The AdS/CFT S-matrix

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Outline

- Introduction
- The S-matrix, integrability and symmetries
- Quantum corrections at strong coupling
- A crossing symmetric phase
- Matching with small coupling
- Conclusions

AdS/CFT Correspondence

$\mathcal{N}=4\ SU(N)$ YM $\Leftrightarrow$ Type IIB strings on $AdS_5 \times S^5$

Maldacena

$$\begin{aligned} 1/N &\Leftrightarrow g_{st} \\ \lambda = g_{YM}^2 N &\Leftrightarrow \lambda = R^4/\alpha'^2 \end{aligned}$$

gauge th. operators (Δ) $\Leftrightarrow$ string spectrum (E)

Suppression of string loops $\Leftrightarrow$ large N limit

- Strong/weak coupling duality:

  $\Delta = \Delta(\lambda)$ λ small, $E = E(\frac{1}{\sqrt{\lambda}})$ λ large

- String sigma model is very involved

How to bridge from small to large λ ?

- BPS quantities — Supergravity approximation

- AdS/CFT at large quantum numbers Polyakov

Op. with a large R-charge $\Leftrightarrow$ Strings on pp-waves

Berenstein, Maldacena, Nastase

Long operators: $\mathrm{Tr}(\phi_1\phi_2\cdots\phi_J) \Leftrightarrow$ Semiclassical strings

- Controlled quantum corrections: $\frac{\Delta-\Delta_0}{\Delta_0}$ small
- Operator mixing

How to bridge from small to large λ ?

- Integrability

Integrable structures in perturbative $\mathcal{N}=4$

Lipatov; Minahan, Zarembo
Beisert, Staudacher

$AdS_5 \times S^5$ classical string is integrable

Bena, Polchinski, Roiban

Hyphothesis: Integrability holds for any λ

$\mathcal{N}=4$ Yang-Mills and spin chains

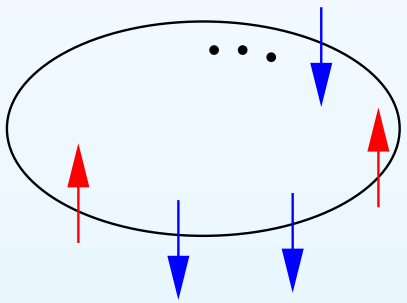
Dilatation operator: $D\mathcal{O} = \Delta\mathcal{O}$

At large N, enought with $\mathcal{O}$ single trace

Equivalent problem: Spin chain dynamics

$\mathcal{O} = \mathrm{Tr}(XYYXY\cdots) \longrightarrow$

$X = \uparrow \quad Y = \downarrow$



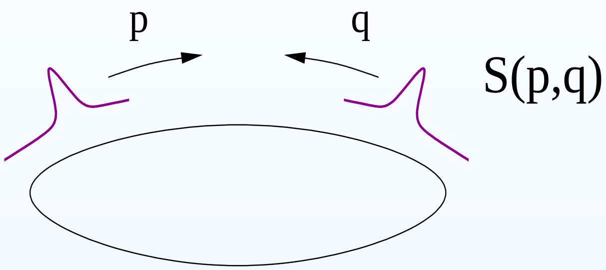
D: spin chain integrable Hamiltonian

- $D_{XY,1-loop}$: Heisenberg ferro. spin chain
- Long range chain: Interaction range $\Leftrightarrow$ Loop order
- Dynamical chain: $XYZ \rightarrow \psi_1\psi_2$

Integrability and asymptotic Bethe ansatz

Integrability $\Leftrightarrow$ Factorized scattering

Central object: $2 \to 2$ scattering matrix



$|p\rangle = \sum_{l=1}^{J} e^{ilp} |\uparrow \cdots \uparrow \underset{l}{\downarrow} \uparrow \cdots \uparrow\rangle$

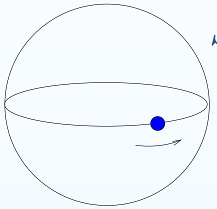
Spectrum: periodicity conditions on wavefunctions

$$e^{ip_j J} = \prod_{k \neq j} S(p_j, p_k)$$

Asymptotic Bethe ansatz Staudacher

- Infinite chain, asymptotic states $\longrightarrow$ S-matrix
- Periodicity conditions
- Spectrum accurate to order λ^J

The AdS/CFT S-matrix

	Gauge th.	Strings	
Vacuum	$\mathrm{Tr}\, X^{J\to\infty} = \, = \cdots \uparrow\uparrow \cdots \uparrow \cdots$	 S^5 $J \to \infty$ (λ fixed)	
Excitations	$\phi_i, \partial_\mu X, \psi_k$	8b + 8f	BMN
Symm. algebra	$psu(2,2\|4)$		
Residual symm.	$psu(2\|2)^2 \times R$		
Enlarged symm. (introduce momentum $\|p\rangle$)	$psu(2\|2)^2 \times R^3$		

