Universal identity in second order fluid dynamics

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Based on:
S.Grozdanov and A.Starinets:
Teor. Mat. Fiz. 182, no. 1, 76 (2014)
arXiv: 1412.5685 [hep-th]
arXiv: 1504.XXXX [hep-th]

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Firenze, Italy
3 April 2015

Fluid dynamics is an effective theory valid in the long-wavelength, long-time limit

Fundamental degrees of freedom = densities of conserved charges

Equations of motion = conservation laws + constitutive relations^*

Example I

$$\left.\begin{aligned} &\partial_a J^a = 0 \\ &J^a = -D\nabla^a J^0 + \cdots \end{aligned}\right\} \quad \partial_t J^0 = D\,\nabla^2 J^0 + \cdots$$

Example II

$$\left.\begin{aligned} &\partial_a T^{ab} = 0 \\ &T^{ab} = \varepsilon u^a u^b + P(\varepsilon)\left(g^{ab} + u^a u^b\right) + \Pi^{ab} + \cdots \end{aligned}\right\}$$

Navier-Stokes eqs
Burnett eqs
.....

* Modulo assumptions e.g. analyticity

** E.o.m. universal, transport coefficients depend on underlying microscopic theory

Consider relativistic neutral conformal fluid in a d-dimensional (curved) space-time

$$T^{ab} = \varepsilon u^a u^b + P(\varepsilon)\left(g^{ab} + u^a u^b\right) + \Pi^{ab} + \cdots$$

Including only terms with first and second derivatives of fluid velocity:

$$\begin{aligned}\Pi^{ab} = & -\eta\sigma^{ab} \\ & + \eta\tau_\Pi\left[{}^{\langle}D\sigma^{ab\rangle} + \frac{1}{d-1}\sigma^{ab}\left(\nabla\cdot u\right)\right] \\ & + \kappa\left[R^{\langle ab\rangle} - (d-2)u_c R^{c\langle ab\rangle d}u_d\right] \\ & + \lambda_1\sigma^{\langle a}{}_c\sigma^{b\rangle c} + \lambda_2\sigma^{\langle a}{}_c\Omega^{b\rangle c} + \lambda_3\Omega^{\langle a}{}_c\Omega^{b\rangle c}\end{aligned}$$

Transport coefficients (in conformal case): $\eta\,,\tau_\Pi\,,\kappa\,,\lambda_1\,,\lambda_2\,,\lambda_3$

In general: 2 first order coefficients, 15 (10) second order coefficients
(see S.Bhattacharyya, 1201.4654 [hep-th])

Notations used in the derivative expansion

$$D \equiv u^a \nabla_a$$

$$\Delta^{ab} \equiv g^{ab} + u^a u^b$$

$$A^{\langle ab \rangle} \equiv \frac{1}{2}\Delta^{ac}\Delta^{bd}\left(A_{cd} + A_{dc}\right) - \frac{1}{d-1}\Delta^{ab}\Delta^{cd}A_{cd} \equiv \langle A^{ab} \rangle$$

$$\sigma^{ab} = 2^{\langle}\nabla^a u^{b\rangle}$$

$$\Omega^{ab} = \frac{1}{2}\Delta^{ac}\Delta^{bd}\left(\nabla_c u_d - \nabla_d u_c\right)$$

*Hydro definitions differ in the literature – see footnote 91 on page 128 of M.Haehl, R.Loganayagam, M.Rangamani, 1502.00636 [hep-th]

Transport coefficients $\eta\,,\tau_\Pi\,,\kappa\,,\lambda_1\,,\lambda_2\,,\lambda_3$ are non-trivial functions of the parameters of the underlying microscopic theory, even in the simplest case of conformal liquids

$$\frac{\eta}{s} = F\left(N_c, N_f, \frac{m}{T}, \frac{\Lambda}{T}, \ldots\right)$$

Some information can be obtained from kinetic theory at weak coupling and from gauge-string duality at strong coupling (for theories with string or gravity duals). In the latter case, it is natural to look for "universal" (independent of the specific string construction) results.

