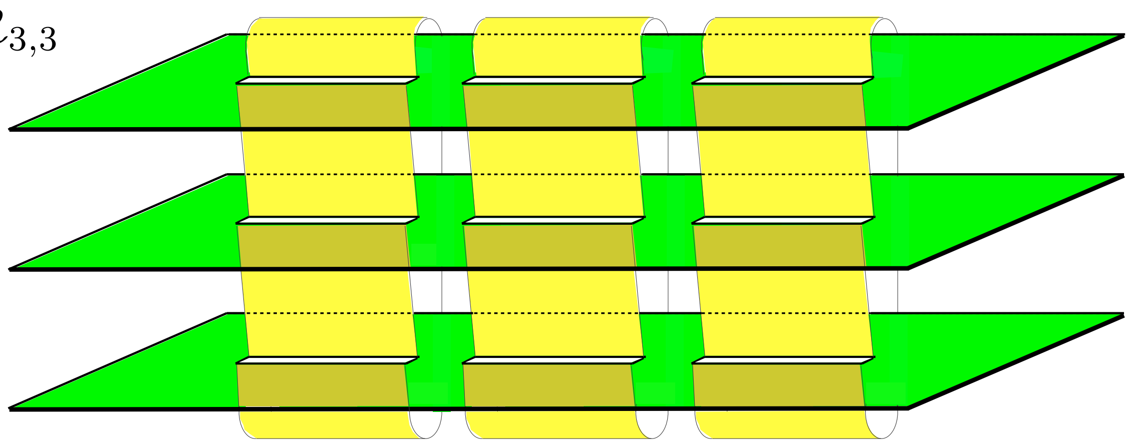
- $N$ disjoint intervals $\Longrightarrow$ $2N$ point function of twist fields



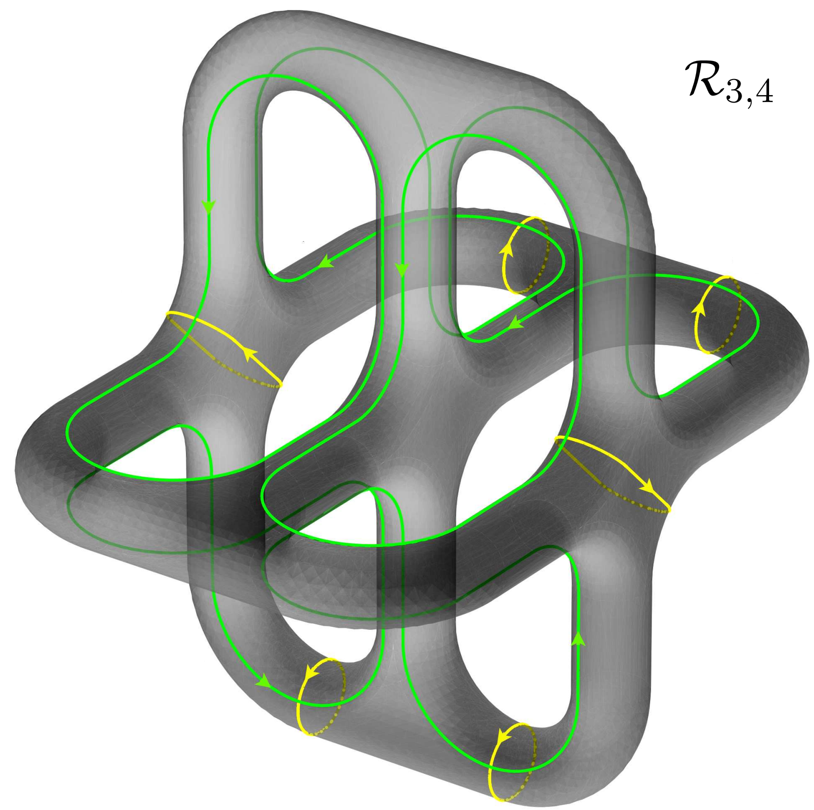

$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n} = \langle \prod_{i=1}^{N} \mathcal{T}_n(u_i)\bar{\mathcal{T}}_n(v_i)\rangle = c_n^N \left| \frac{\prod_{i<j}(u_j-u_i)(v_j-v_i)}{\prod_{i,j}(v_j-u_i)} \right|^{2\Delta_n} \mathcal{F}_{N,n}(\boldsymbol{x})$$

- $\mathcal{Z}_{N,n}$ partition function of $\mathcal{R}_{N,n}$, a particular Riemann surface of genus $g=(N-1)(n-1)$ obtained through replication

$\mathcal{R}_{3,3}$



$\mathcal{R}_{3,4}$



# N intervals: free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $$y^n = \prod_{\gamma=1}^{N} (z - x_{2\gamma-2}) \left[ \prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1}$$ $g = (N-1)(n-1)$ [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

  Riemann theta function with characteristic $$\Theta[\boldsymbol{e}](\boldsymbol{0}|\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^p} \exp\left[ \mathrm{i}\pi(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi\mathrm{i}(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta} \right]$$

- Free compactified boson ($\eta \propto R^2$) [Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}_{N,n}(\boldsymbol{x}) = \frac{\Theta(\boldsymbol{0}|T_\eta)}{|\Theta(\boldsymbol{0}|\tau)|^2} \qquad T_\eta = \begin{pmatrix} \mathrm{i}\,\eta\,\mathcal{I} & \mathcal{R} \\ \mathcal{R} & \mathrm{i}\,\mathcal{I}/\eta \end{pmatrix} \qquad \tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$$

period matrix

- Ising model $$\mathcal{F}^{\text{Ising}}_{N,n}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\boldsymbol{0}|\tau)|}{2^g\, |\Theta(\boldsymbol{0}|\tau)|}$$

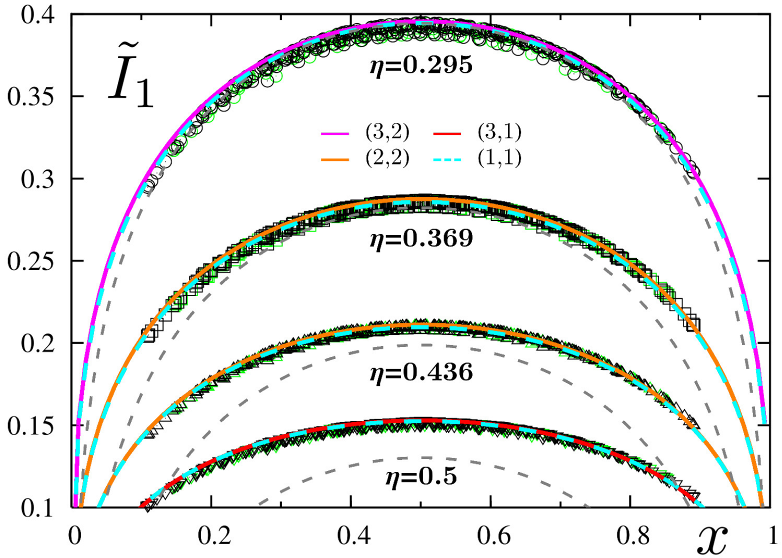
Nasty $n$ dependence

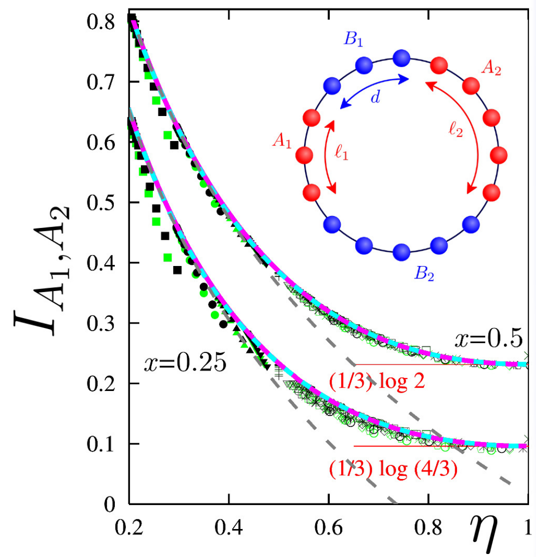
- Two intervals case: [Caraglio, Gliozzi, (2008)] [Furukawa, Pasquier, Shiraishi, (2009)] [Calabrese, Cardy, E.T., (2009), (2011)] [Fagotti, Calabrese, (2010)] [Alba, Tagliacozzo, Calabrese, (2010), (2011)]

# *Two disjoint intervals: numerical extrapolations*

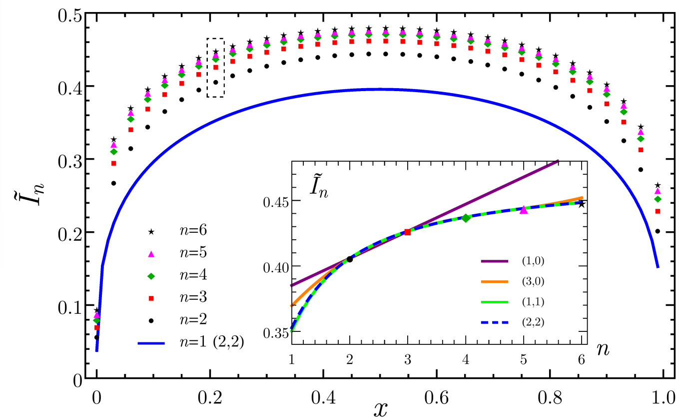
## Mutual information in XXZ model

(exact diagonalization) [Furukawa, Pasquier, Shiraishi, (2009)]