Beisert

$$\boxed{S_{su(2|2)} = \bar{S}_0 \, \hat{S}_{su(2|2)}} \longrightarrow \boxed{S_{\mathcal{N}=4} = S_0 \, \hat{S}_{su(2|2)} \, \hat{S}'_{su(2|2)}}$$

$\hat{S}_{su(2|2)}$: uniquely fixed flavour structure $\bar{S}_0, S_0$: scalar factors

The dressing phase

$$S_0(p,q;\lambda) = e^{i\theta(p,q;\lambda)}$$

where

Beisert, Klose

$$\theta = \sqrt{\lambda} \sum_{r=2}^{\infty} \sum_{s>r}^{\infty} c_{r,s}(\lambda) \Big(q_r(p)\, q_s(q) - q_r(q)\, q_s(p) \Big)$$

q_r : tower of conserved charges ($q_1(p) = p$, $q_2(p) \sim E - J$)

- Strong coupling

$$c_{r,s}(\lambda) = c_{r,s}^{(0)} + \frac{1}{\sqrt{\lambda}}\, c_{r,s}^{(1)} + \cdots$$

Classical strings $\longrightarrow$ $c_{r,s}^{(0)} = \frac{1}{2\pi}\, \delta_{r+1,s}$ Arutyunov, Frolov, Staudacher

- Small coupling: $\theta = 0$ up to three-loops

Quantum corrections at strong λ

$$E_{st} = \sqrt{\lambda}\,\epsilon_{cl}(\mathcal{J}) + \delta E(\mathcal{J}) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\,, \qquad \mathcal{J} = \frac{J}{\sqrt{\lambda}}$$

δE: sum over fluctuations around the classical solution

$$\delta E = \frac{1}{2}\sum(\omega_{B,n} - \omega_{F,n})$$

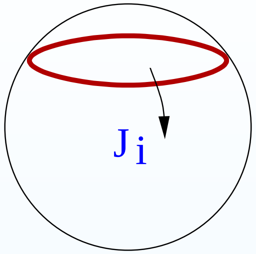
Bethe equations: $e^{ip_j J} = \prod_{k\neq j} S(p_j, p_k) \longrightarrow$ spectrum

- E_{cl}: thermodynamic limit of BE ($J, \#\text{excitations} \to \infty$)
- Two souces of contribution to δE
  - Finite size corrections: $\frac{1}{J}$
  - Quantum correction to the S-matrix: $\frac{1}{\sqrt{\lambda}} \longleftrightarrow c^{(1)}_{r,s}$

Circular strings

2-spin rigid circular strings on $R_t \times S^3$ or $AdS_3 \times S^1$

Frolov, Tseytlin



The frequency sum can be divide in two pieces at large $\mathcal{J}$

$$\frac{1}{2}(\omega_{B,n} - \omega_{F,n}) \to e_1(n),\; e_2(n/\mathcal{J})$$

Beisert, Tseytlin; Schäfer-Nameki

- Fluctuations with finite mode number n

$$\delta E_1 = \sum e_1(n)$$: finite size correction, $\mathcal{O}(\frac{1}{J})$

($p = \frac{n}{J} \to 0$ as $J \to \infty$, n fixed)

- Fluctuations with finite $z = \frac{n}{\mathcal{J}} = \sqrt{\lambda}\, p$

$$\delta E_2 = \mathcal{J} \int dz\, e_2(z)$$: quantum correction, $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$

Quantum corrections to the dressing phase

$$c_{r,s}^{(1)} = \frac{(-1)^{r+s} - 1}{\pi} \frac{2(r-1)(s-1)}{(r+s-2)(s-r)}$$

Hernandez, EL

Tests:

- 2-spin circular strings

  On S^3: checked up to $1/\mathcal{J}^{101}$!!

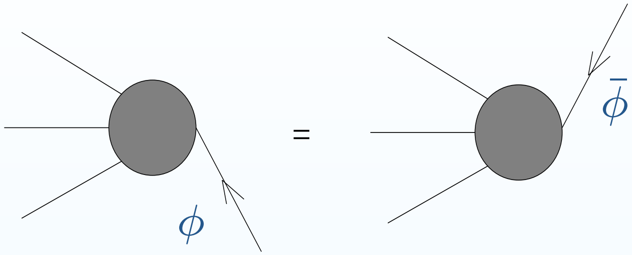
$$\delta E_2 = -\frac{m^6}{3\,\mathcal{J}^5} + \frac{m^8}{3\,\mathcal{J}^7} - \frac{49\,m^{10}}{120\,\mathcal{J}^9} + \frac{2\,m^{12}}{5\,\mathcal{J}^{11}} - \ \ldots$$

  On $AdS_3 \times S^1$: up to $1/\mathcal{J}^{15}$

$$\delta E_2 = \frac{(m-k)^3\,m^3}{3\,\mathcal{J}^5}\left[1 - \frac{3k^2-8km}{2\,\mathcal{J}^2} + \frac{75k^4-455k^3m+679k^2m^2-153km^3+29m^4}{40\,\mathcal{J}^4} - \right.$$