In the limit of infinite N (gauge group rank) and infinite coupling (i.e. in the supergravity approximation from dual string theory point of view)

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Policastro, Kovtun, Son, AOS, 2001-2008
Buchel, J.Liu, 2003; Buchel, 2004
Iqbal, H.Liu, 2008

$$\frac{\sigma}{\chi} = \frac{1}{4\pi T}\frac{d}{d-2}$$

[for conformal relativistic fluids]: Kovtun, Ritz, 2008

$$2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$$

[for conformal relativistic fluids]: Haack, Yarom, 2008

More speculative statements inspired by holography

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

Kovtun, Son, AOS, 2004 (violated by some models with Higher derivative gravity, Kats and Petrov, 2007; Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008)

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{d-1} - c_s^2\right)$$

Buchel, 2008 (counterexample: Buchel, 2012)

$$c_s^2 \leq c_{s,conf}^2 = \frac{1}{d-1}$$

Hohler and Stephanov, 2009;
Cherman, Cohen, Nellore, 2009

$$\frac{\sigma}{\chi} \geq \frac{\hbar v^2}{4\pi T}\frac{d}{d-2}$$

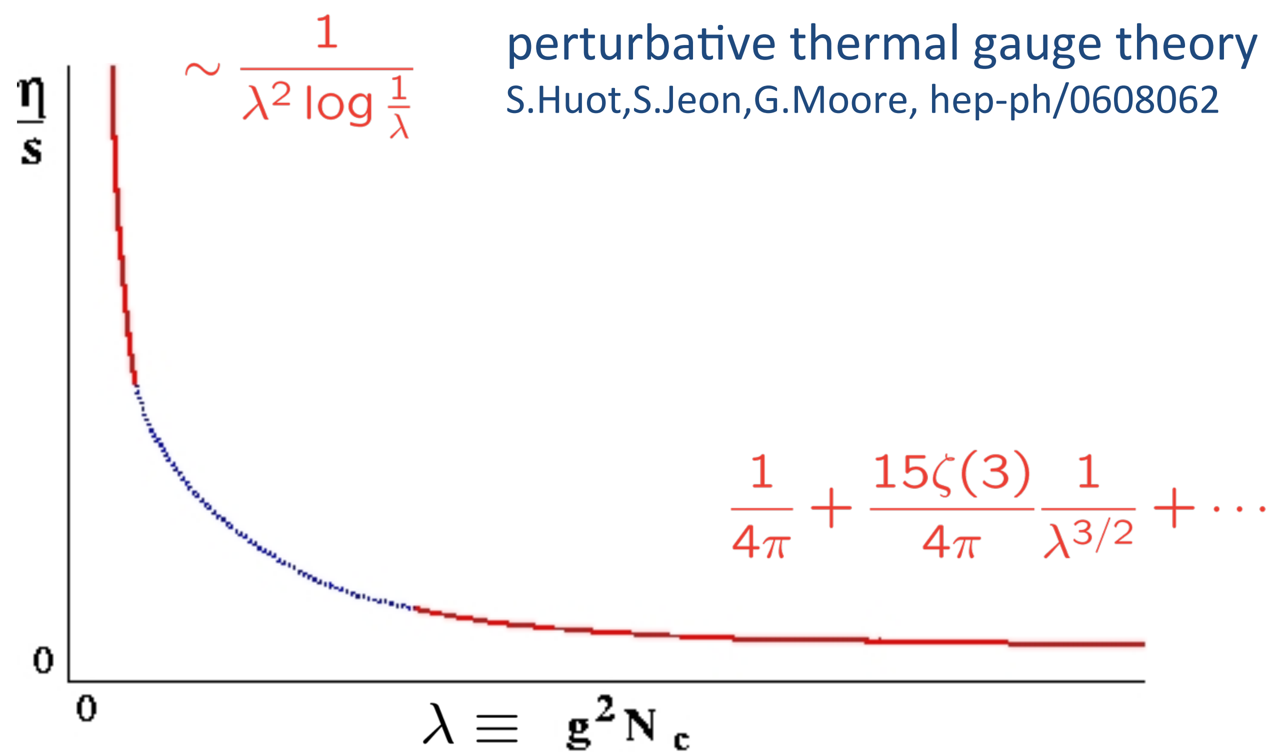
Kovtun and Ritz, 2008

$$D \gtrsim \frac{\hbar v_F}{k_B T}$$

Hartnoll, 2014

See also "Fluctuation bounds…" by Kovtun 1407.0690 [hep-th]

Shear viscosity in $\mathcal{N} = 4$ SYM



Correction to $1/4\pi$: Buchel, Liu, A.S., hep-th/0406264

Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]