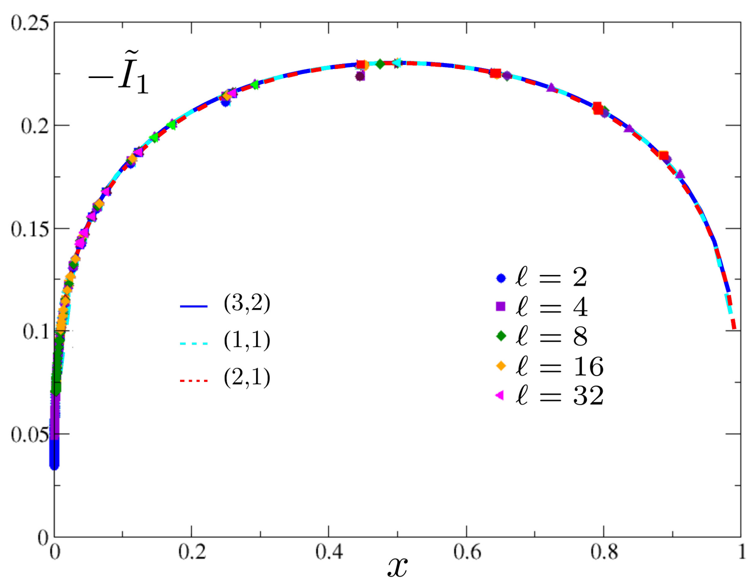




Rational interpolation: an example




## Mutual information in critical Ising chain

(Tree Tensor Network) [Alba, Tagliacozzo, Calabrese, (2010)]




## Rational interpolation:

[De Nobili, Coser, E.T., (2015)]

$$W^{(n)}_{(p,q)}(x) \equiv \frac{a_0(x) + a_1(x)n + \cdots + a_p(x)n^p}{b_0(x) + b_1(x)n + \cdots + b_q(x)n^q}$$

Method first employed in $2+1$ dimensions

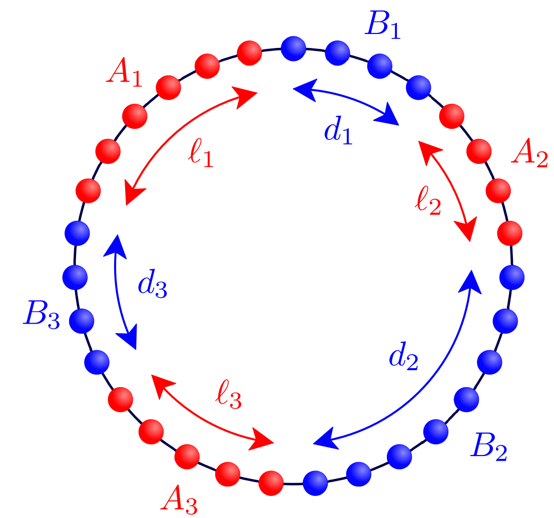
[Agón, Headrick, Jafferis, Kasko, (2014)]

# Periodic harmonic chain: three disjoint blocks

- Harmonic chain on a circle (critical for $\omega = 0$)

$$H = \frac{1}{2} \sum_{j=1}^{L} \left[ p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2 \right]$$

[Peschel, Chung, (1999)] [Botero, Reznik, (2004)]
[Audenaert, Eisert, Plenio, Werner,(2002)]



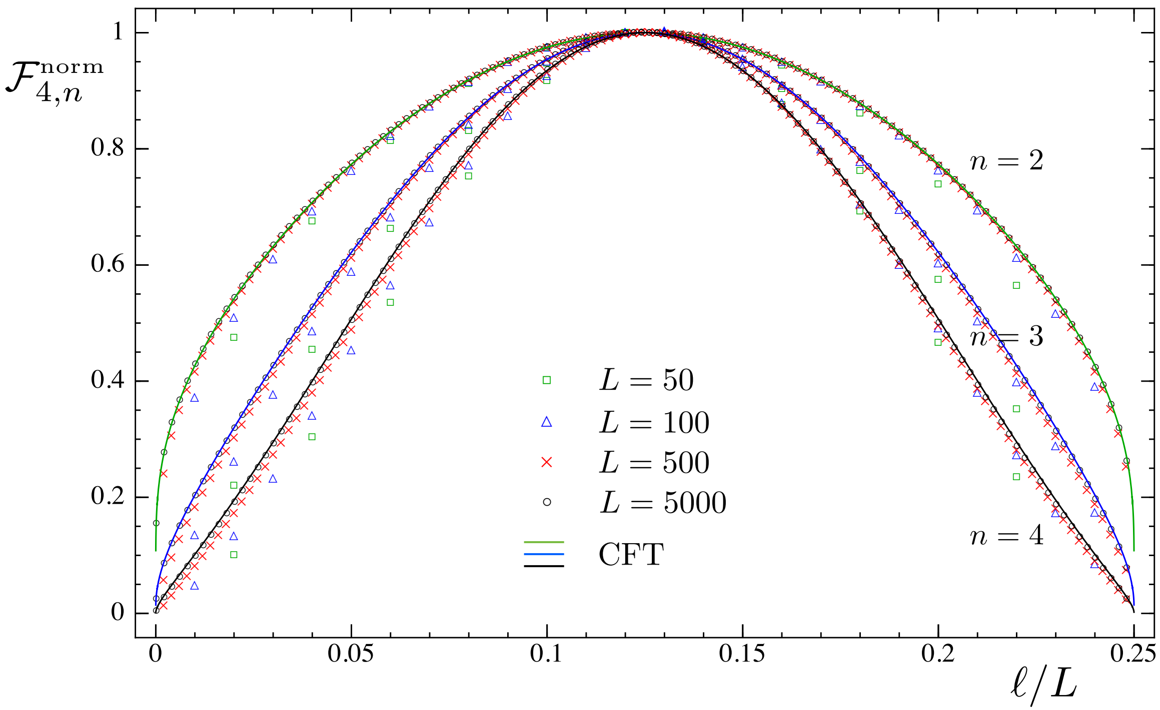

- Decompactification regime

[Dijkgraaf, Verlinde, Verlinde, (1988)] [. . . ]
[Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}^{\text{dec}}_{N,n}(\boldsymbol{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})}\, |\Theta(\mathbf{0}|\tau)|^2}$$

- period matrix $\tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$
  [Enolski, Grava, (2003)]
- Riemann theta function $\Theta$

