Chaos and Cohomogeneity One

G W Gibbons

Cambridge

wk. in progress with C N Pope, College Station

In the early '70's Belinsky. Khalatnikov and Lifshitz (BKL) developed an ultra local or *Carrollian* approximation scheme for Einstein's Equations in 3+1 dimensions,in which the metric at every point of *space* evolves independently of every other point of *space*, thus reducing a finite set of PDE's to a finite set of ODES in time t whose coefficients depend on the spatial variables x^i.

The ODE's are obtained by either ignoring spatial derivatives or replacing them by a suitable function of the spatial metric variables g_{ij}.

An *exact special case* of this procedure is to assume spatial homogeneity, i.e. that a three dimensional Bianchi Lie group G acts by isometries on the spatial hypersurfaces $t =$ constant . If G is abelian (Bianchi I) the metric coefficients evolve as as powers of time

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \tag{1}$$

$$p_1 + p_2 + p_3 = 1 \,, \qquad p_1^2 + p_2^2 + p_3^2 = 1 \,. \tag{2}$$

The Kasner motion is linear, i.e. free in $\ln t$.

In the BKL approximation this exact Kasner metric is modified by allowing the Kasner exponents p_i to depend arbitrarily on the spatial variables. $p_i = p_i(x^k)$.

A more interesting *exact* model is when $G = SU(2)$ or $SO(3)$ (Bianchi IX) for which

$$ds^2 = -dt^2 + g_{ij}(t)\sigma^i \otimes \sigma^j \,, \tag{3}$$

where $\sigma^i(x^k)$ are left-invariant one forms for $g = su(2)$.

The exact ode's are not explicitly integrable but BKL and Misner et al. developed an approximation scheme according to which the motion consist of of a chaotic quasi-periodic sequence of Kasner (i.e. piecewise linear or free exact solutions of Bianchi I type) epochs and eras, called by Misner the Mix-master Universe.

On this basis, BKL conjectured that the generic behaviour near a spacetime singularity was of this type.

The claim remains controversial, but there is considerable body of evidence that a open subset of Cauchy data will evolve in this way.

Subsequently, in the '80's and 90's many people investigated BKL behaviour in higher dimensions. For pure gravity, Henneaux et al. claimed that there could be no chaotic behaviour of BKL type in higher than 9+1.

In intermediate dimensions, they claimed that chaos is in principle possible but few if any explicit exact examples, analogous to the Bianchi IX case, in which a group G acts simply transitively on spacelike surfaces appear to be known.

In supergravity one must add sources, the various p-form fields, e.g. $p = 4$, in 10+1.

Damour, Henneaux,Julia, and Nicolai have analysed this case and discovered that chaos, in the BKL sense, persists in 10+1 and moreover is characterised algebraically in terms of the Kac-Moody algebra E_{10}.

On the basis of this result they have made the remarkable claim that E_{10} encodes the symmetries, and much else, of M-theory.

The potential importance of this result, means that the chaotic behaviour of higher dimensional spacetimes deserves the closest possible scrutiny.

In particular the DHJN result is based on two different approximations

- Approximating the PDE's by ODE's

- Approximating the resultant ODE's by a piecewise free motion subject to reflections.

The aim of our work is to obtain *exact* ODE's obtained by a *consistent* truncation of the Einstein equations to one time dimension and then to apply the BKL test for chaos, as developed by Damour, Henneaux, Nicolai and others.

A new feature of our work is that we deal with cases when the spatial manifold is not only a group manifold but also the case when it is a coset G/K. If the coset is *reductive*

$$g = p \oplus k \,, \qquad [k, p] \subset p \tag{4}$$

$$ds^2 = -dt^2 + g_{ij}(t) p^i \otimes p^j \tag{5}$$

then $g_{ij}(t)$ is a $k =$ dimensional family of K-invariant metrics on the tangent space p of G/K.