Electrical conductivity in $\mathcal{N} = 4$ SYM

Weak coupling:

$$\sigma \approx 1.28349 \frac{e^2(N_c^2 - 1)T}{\lambda^2[\ln \lambda^{-1/2} + O(1)]}$$

$\lambda \ll 1$

Strong coupling:

$$\sigma = \frac{e^2\, N_c^2\, T}{16\pi} + O\left(\frac{1}{\lambda^{3/2}}\right)$$

$\lambda \gg 1$

* Charge susceptibility can be computed independently: $\Xi = \frac{N_c^2 T^2}{8}$

D.T.Son, A.S., hep-th/0601157

Einstein relation holds: $\frac{\sigma}{e^2 \Xi} = D_{U(1)} = \frac{1}{2\pi T}$

First and second order transport coefficients of *conformal* holographic fluids *to leading order* in supergravity approximation

$$
\begin{aligned}
\eta &= s/4\pi\,, \\
\tau_{\Pi} &= \frac{d}{4\pi T}\left(1+\frac{1}{d}\left[\gamma_E+\psi\left(\frac{2}{d}\right)\right]\right)\,, \\
\kappa &= \frac{d}{d-2}\,\frac{\eta}{2\pi T}\,, \\
\lambda_1 &= \frac{d\eta}{8\pi T}\,, \\
\lambda_2 &= \left[\gamma_E+\psi\left(\frac{2}{d}\right)\right]\,\frac{\eta}{2\pi T}\,, \\
\lambda_3 &= 0
\end{aligned}
$$

Bhattacharyya et al, 2008

Note: $2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0$

How to compute second order transport coefficients?

Fluid-gravity correspondence [Bhattacharyya et al, 2007]

Quasinormal spectrum [Baier et al, 2007]

Kubo formulas & three-point functions
[Moore, Sohrabi, Saremi, 2010, 2011; Arnold, Vaman, Wu, Xiao, 2011]

Coupling constant corrections to N=4 SYM transport coefficients

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(R + \frac{12}{L^2} + \gamma\mathcal{W}\right) \qquad \gamma = \lambda^{-3/2}\zeta(3)/8$$

$$\eta = \frac{\pi}{8}N_c^2 T^3\,(1 + 135\gamma + \dots)$$

$$\tau_\Pi = \frac{(2-\ln 2)}{2\pi T} + \frac{375\gamma}{4\pi T} + \dots$$

$$\kappa = \frac{N_c^2 T^2}{8}(1 - 10\gamma + \dots)$$

$$\lambda_1 = \frac{N_c^2 T^2}{16}(1 + 350\gamma + \dots)$$

$$\lambda_2 = -\frac{N_c^2 T^2}{16}(2\ln 2 + 5\,(97 + 54\ln 2)\,\gamma + \dots)$$

$$\lambda_3 = \frac{25 N_c^2 T^2}{2}\gamma + \dots$$

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Kubo formulas for second order transport coefficients

First order transport coefficients can be computed from two-point functions of the corresponding operators using Kubo formulas

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt\, d^3x e^{i\omega t} \langle \left[T_{xy}(t,x), T_{xy}(0,0) \right] \rangle$$

Similarly, second transport coefficients can be computed from three-point functions

$$\lambda_2 = 2\eta\tau_\Pi - 4 \lim_{p,q\to 0} \frac{\partial^2}{\partial p^0 \partial q^z} G_{RAA}^{xy,ty,xz}(p,q)$$

Moore, Sohrabi, Saremi, 2010, 2011; Arnold, Vaman, Wu, Xiao, 2011

Schwinger-Keldysh generating functional

$$G_{RA\ldots}^{ab,cd,\ldots}(0,x,\ldots) = \left. \frac{(-i)^{n-1}(-2i)^n \delta^n W}{\delta h_{A\;ab}(0) \delta h_{R\;cd}(x)\ldots} \right|_{h=0} = (-i)^{n-1} \langle T_R^{ab}(0) T_A^{cd}(x)\ldots \rangle$$

$$W\left[h^+, h^-\right] = \ln \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \mathcal{D}\varphi \exp\left\{ i \int d^4x^+ \sqrt{-g^+} \mathcal{L}\left[\phi^+(x^+), h^+\right] \quad - \int_0^\beta d^4y \mathcal{L}_E\left[\varphi(y)\right] - i \int d^4x^- \sqrt{-g^-} \mathcal{L}\right.$$