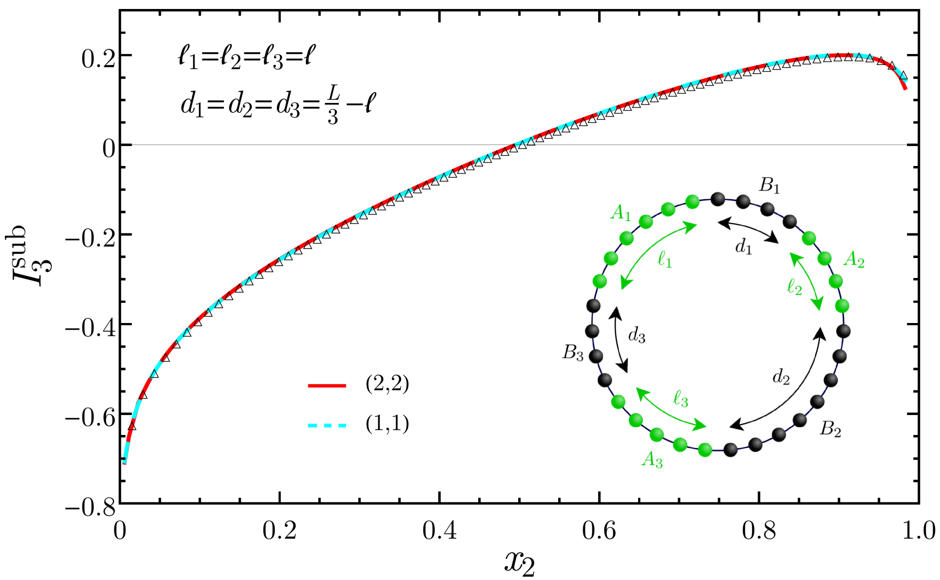
→ Nasty $n$ dependence

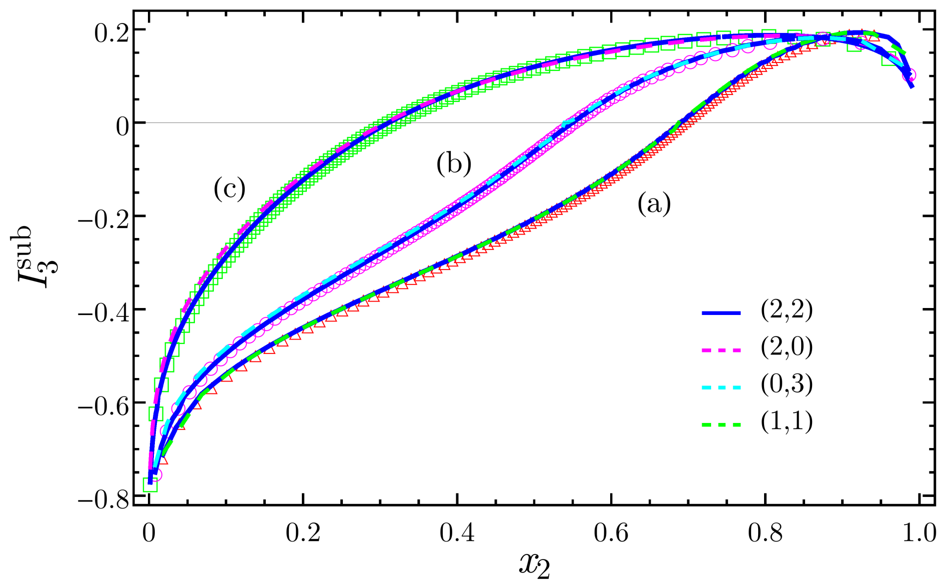



- Numerical checks for the Ising model through Matrix Product States

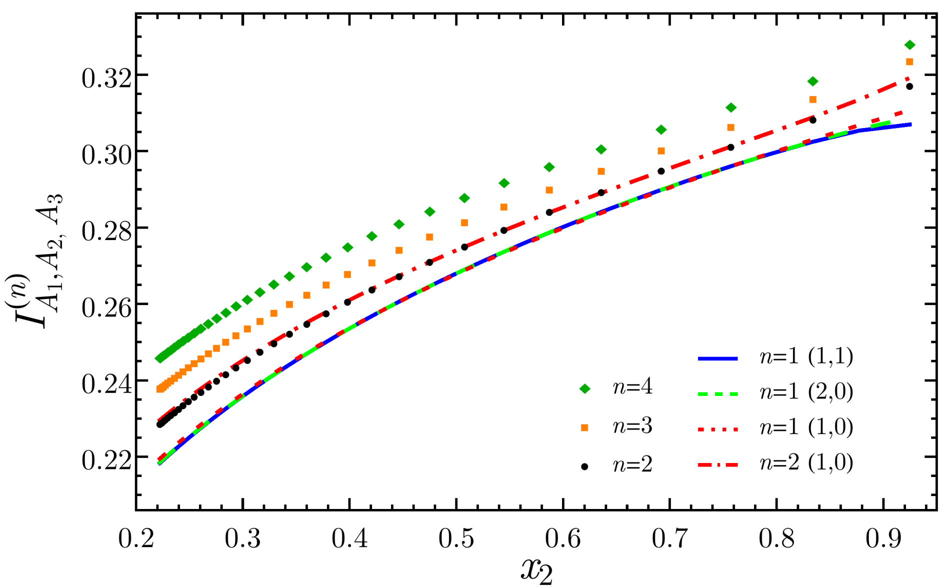
# Three disjoint intervals: Numerical extrapolations

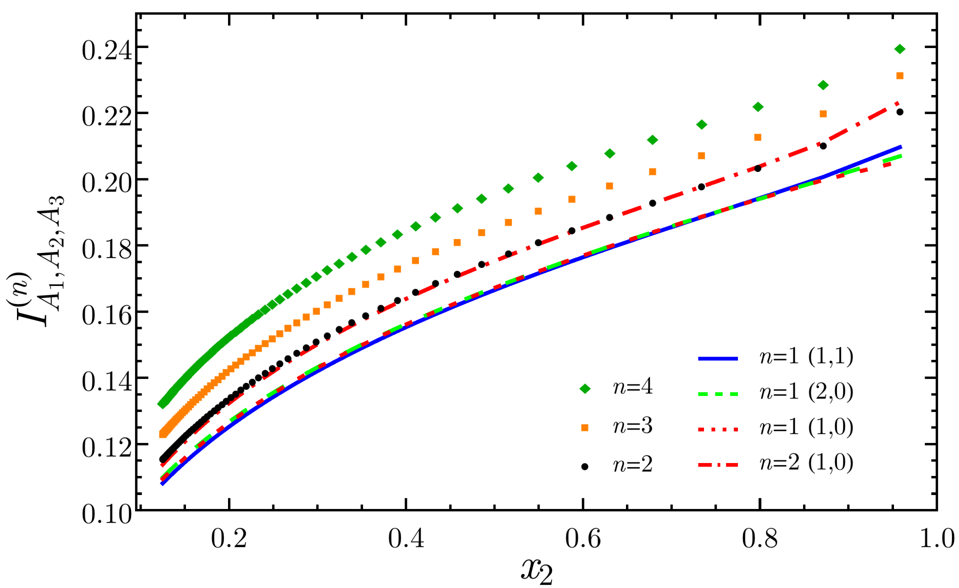
## Non compact boson (data from the periodic harmonic chain)



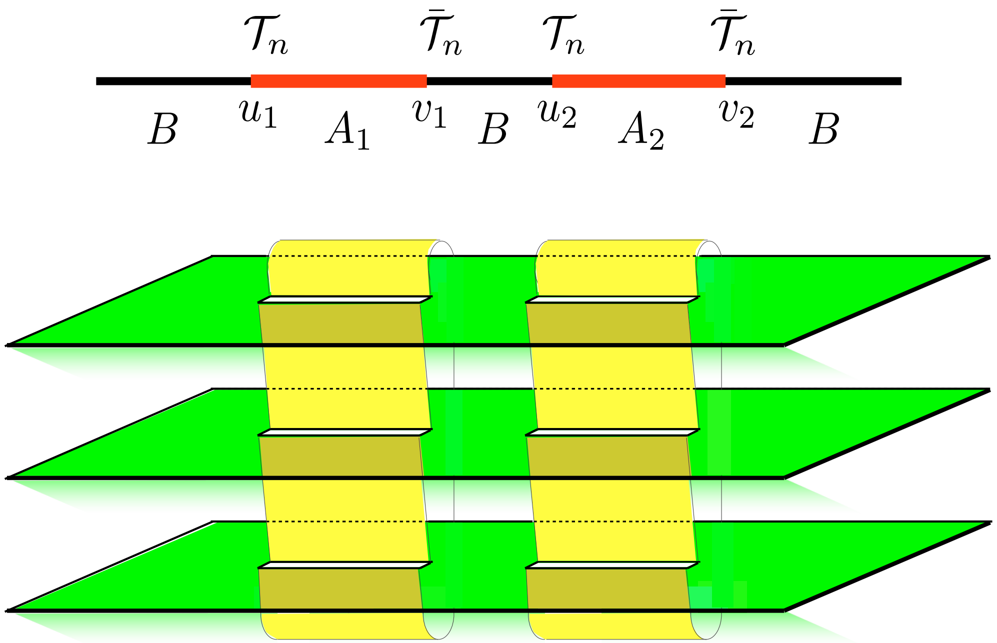

## Ising model



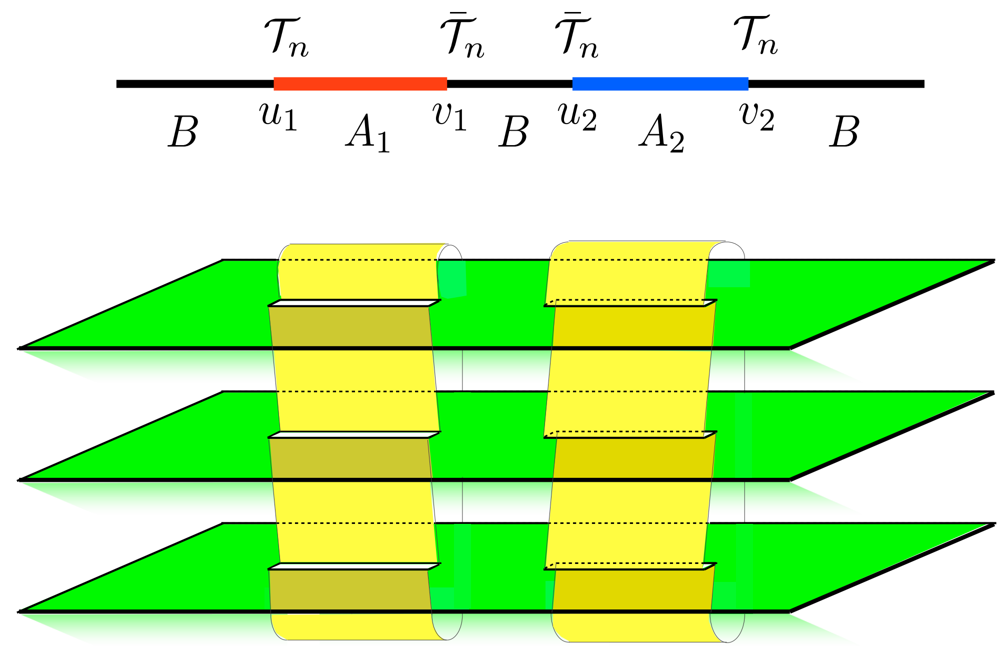
# Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$




$$\mathrm{Tr}\rho^n_A = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$$



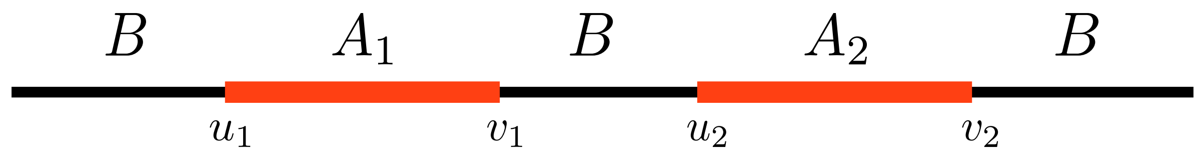

$$\mathrm{Tr}(\rho^{T_2}_A)^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