The Einstein equations become a Lagrangian system with a constraint

$$L_L = \frac{1}{2} G_{\mu\nu}(\alpha) \alpha^{\mu\prime} \alpha^{\nu\prime} - V_L(\alpha^\mu) \tag{6}$$

$$H_L = \frac{1}{2} G_{\mu\nu} \alpha^{\mu\prime} \alpha^{\nu\prime} + V_L(\alpha^\mu) \tag{7}$$

$$\prime = \frac{1}{\det g_{ij}} \frac{d}{dt} . \tag{8}$$

Here α^μ, $\mu = 1, 2, \ldots k$ are coordinates on a k -dimensional family of K-invariant metrics on G/K, and $G_{\mu\nu}(\alpha)$ is a (Lorentzian) metric on that space with signature $k-1, 1$. The timelike direction corresponds to rescaling the metric g_{ij}.

A closely related set of equations arises when considering Riemannian co-homogeneity one metrics (particularly of reduced holonomy)

$$ds = d\tau^2 + g_{ij}(\tau) p^i \otimes p^j \,, \tag{9}$$

$$L_R = \frac{1}{2} G_{\mu\nu}(\alpha) \alpha^{\mu\prime} \alpha^{\nu\prime} + V_L(\alpha^\mu) \tag{10}$$

$$H_R = \frac{1}{2} G_{\mu\nu} \alpha^{\mu\prime} \alpha^{\nu\prime} - V_L(\alpha^\mu) \tag{11}$$

and now

$$' = \frac{1}{\det g_{ij}} \frac{d}{d\tau} \,. \tag{12}$$

In the Riemannian case one often finds a subset of solutions with reduced holonomy given by BPS equations of the form

$$\alpha^{\mu\prime} = \pm G^{\mu\nu}\partial_\nu W \,, \tag{13}$$

$$V_R = -V_L = -\frac{1}{2}G^{\mu\nu}\partial_\mu W \partial_\nu W \,, \tag{14}$$

for some ‘superpotential‘$W(\alpha^\mu)$.

A gradient flow like this is not very chaotic and we suspect that *all Riemannian solutions, whether BPS or not, are non-chaotic*.

This is consistent with our experience applying the BKL-DHN criteria to all our explicit examples.

In $n+1$ Lorentzian dimensions the metric

$$G_{\mu\nu} d\alpha^{\mu} d\alpha^{\nu} \tag{15}$$

is essentially a restriction of a conformal rescaling of De-Witt metric on $GL(n.\beta R)/SO(n) \subset \beta R^{\frac{1}{2}n(n+1)}$, the space of positive definite $n \times n$ matrices

$$\frac{1}{4} g^{ij} g^{rs} dg_{ir} dg_{js} - g^{ij} dg_{ij} g^{rs} dg_{rs} \,. \tag{16}$$

For diagonal metrics it is the flat Minkowski metric on $\beta E^{n-1,1}$, but for non-diagonal metrics it is curved. For diagonal metrics, the free Kasner motion (obtained by setting $V_L = 0$) is that of a light ray in Minkowski spacetime $\beta E^{n-1,1}$.

In the diagonal case, we set g_{ij} = diag$(e^{-2\beta_\mu})$ $\mu = 1, 2, \ldots n$. The Lorentzian potential is then given by the Ricci scalar of G/K with the invariant metric g_j. It is given in terms of the structure constants of G and K and is then of the form

$$V_L = \sum \pm \exp -2\pi_\mu \beta^\mu \tag{17}$$

The hyperplanes through the origin of $\beta E^{n-1,1}$.

$$\pi_\mu \beta^\mu = 0\,, \tag{18}$$

determine positive or negative walls which may be timelike, spacelike or lightlike w.r.t. the metric $G_{\mu\nu}$.

In the BKL approximation, the free motion comes to an end when the light ray bounces off a timelike positive wall

If there are sufficiently many timelike walls, then the lightray will be confined within a closed polyhedral billiard and the motion is chaotic.

Following Chitre, one may project onto the ‘mass shell ’in $\beta E^{n-1,1}$, that is hyperbolic space H^{n-1} and one seeks a hyperbolic billiard

For any given model, checking whether the positive timelike walls enclose a closed polyhedral billiard (possibly with vertices on the ‘absolute ’or sphere at infinity, i.e. on the conformal boundary of H^{n-1}, is straightforward,although possibly involved.