Curvature squared corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{R^2} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - 2\Lambda + L^2\left(\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\eta = \frac{r_+^3}{2\kappa_5^2}\left(1 - 8\left(5\alpha_1 + \alpha_2\right)\right) + \ldots$$

$$\eta\tau_\Pi = \frac{r_+^2\left(2 - \ln 2\right)}{4\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2\left(23 + 5\ln 2\right)}{12\kappa_5^2}\alpha_3 + \ldots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{25 r_+^2}{6\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2}{12\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_2 = -\frac{r_+^2 \ln 2}{2\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2\left(21 + 5\ln 2\right)}{6\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_3 = -\frac{28 r_+^2}{\kappa_5^2}\alpha_3 + \ldots$$

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\eta = \frac{r_+^2}{2\kappa_5^2}\left(1 - 4\lambda_{GB}\right)$$

$$\eta\tau_\Pi = \frac{r_+^2\left(2 - \ln 2\right)}{4\kappa_5^2} - \frac{r_+^2\left(25 - 7\ln 2\right)}{8\kappa_5^2}\lambda_{GB} + \ldots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2} - \frac{17r_+^2}{4\kappa_5^2}\lambda_{GB} + \ldots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2} - \frac{9r_+^2}{8\kappa_5^2}\lambda_{GB} + \ldots$$

$$\lambda_2 = -\frac{r_+^2\ln 2}{2\kappa_5^2} - \frac{7r_+^2\left(1 - \ln 2\right)}{4\kappa_5^2}\lambda_{GB} + \ldots$$

$$\lambda_3 = -\frac{14r_+^2}{\kappa_5^2}\lambda_{GB} + \ldots$$

Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008

Shaverin, Yarom, 2012

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Non-perturbative Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\eta = \frac{s}{4\pi}\gamma^2 = \frac{s}{4\pi}(1 - 4\lambda_{GB})$$

Brigante et al, 2008

$$\tau_\Pi = \frac{1}{2\pi T}\left(\frac{1}{4}(1+\gamma)\left(5 + \gamma - \frac{2}{\gamma}\right) - \frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right)$$

Banerjee and Dutta, 2011

$$\lambda_1 = \frac{\eta}{2\pi T}\left(\frac{(1+\gamma)(3 - 4\gamma + 2\gamma^3)}{2\gamma^2}\right)$$

Grozdanov and AOS, 2014

$$\lambda_2 = -\frac{\eta}{\pi T}\left(-\frac{1}{4}(1+\gamma)\left(1 + \gamma - \frac{2}{\gamma}\right) + \frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right)$$

Grozdanov and AOS, 2014

$$\lambda_3 = -\frac{\eta}{\pi T}\left(\frac{(1+\gamma)(3 + \gamma - 4\gamma^2)}{\gamma^2}\right)$$

Grozdanov and AOS, 2014

$$\kappa = \frac{\eta}{\pi T}\left(\frac{(1+\gamma)(2\gamma^2 - 1)}{2\gamma^2}\right)$$

Banerjee and Dutta, 2011

$$H(\lambda_{GB}) = 2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T}\frac{(1-\gamma_{GB})(1-\gamma_{GB}^2)(3 + 2\gamma_{GB})}{\gamma_{GB}^2} = -\frac{40\lambda_{GB}^2\eta}{\pi T} + \ldots$$

Gauss-Bonnet gravity and dissipationless fluids

$$S_{GB} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\frac{\eta}{s} = \frac{1-4\lambda_{GB}}{4\pi}$$

Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2007

The existence of dissipationless fluids (fluids with zero entropy production and yet non-trivial second order transport coefficients) was conjectured by Bhattacharyya et al, 1211.1020 [hep-th]. Holographic Gauss-Bonnet fluid is a natural candidate (modulo problems with UV)

Conditions: $\kappa = 2\lambda_1, H(\lambda_{GB}) = 0$

In the limit $\lambda_{GB} \to 1/4$

$$\eta\tau_\Pi = 0, \quad \lambda_1 = \frac{3\pi^2T^2}{2\sqrt{2}\kappa_5^2}, \quad \lambda_2 = 0, \quad \lambda_3 = -\frac{3\sqrt{2}\pi^2T^2}{\kappa_5^2}, \quad \kappa = -\frac{\pi^2T^2}{\sqrt{2}\kappa_5^2}$$

Thus, GB fluid is not dissipationless…

From strong to weak coupling

Is H(λ) a non-trivial function of the coupling?