[Calabrese, Cardy, E.T., (2012)]

# Partial Transposition for bipartite systems: pure states

$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$

- $\lim_{B \to \emptyset}$ (  )



$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2)\bar{\mathcal{T}}_n^2(v_2)\rangle$$

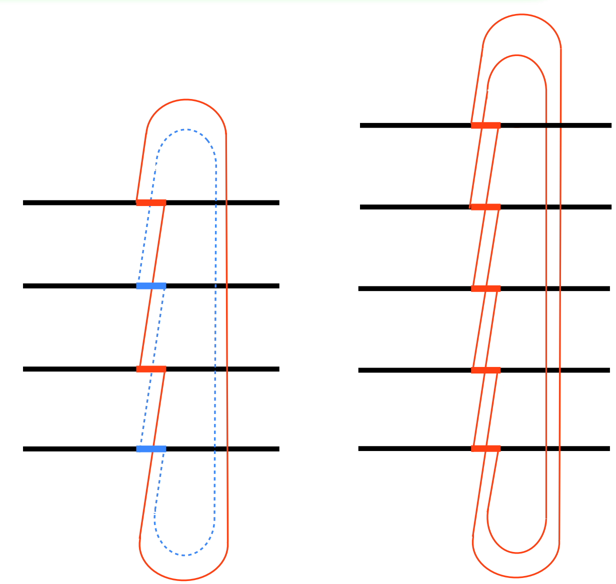
Partial Transposition = exchange two twist fields

- $\mathcal{T}_n^2$ connects the $j$-th sheet with the $(j+2)$-th one

Even $n = n_e \implies$ decoupling

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \left(\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle\right)^2 = \left(\mathrm{Tr}\,\rho_{A_2}^{n_e/2}\right)^2$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\,\rho_{A_2}^{n_o}$$



$n = 4$ $\quad$ $n = 5$

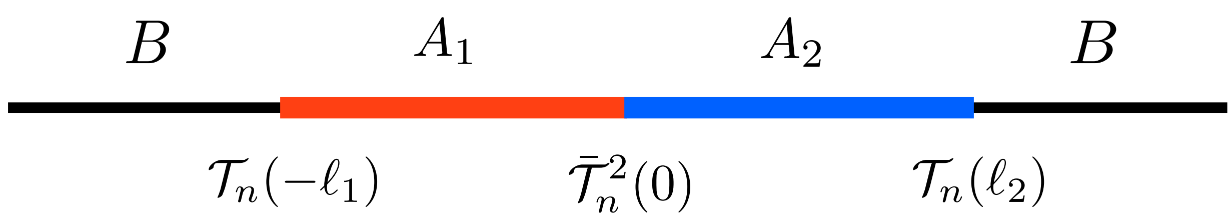
- Two dimensional CFTs

$$\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right) = \Delta_{\mathcal{T}_{n_o}}$$

$$\Delta_{\mathcal{T}_{n_e}^2} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)$$

$$\mathcal{E} = \frac{c}{2}\ln \ell + \text{const}$$

# *Partial Transpose in 2D CFT: two adjacent intervals*




- Three point function

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} \propto (\ell_1\ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2}-\frac{2}{n_e}\right)}(\ell_1+\ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2}+\frac{1}{n_e}\right)}$$

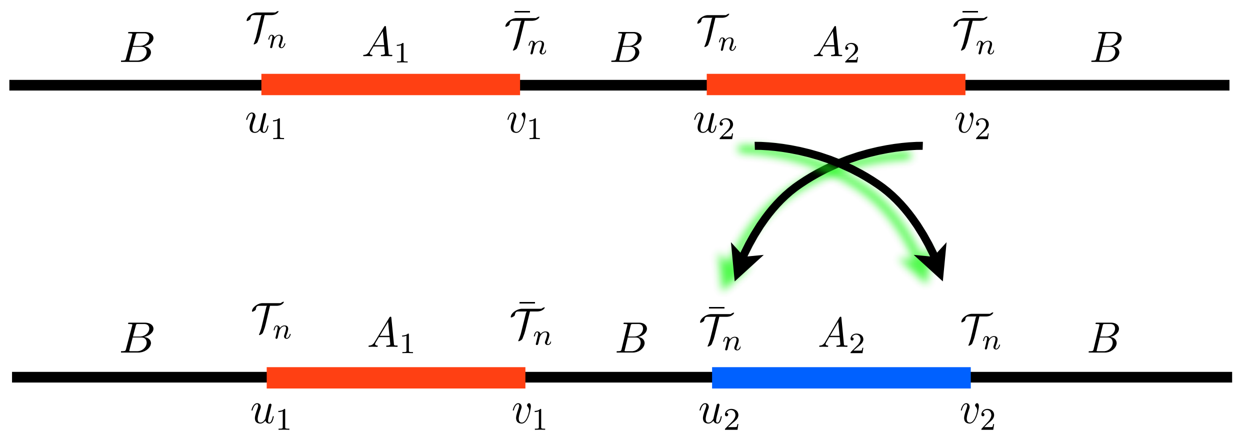
$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} \propto \left(\ell_1\ell_2(\ell_1+\ell_2)\right)^{-\frac{c}{12}\left(n_o-\frac{1}{n_o}\right)}$$

- $$\mathcal{E} = \frac{c}{4}\ln\left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right) + \mathrm{const}$$

# *Partial Transpose in 2D CFT: two disjoint intervals*

$\mathrm{Tr}\rho^n_{A_1\cup A_2}$

$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$




$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n = c_n^2\left[\ell_1\ell_2(1-y)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\mathcal{G}_n(y)$$

- $\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$ is obtained from $\mathrm{Tr}\rho^n_{A_1\cup A_2}$ by exchanging two twist fields

$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}\left(n-\frac{1}{n}\right)}\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

- $$\mathcal{E}(y) = \lim_{n_e\to 1}\mathcal{G}_{n_e}(y) = \lim_{n_e\to 1}\left[\mathcal{F}_n\left(\frac{y}{y-1}\right)\right]$$

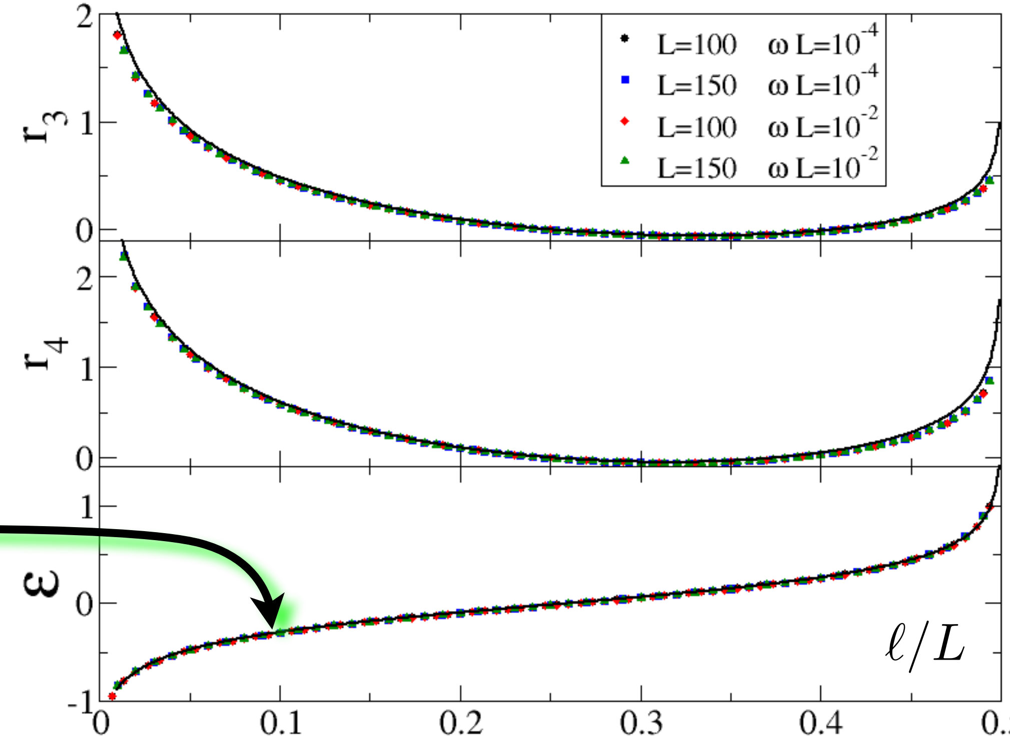
# *Two adjacent intervals: harmonic chain & Ising model*