The simplest case is when there are just n positive walls π^i_μ and the billiard is an n-symplex The diagonal elements of the Grammian

$$G^{ij} = G^{\mu\nu}\pi^i_\mu \pi^j_\nu \,. \tag{19}$$

must all be be positive in order that the n walls be timelike. By Legendre duality, the inverse Grammian gives the Grammian of the vertices

$$G_{ij} = G_{\mu\nu} v^\mu_i v^\nu_j \qquad \pi^i_\mu v^\mu_j = \delta^i_j \,. \tag{20}$$

If the hyperbolic billiard is to have finite volume, the vertices must lie inside or on the absolute, i.e. the vertices must be timelike or lightlike which means that the diagonal elements of G_{ij} must all be non-positive. If there are more than n positive walls the analysis is more complicated because some walls may lie behind, or be 'dominated 'by other walls.

For non-diagonal metrics g_{ij} the analysis is much more complicated.One introduces an Iwasawa decomposition

$$g_{ij} = (N^t \Delta N)_{ij} \tag{21}$$

where $\Delta = \text{diag}(\exp -2\beta_\mu)$ and N has ones on the diagonal and zeros beneath it. The full curved De-Witt metric on $SL(n.\mathbf{R})/SO(n)$ is given by

$$G_{\mu\nu} d\alpha^\mu d\alpha^\nu = \sum_\mu (d\beta^\mu)^2 - (\sum_\mu d\beta^\mu)^2 + \frac{1}{2} \sum_{\mu<\nu} e^{2(\beta^\mu - \beta^\nu)} (dNN^{-1})^2_{\mu\nu} \tag{22}$$

The potential V_L is also modified by additional wall terms of the form

$$\pm N^2 \exp -2\pi_\mu \beta^\mu \qquad \text{and} \qquad \pm N^4 \exp -2\pi_\mu \beta^\mu \,. \tag{23}$$

The strategy of DHN is to project the motion onto the space of diagonal variables β_μ and to regard the Isawawa variables N as ‘frozen ’or slowly varying. The term in the kinetic energy is $\frac{1}{2}\sum_{\mu<\nu} e^{2(\beta^\mu-\beta^\nu)}(N'N^{-1})^2_{\mu\nu}$ gives rise to a further ‘centrifugal potential ’in the effective Lorentzian potential

$$V_L^{\text{eff}} = V_L + \frac{1}{2}\sum_{\mu<\nu} e^{-2(\beta^\mu-\beta^\nu)}(PN)^2_{\mu\nu} \tag{24}$$

where P is the momentum canonically conjugate to N. Freezing P means introduces *additional* symmetry walls of the form

$$\beta_\mu - \beta_\nu = 0\,. \tag{25}$$

If one accepts the strategy of DHN, one may now, in any given case, test for chaos in the sense of BKL. For coset models, G/K the number of Iwasawa and diagonal variables is reduced since we must respect K-invariance. Thus the results of DHN and others are not directly applicable and we must test anew.

So far, in all the examples we have examined above $3+1$ dimensions, we have not found a single chaotic example (Lorentzian or Riemannian).

Our special coset examples were motivated by our previous work on reduced holonomy and contain no form fields. Thus there is no contradiction with the work of DNH.

However it is disappointing since we have no simple explicit examples of BKL behaviour to examine in depth.

An example, a three function ansatz for $SO(n+2)/SO(n)$ in $(2n+1)+1$ dimensions

$$ds^2 = dt^2 + a^2\sigma_i^2 + b^2\,\tilde{\sigma}_i^2 + c^2\,\nu^2\,, \tag{26}$$

Geometrically the orbits are the bundle of unit tangent vectors on S^{n+1}. The case $n=1$ is Bianchi IX. Riemannian solutions with reduced holonomy include $n=1$ Eguchi-Hanson and $n=2$ the Deformed Conifold. For arbitrary n there is a non-compact Calabi-Yau metric with holonomy $SU(n+1)$ on $T^\star S^{n+1}$ called the Stenzel metric.