$$H(\lambda_{GB}) = 2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T}\frac{(1-\gamma_{GB})\left(1-\gamma_{GB}^2\right)(3+2\gamma_{GB})}{\gamma_{GB}^2} \leq 0$$

Conformal kinetic theory: $2\eta\tau_\Pi + \lambda_2 = 0$ and $\lambda_1/\eta\tau_\Pi \approx 1$

[Baier et al, 2007; York and Moore, 2008; Betz et al, 2008]

Also, in (perturbative) QED, QCD and in conformal kinetic theory: $\kappa = 0\,,\ \lambda_3 = 0$

[York and Moore, 2008]

Recall that at strong coupling $\lambda_3 = 0$ but $\kappa \neq 0$

Breaking conformal invariance

Bigazzi, Cotrone, 1006.4634 [hep-th]

$\frac{\eta}{s}$	$\frac{1}{4\pi}$	$T\tau_\pi$	$\frac{2-\log 2}{2\pi} + \frac{3(16-\pi^2)}{64\pi}\delta$	$\frac{T\kappa}{s}$	$\frac{1}{4\pi^2}\left(1 - \frac{3}{4}\delta\right)$
$\frac{T\lambda_1}{s}$	$\frac{1}{8\pi^2}\left(1 + \frac{3}{4}\delta\right)$	$\frac{T\lambda_2}{s}$	$-\frac{1}{4\pi^2}\left(\log 2 + \frac{3\pi^2}{32}\delta\right)$	$\frac{T\lambda_3}{s}$	0
$\frac{T\kappa^*}{s}$	$-\frac{3}{8\pi^2}\delta$	$T\tau_\pi^*$	$-\frac{2-\log 2}{2\pi}\delta$	$\frac{T\lambda_4}{s}$	0
$\frac{\zeta}{\eta}$	$\frac{2}{3}\delta$	$T\tau_\Pi$	$\frac{2-\log 2}{2\pi}$	$\frac{T\xi_1}{s}$	$\frac{1}{24\pi^2}\delta$
$\frac{T\xi_2}{s}$	$\frac{2-\log 2}{36\pi^2}\delta$	$\frac{T\xi_3}{s}$	0	$\frac{T\xi_4}{s}$	0
$\frac{T\xi_5}{s}$	$\frac{1}{12\pi^2}\delta$	$\frac{T\xi_6}{s}$	$\frac{1}{4\pi^2}\delta$		

Table 1: The transport coefficients, in the notation of (1)-(3), for a marginally (ir)relevant deformation of a conformal theory, at leading order in the deformation parameter $\delta \equiv (1 - 3c_s^2)$. The holographic equation of state is $\varepsilon = 3(1+\delta)p$.

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Near-equilibrium entropy production and black holes

$$\frac{\nabla_a s^a}{s} = \frac{\eta}{2sT}\sigma_{ab}\sigma^{ab} + \frac{\kappa - 2\lambda_1}{4sT}\sigma_{ab}\sigma^a_c\sigma^{bc} + \left(\frac{A_1}{2s} + \frac{\kappa - \eta\tau_\Pi}{2sT}\right)\sigma_{ab}\left[\langle D\sigma^{ab}\rangle + \frac{1}{3}\sigma^{ab}\left(\nabla\cdot u\right)\right]$$

[Romatschke, 2009]

For dissipationless liquids: $\kappa = 2\lambda_1$ $\quad 2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Weakly coupled N=4 SYM: $\eta/s \sim 1/\lambda^2 \ln\lambda^{-1}\,,\ \lambda_1 \sim T^2/\lambda^4\ln^2\lambda^{-1}\,,\ \kappa \sim T^2/\lambda^2$

[York and Moore, 2008]

Strongly coupled N=4 SYM: $\eta/s = O(\lambda^{-3/2})\,,\ \kappa - 2\lambda_1 = O(\lambda^{-3/2})\,, H(\lambda) = 0$

Conjecture: near-equilibrium hydrodynamic entropy production is generically suppressed at strong coupling

[See also Haehl, Loganayagam, Rangamani, 1412.1090 [hep-th]]

Open questions

Can we prove (holographically or otherwise – in conformal perturbation theory?) that $H(\lambda) = 0$ at NLO at strong coupling?

What happens for non-conformal theories?

Are black holes minimizing entropy production?

Consequences for the entropy production in heavy ion collisions?

THANK YOU!