- Critical periodic harmonic chain

Finite system: $\ell \longrightarrow (L/\pi)\sin(\pi\ell/L)$

$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

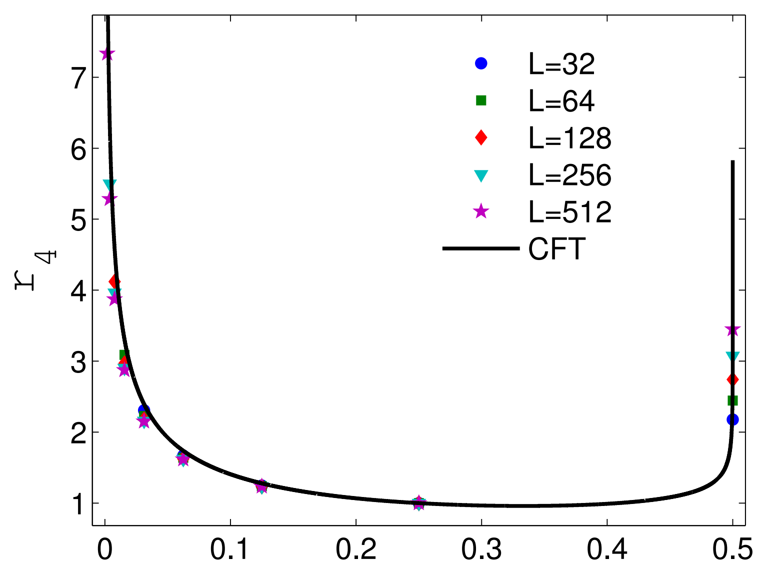
$$\frac{1}{4}\log\frac{\sin(\pi\ell_1/L)\sin(\pi\ell_2/L)}{\sin(\pi[\ell_1+\ell_2]/L)} + \mathrm{cnst}$$

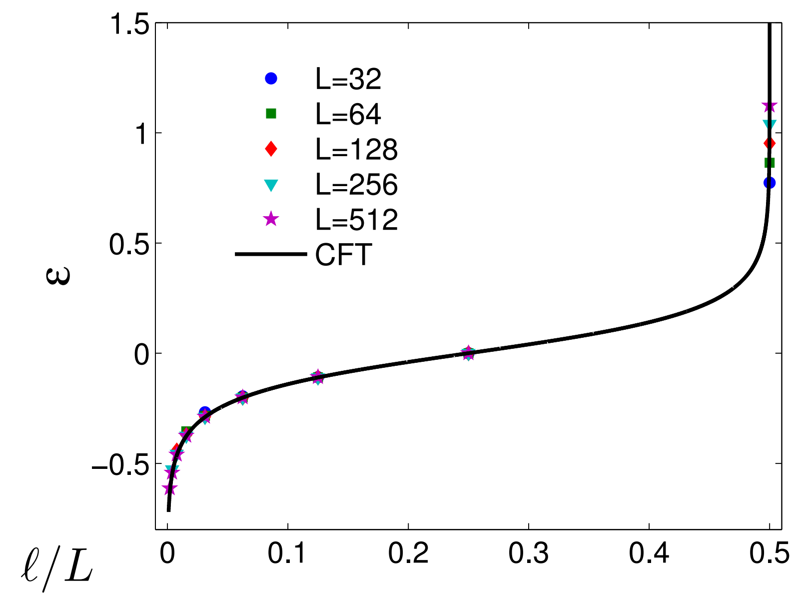



- Ising model:

Monte-Carlo analysis [Alba, (2013)]

Tree Tensor Network [Calabrese, Tagliacozzo, E.T., (2013)]



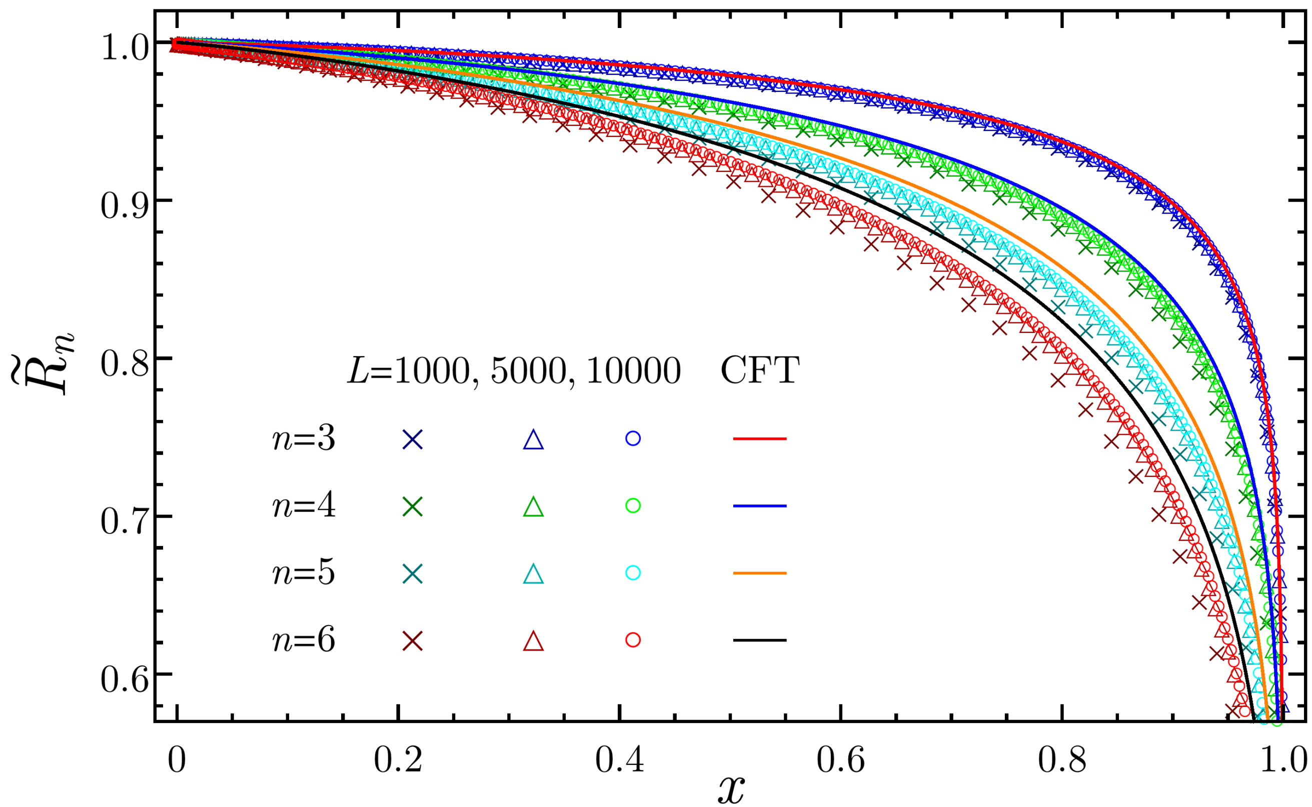
# *Two disjoint intervals: periodic harmonic chains*

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains [Wichterich, Molina-Vilaplana, Bose, (2009)] [Marcovitch, Retzker, Plenio, Reznik, (2009)]

- Non compact free boson [Calabrese, Cardy, E.T., (2012)]

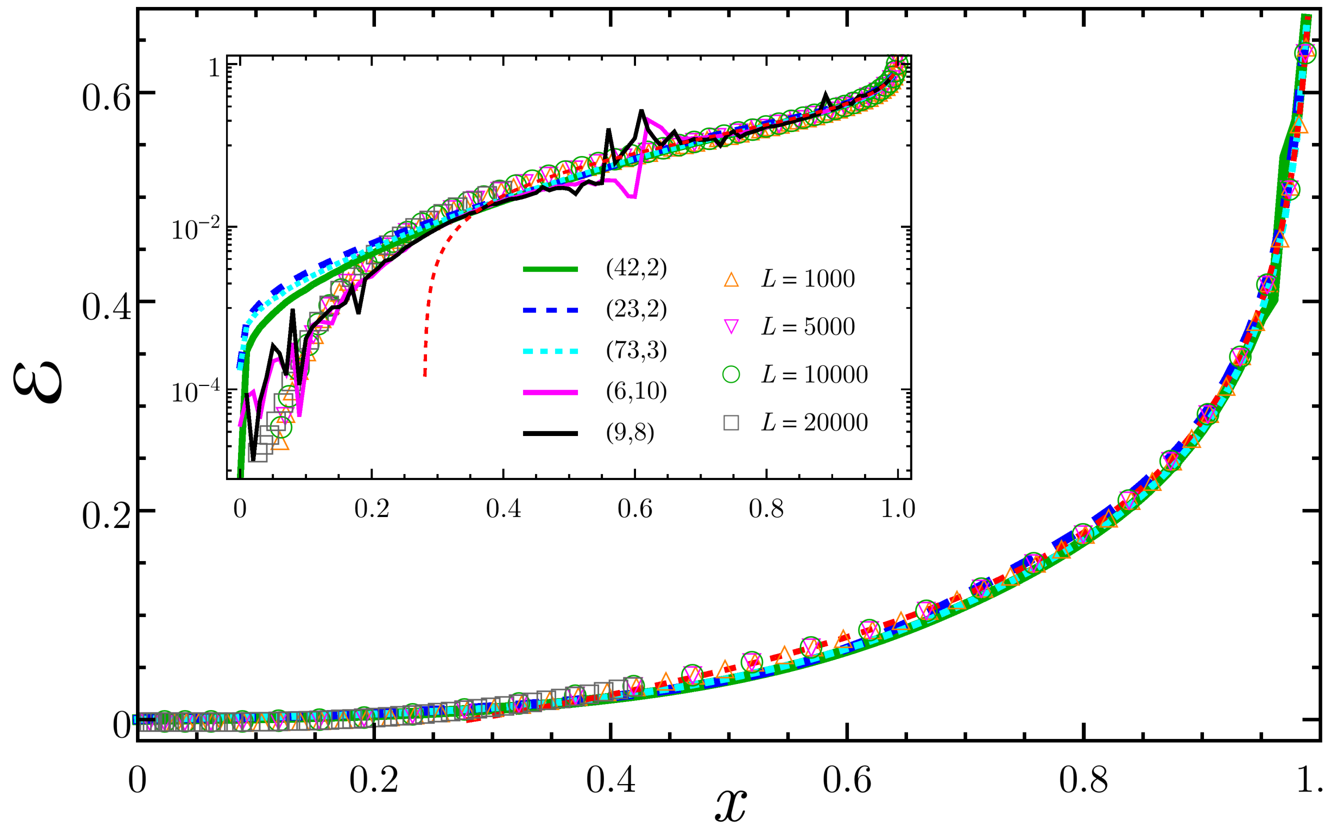
$$\widetilde{R}_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}$$