Defining $a = e^\alpha$, $b = e^\beta$, $c = e^\gamma$, and introducing the new coordinate η by $a^n\, b^n\, c\, d\eta = dt$, we find that the Riemannian Ricci-flat equations can be derived from the Lagrangian $L_E = T - V_E$, where

$$\begin{aligned} T &= -\alpha'\,\gamma' - \beta'\,\gamma' - n\,\alpha'\,\beta' - \tfrac{1}{2}(n-1)\,\alpha'^2 - \tfrac{1}{2}(n-1)\,\beta'^2\,, \\ V_E &= -\tfrac{1}{4}(a\,b)^{2n-2}\,(a^4 + b^4 + c^4 - 2a^2\,b^2 - 2n\,a^2\,c^2 - 2n\,b^2\,c^2)\,, \end{aligned} \tag{27}$$

where a prime means $d/d\eta$, together with the constraint that the Hamiltonian vanishes, $H_R = T + V_E = 0$. The DeWitt metric and its inverse are thus given by

$$G_{\mu\nu} = -\begin{pmatrix} n-1 & n & 1 \\ n & n-1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad G^{\mu\nu} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2n-1 \end{pmatrix}. \tag{28}$$

Writing the Lagrangian as $L_E = \frac{1}{2} G_{\mu\nu}\, (d\alpha^\mu/d\eta)\, (d\alpha^\nu/d\eta) - V$, where $\alpha^\mu = (\alpha, \beta, \gamma)$, we find that the potential can be written in terms of a superpotential, as

$$V_E = -\tfrac{1}{2} G^{\mu\nu}\, \frac{\partial W}{\partial\alpha^\mu}\, \frac{\partial W}{\partial\alpha^\nu} \tag{29}$$

with

$$W = \tfrac{1}{2}(a\, b)^{n-1}\, (a^2 + b^2 + c^2)\,. \tag{30}$$

It follows that the Lagrangian can be written, after dropping a total derivative, as

$$L = \tfrac{1}{2} G_{\mu\nu}\, (\frac{d\alpha^\mu}{d\eta} \pm G^{\mu\rho}\, \partial_\rho W)\, (\frac{d\alpha^\nu}{d\eta} \pm G^{\nu\sigma}\, \partial_\sigma W)\,, \tag{31}$$

where $\partial_\mu W \equiv \partial W/\partial\alpha^\mu$. This implies that the second-order equations for Ricci-flatness are satisfied if the first-order equations $d\alpha^\mu/d\eta =$

$\mp G^{\mu\nu}\,\partial_\nu W$ are satisfied. Thus we arrive at the first-order equations

$$\begin{aligned}
\dot\alpha &= \tfrac{1}{2}e^{-\alpha-\beta-\gamma}\,(e^{2\beta}+e^{2\gamma}-e^{2\alpha})\,,\\
\dot\beta &= \tfrac{1}{2}e^{-\alpha-\beta-\gamma}\,(e^{2\alpha}+e^{2\gamma}-e^{2\beta})\,,\\
\dot\gamma &= \tfrac{1}{2}n\,e^{-\alpha-\beta-\gamma}\,(e^{2\alpha}+e^{2\beta}-e^{2\gamma})\,,
\end{aligned} \tag{32}$$

For the Lorentzian motion there are three positive walls, of the form

$$\pi^{+1}_\mu=(n+1,n-1,0): \qquad 2\alpha+(n-1)(\alpha+\beta)=0 \tag{33}$$

$$\pi^{+2}_\mu=(n-1,n+1,0): \qquad 2\beta+(n-1)(\alpha+\beta)=0 \tag{34}$$

$$\pi^{+3}_\mu=(n-1,n-1,2): \qquad 2\gamma+(n-1)(\alpha+\beta)=0 \tag{35}$$

These have the intersections

$$\pi^1\cap\pi^2: \qquad v^\mu_{+3}=\lambda(0,0,1)^t \tag{36}$$

$$\pi^2 \cap \pi^3 : \qquad v^\mu_{+1} = \lambda(1+n, 1-n, 1-n)^t \tag{37}$$

$$\pi^3 \cap \pi^1 : \qquad v^\mu_{+2} = \lambda(1-n, 1+n, 1-n)^t \tag{38}$$

The first is a lightlike vector and the last two are spacelike. All three planes are timelike. Although one of the intersections is lightlike, the other two are spacelike:

$$\begin{aligned} &G^{\mu\nu}\pi^{+1}_\mu \partial^{+1}_\nu = G^{\mu\nu}\pi^{+2}_\mu \partial^{+2}_\nu = G^{\mu\nu}\pi^{+3}_\mu \partial^{+3}_\nu = 2\,, \\ &G_{\mu\nu}v^\mu_{+1}v^\nu_{+1} = G_{\mu\nu}v^\mu_{+2}v^\nu_{+2} = 2\lambda^2(n^2-1)\,, \qquad G_{\mu\nu}v^\mu_{+3}v^\nu_{+3} = 0\,. \end{aligned} \tag{39}$$

Thus the triangular billiard has one vertex on the absolute and two outside it, implying that the motion is not chaotic in the BKL sense.

For the Riemannian motion, the relevant walls are negative (as terms in $V_L = -V_E$), and they take the form

$$\pi^{-1}_\mu = (n, n-1, 1)\,, \qquad \pi^{-2}_\mu = (n-1, n, 1)\,, \qquad \pi^{-3}_\mu = (1, 1, 0)\,. \tag{40}$$

These have the intersections

$$\pi^1 \cap \pi^2 : \qquad v^{\mu}_{-3} = \lambda(1, 1, 1 - 2n)^t \tag{41}$$
$$\pi^2 \cap \pi^3 : \qquad v^{\mu}_{-1} = \lambda(1, -1, 1)^t \tag{42}$$
$$\pi^3 \cap \pi^1 : \qquad v^{\mu}_{-2} = \lambda(-1, 1, 1)^t \tag{43}$$

The third plane, π_{μ}^{-3}, is lightlike and if $n > 1$, the other two planes are spacelike. All of the intersections are spacelike:

$$\begin{gathered} G^{\mu\nu}\pi_{\mu}^{-1}\partial_{\nu}^{-1} = G^{\mu\nu}\pi_{\mu}^{-2}\partial_{\nu}^{-2} = 1 - n\,, \qquad G^{\mu\nu}\pi_{\mu}^{-3}\partial_{\nu}^{-3} = 0\,, \\ G_{\mu\nu}v^{\mu}_{-1}v^{\nu}_{-1} = G_{\mu\nu}v^{\mu}_{-2}v^{\nu}_{-2} = 2\lambda^2\,, \qquad G_{\mu\nu}v^{\mu}_{-3}v^{\nu}_{-3} = 2\lambda^2(2n - 1)\,. \end{gathered} \tag{44}$$

Thus there is no chaos for the Riemannian equations.

Introduction of off-diagonal terms. To be consistent with $K = SO(n)$ invariance we can only consider a single Iwasawa variable.

$$ds^2 = dt^2 + a^2(\sigma_i + N\tilde{\sigma}_i)^2 + b^2\,\tilde{\sigma}_i^2 + c^2\,\nu^2\,, \tag{45}$$

For concreteness we set $n = 2$. One finds that

$$T \to T + e^{2\alpha - 2\beta}(N')^2 \tag{46}$$

$$V_L \to V_L + 2N^2 e^{6\alpha+2\beta} + 2N^2 e^{4\alpha+4\beta} + 2N^4 e^{6\alpha+2\beta} - 4N^2 e^{4\alpha+2\beta+2\gamma} \tag{47}$$

Including the symmetry wall we obtain a total of 5 positive walls.

$$
\begin{aligned}
3\alpha+\beta &= 0 && (48)\\
\alpha+3\beta &= 0 && (49)\\
\alpha+\beta+2\gamma &= 0 && (50)\\
\alpha+\beta &= 0 && (51)\\
\beta-\alpha &= 0\,. && (52)
\end{aligned}
$$

Eliminating redundant walls we obtain just three walls

$$
\begin{aligned}
i) \qquad \beta-\alpha &= 0 && (53)\\
ii) \qquad \alpha+\beta+2\gamma &= 0\gamma && (54)\\
iii) \qquad 3\alpha+\beta &= 0 && (55)
\end{aligned}
$$

and the simplex method applies. All three walls are timelike. The intersection of the second and third is spacelike while the other two intersections lie on the light cone. No chaos in the sense of BKL is possible in this case.