$$\widetilde{R}_n = \left[\frac{(1-x)^{\frac{2}{3}(n-\frac{1}{n})}\prod_{k=1}^{n-1} F_{\frac{k}{n}}(x)\, F_{\frac{k}{n}}(1-x)}{\prod_{k=1}^{n-1}\mathrm{Re}\big(F_{\frac{k}{n}}(\frac{x}{x-1})\,\bar{F}_{\frac{k}{n}}(\frac{1}{1-x})\big)}\right]^{\frac{1}{2}}$$




[De Nobili, Coser, E.T., (2015)]

# Two disjoint intervals: periodic harmonic chains




- Analytic continuation for $x \sim 1$ [Calabrese, Cardy, E.T., (2012)]

$$\mathcal{E} = -\frac{1}{4}\log(1-x) + \log K(x) + \text{cnst}$$

- Analytic continuation $n_e \to 1$ for $0 < x < 1$ not known
- $\mathcal{E}(x)$ for $x \sim 0$ vanishes faster than any power
- Numerical extrapolations (rational interpolation method) [De Nobili, Coser, E.T., (2015)]

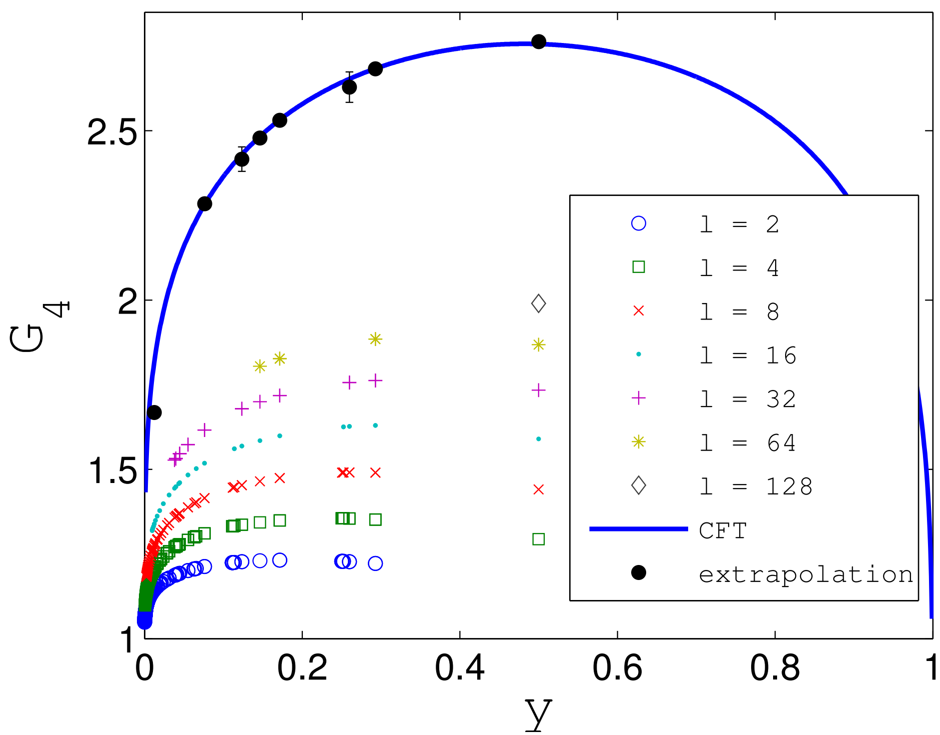
# *Two disjoint intervals: Ising model*

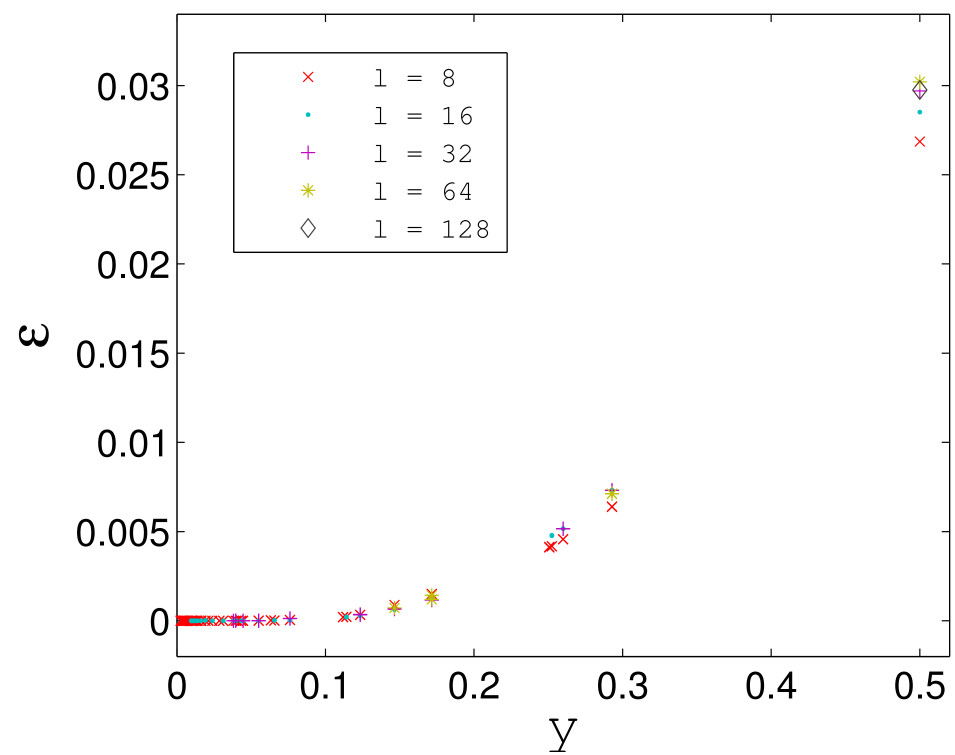
[Calabrese, Tagliacozzo, E.T., (2013)]

- CFT

$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1}|F_{k/n}(\frac{y}{y-1})|^{1/2}} \qquad 0<y<1$$

- Tree tensor network:



# *One interval at finite temperature: a naive approach*

[Calabrese, Cardy, E.T., (2014)]

- Logarithmic negativity $\mathcal{E}$ of one interval at finite $T = 1/\beta$

- A naive approach: compute $\langle \mathcal{T}_n^2(u)\bar{\mathcal{T}}_n^2(v)\rangle_\beta$ through the conformal map relating the cylinder to the complex plane

$$\mathcal{E}_{\text{naive}} = \frac{c}{2}\ln\left(\frac{\beta}{\pi a}\sinh\frac{\pi \ell}{\beta}\right) + 2\ln c_{1/2}$$

Problems:

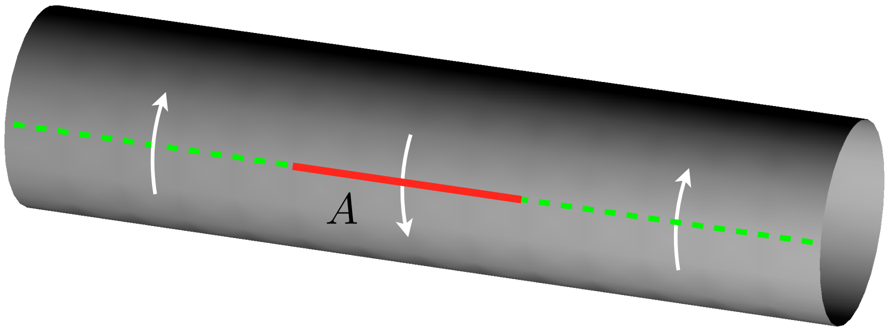
→ The Rényi entropy $n = 1/2$ is not an entanglement measure at finite $T$

→ $\mathcal{E}_{\text{naive}}$ is an increasing function of $T$, linearly divergent at high $T$

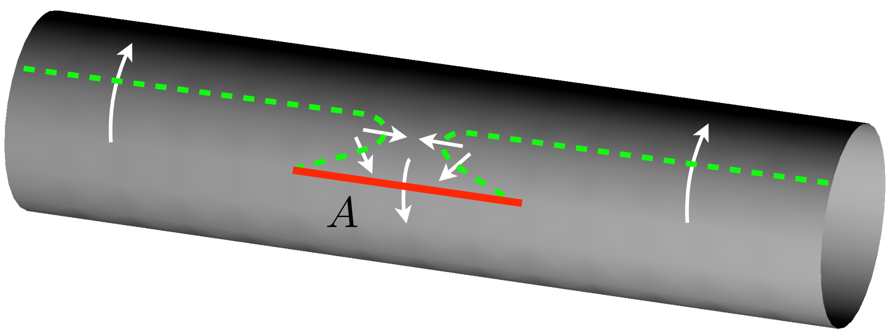
Entanglement should decrease as the system becomes classical

# *One interval at finite temperature in the infinite line*

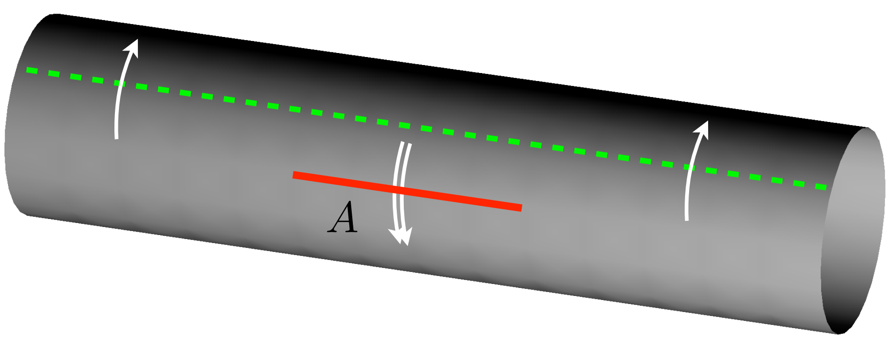
(connection to the $(j+1)$-th cylinder following the arrows)




Single copy of $\rho_\beta^{T_A}$ $\Longrightarrow$ $\mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$




Deformation of the cut along $B$

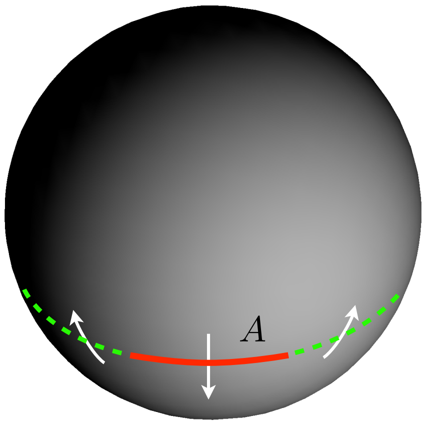



A cut remains connecting consecutive copies

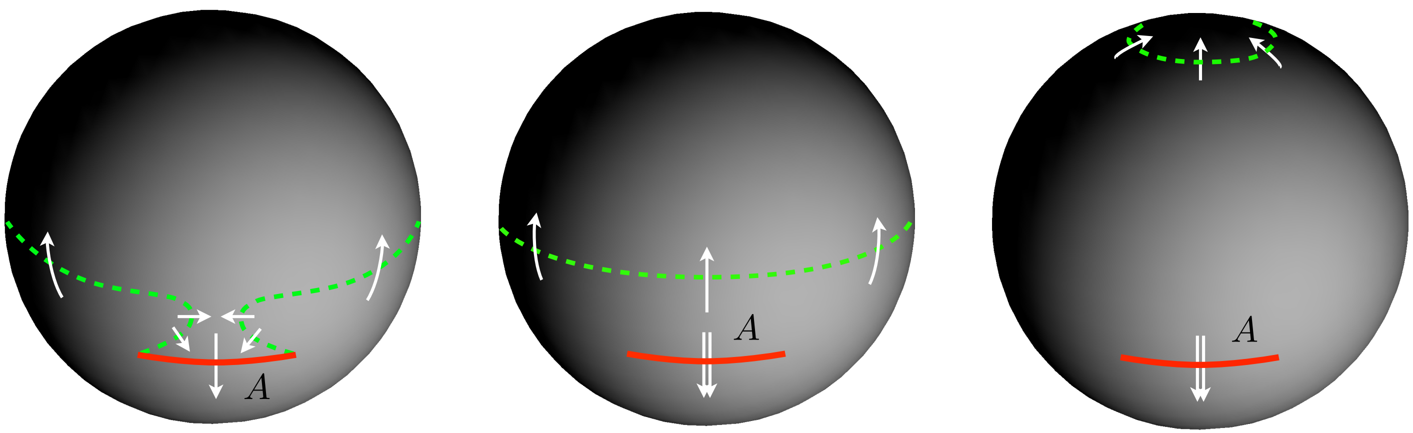
$\Longrightarrow$ No factorization for even $n$

(The double arrow indicates the connection to the $(j+2)$-th copy)

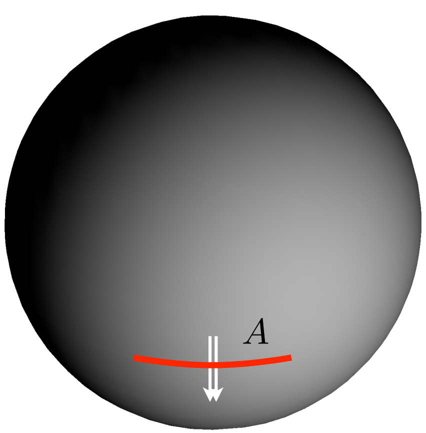
# *Deforming the cut at zero temperature*




Single copy of $\left(|\psi\rangle\langle\psi|\right)^{T_A} \implies \mathrm{Tr}\left[\left(|\psi\rangle\langle\psi|\right)^{T_A}\right]^n$

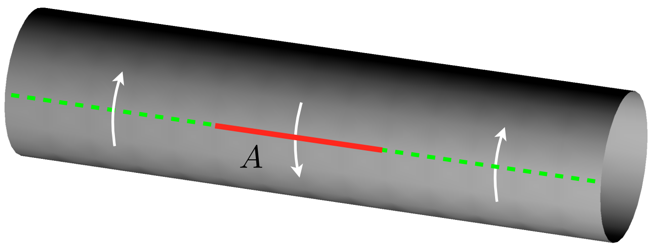



Deformation of the cut along $B$




The cut connecting consecutive copies shrinks to a point
Only the connection to the $j \pm 2$ copies along $A$ remains
$\implies$ Factorization for even $n$

# One interval at finite temperature in the infinite line




Two auxiliary twist fields at $\mathrm{Re}(w) = \pm L$, then $L \to \infty$

$$\mathcal{E}_A = \lim_{L\to\infty} \lim_{n_e\to 1} \ln \langle \mathcal{T}_{n_e}(-L)\bar{\mathcal{T}}^2_{n_e}(-\ell)\mathcal{T}^2_{n_e}(0)\bar{\mathcal{T}}_{n_e}(L)\rangle_\beta$$

- Conformal map the cylinder into the plane $z = e^{2\pi w/\beta}$

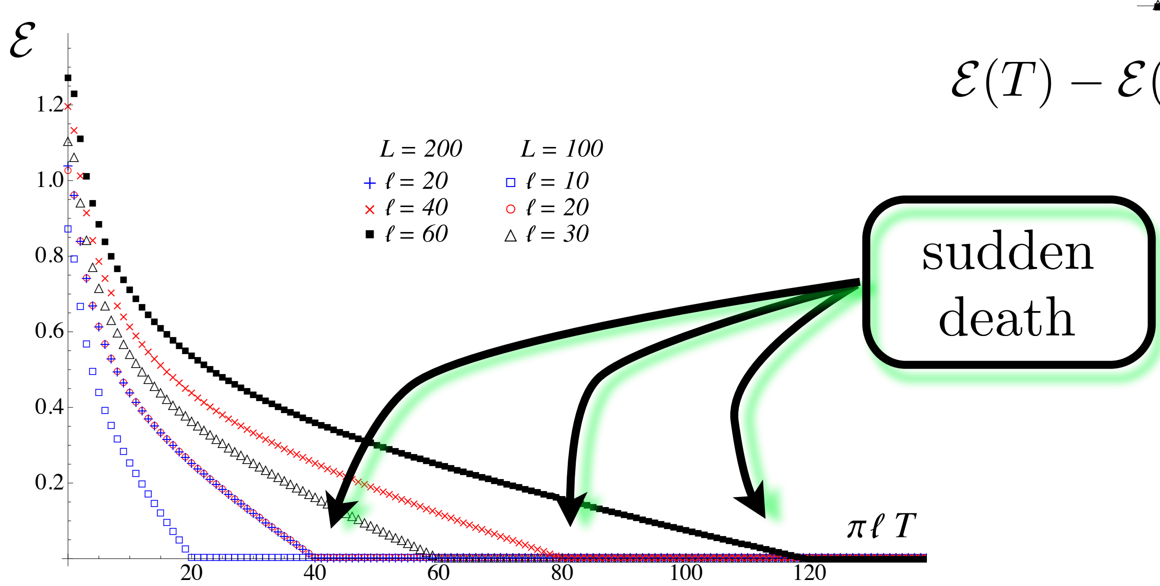
$$\langle \mathcal{T}_n(z_1)\bar{\mathcal{T}}^2_n(z_2)\mathcal{T}^2_n(z_3)\bar{\mathcal{T}}_n(z_4)\rangle = \frac{c_n\, c_n^{(2)}}{z_{14}^{2\Delta_n}\, z_{23}^{2\Delta_n^{(2)}}}\frac{\mathcal{F}_n(x)}{x^{\Delta_n^{(2)}}} \qquad \mathcal{F}_n(1) = 1 \qquad \mathcal{F}_n(0) = \frac{C^2_{\mathcal{T}_n\bar{\mathcal{T}}^2_n\bar{\mathcal{T}}_n}}{c_n^{(2)}}$$

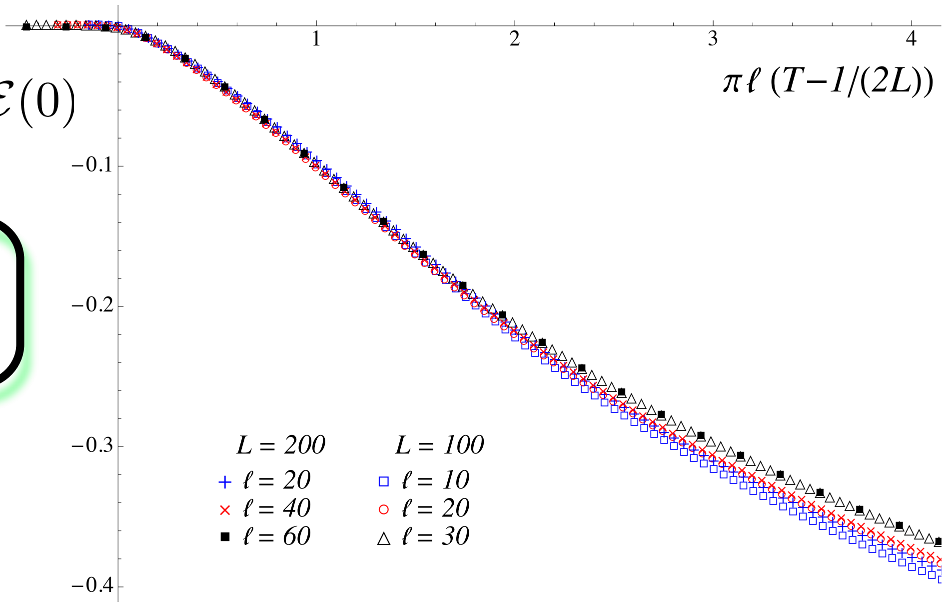
$$x \to e^{-2\pi\ell/\beta} \quad \text{when} \quad L \to \infty \qquad\qquad f(x) \equiv \lim_{n_e\to 1} \ln[\mathcal{F}_{n_e}(x)]$$

$$\mathcal{E}_A = \frac{c}{2}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] - \frac{\pi c\ell}{2\beta} + f(e^{-2\pi\ell/\beta}) + 2\ln c_{1/2}$$

- → $\mathcal{E}_A = \mathcal{E}_B$
- → $\mathcal{E}$ depends on the full operator content of the model
- → large $T$ linear divergence of $\mathcal{E}_{\text{naive}}$ is canceled
- → semi infinite systems $\mathrm{Re}(w) < 0$ (BCFT) have been also studied

# Harmonic chain (Dirichlet b.c.) and finite size setup

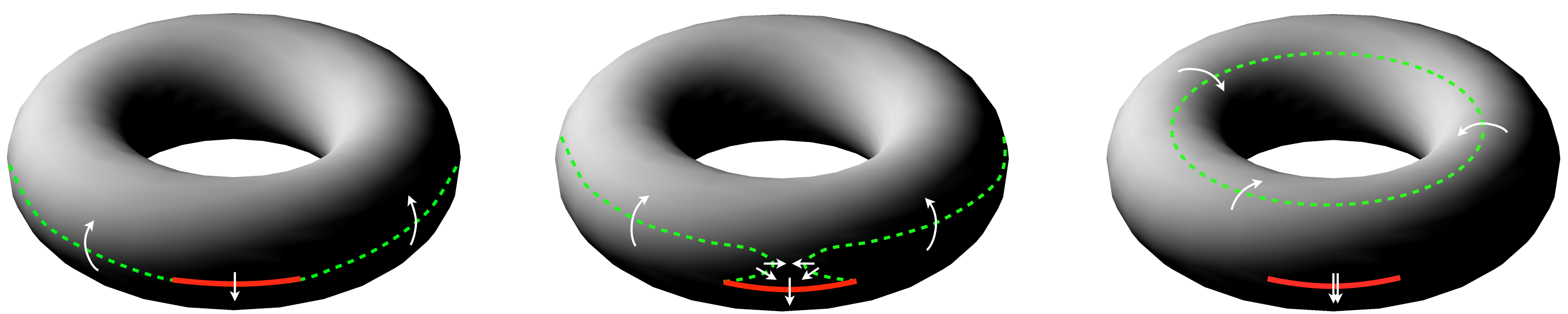



Single copy of $\rho_\beta^{T_A}$
$\Longrightarrow \ \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$

Deformation of the cut along $B$

A cut connects consecutive copies

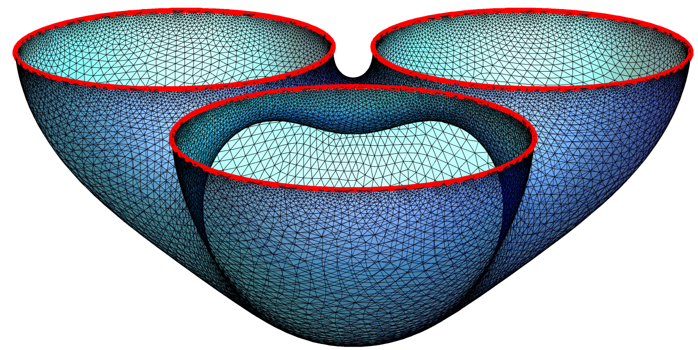


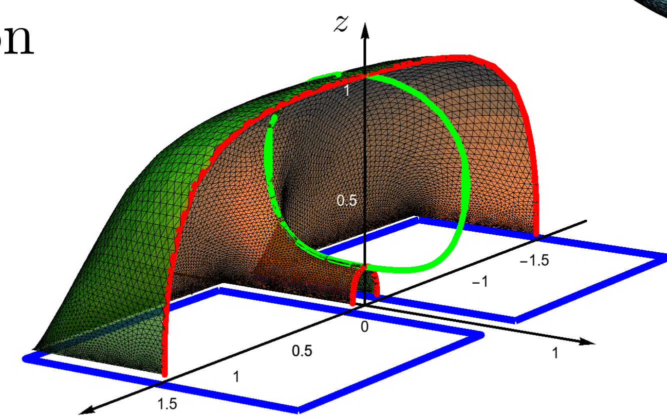
# Conclusions

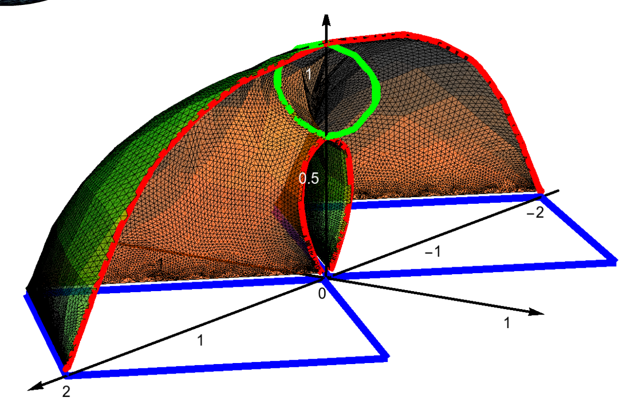
- Shape dependence of holographic entanglement entropy in $AdS_4$.
  - → ellipses and other shapes, including finite polygons
  - → Holographic mutual information for unusual regions
- Numerical extrapolations for the replica limit
- Entanglement for mixed states.
  Entanglement negativity in QFT (1+1 CFTs): $\mathrm{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$
  - → free boson on the line and Ising model
  - → finite temperature
- Entanglement negativity. Some generalizations:
  - → free compactified boson, systems with boundaries and massive case
  - → topological systems (toric code) [Lee, Vidal, (2013)] [Castelnovo, (2013)]
  - → results for holographic models [Rangamani, Rota, (2014)] [Kulaxizi, Parnachev, Policastro, (2014)]
  - → evolution after a quantum quench [Coser, E.T., Calabrese, (2014)] [Eisler, Zimboras, (2014)] [Wen, Chang, Ryu, (2015)]
  - → fermionic Gaussian states [Eisler, Zimboras, (2015)]

# *Open issues*

- Shape dependence of holographic entanglement entropy:
  - Analytic results
  - Black holes
  - Tripartite information
  - HMI for polygons
  - Higher dimensions
  - Time dependent backgrounds






- Entanglement negativity in CFT:
  - Analytic continuations
  - Higher dimensions
  - Interactions
  - Negativity in AdS/CFT

***Thank you!***