- 3-spin circular strings Freyhult, Kristjansen
- Universality Gromov, Viera

Crossing symmetry



In relativistic integrable QFT

- S-matrix determined by symmetries up to a global phase
- The global phase can be fixed by crossing symmetry

AdS/CFT dispersion relation does not have relativistic inv.

$$E(p) = \pm\sqrt{1 + 16g^2 \sin^2\left(\tfrac{1}{2}p\right)}\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

Beisert

But still admits particle/hole interpretation

Hyphothesis: Crossing symmetry holds for $AdS_5 \times S^5$ strings

Janik

Implementation of crossing

Janik

$$\frac{x^+}{x^-} = e^{ip}\,, \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \longrightarrow$$ torus

- Period ω_1: $p \to p + 2\pi$
- Crossing symmetry ($x^\pm \to 1/x^\pm$): half period ω_2

$$(S_0)_{12}\,(S_0)_{\bar{1}2} = \left[\frac{x_2^-}{x_2^+}\,\frac{x_1^- - x_2^+}{x_1^+ - x_2^+}\,\frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-}\right]^2 \equiv h_{12}^2$$

Double crossing: $1 \to \bar{1} \to \bar{\bar{1}} = 1$

$(S_0)_{\bar{\bar{1}}2} = (h_{\bar{1}2}/h_{12})^2 \neq (S_0)_{12}$: non-trivial monodromy

Define: $\theta = \theta^{\rm odd} + \theta^{\rm even}$, $S_0 = e^{i\theta}$

$$\theta^{\rm odd}_{12} + \theta^{\rm odd}_{\bar{1}2} = \log\frac{h_{12}}{h_{\bar{1}2}}\,, \qquad \theta^{\rm even}_{12} + \theta^{\rm even}_{\bar{1}2} = \log h_{12}\,h_{\bar{1}2}$$

A crossing symmetric phase

$$\theta_{12} = \sum g^{1-n}\,\theta_{12}^{(n)}\,, \qquad \theta_{12}^{(n)} = \sum_{r<s} c_{rs}^{(n)} (q_{r1}\,q_{s2} - q_{r2}\,q_{s1})$$

$$c_{rs}^{(n)} = \frac{((-1)^{r+s}-1)(r-1)(s-1)\,B_n}{2\cos(\frac{1}{2}\pi n)\Gamma[n-1]\Gamma[n+1]}\;\frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]}\;\frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]}$$

Beisert, Hernandez, EL

- Even crossing

$$\theta^{\text{even}} = \sum g^{1-2n}\,\theta^{(2n)} \qquad \left(c_{rs}^{(0)} = 2\delta_{r+1,s}\right)$$

- Odd crossing

$$\theta^{\text{odd}} = \theta^{(1)} \qquad c_{rs}^{(1)} = \frac{(-1)^{r+s}-1}{\pi}\;\frac{2(r-1)(s-1)}{(r+s-2)(s-r)}$$

Odd Bernoulli numbers: $B_1 = -\frac{1}{2}\,,\;\; B_{n>1} = 0$

Problems at small coupling

- $c_{rs}(g)$ has a finite expansion in $1/g$ $\quad (c_{rs}^{(n)} = 0,\ n \geq s - r + 3)$
- At small coupling $q_r \to g^{r-1} q_r$

Regular extrapolation to small g

$$\theta^{\text{even}} = \mathcal{O}(g^2)\ , \qquad \theta^{\text{odd}} = \mathcal{O}(g^3)$$

On the gauge theory side

- Trivial phase up to 3-loops: $\theta = \mathcal{O}(g^6)$
- Analytical in $\lambda \sim g^2$

Worsens the 3-loop discrepancy of $\theta^{(0)} = \mathcal{O}(g^4)$

A homogeneuos solution

Crossing determines the dressing phase up to

$$\theta_{12}^{\text{hom}} + \theta_{\bar{1}2}^{\text{hom}} = 0$$

Using $c_{rs}^{(n)} \sim \frac{B_n}{\cos(\frac{1}{2}\pi n)} = -\frac{2\Gamma[n+1]\zeta(n)}{(-2\pi)^n}$

$$c_{rs}^{(n)} = \frac{(1-(-1)^{r+s})(r-1)(s-1)\,\zeta(n)}{(-2\pi)^n\Gamma[n-1]}\,\frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]}\,\frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]}$$

$$\theta^{\text{hom}} = \sum_{n>1} g^{-2n}\,\theta^{(2n+1)}$$

$c_{rs}^{\text{hom}}(g)$ does not have a finite expansion $\longrightarrow$

$\longrightarrow$ adding θ^{hom} will alter the small coupling behaviour

Connecting to small coupling

$$c_{rs}(g) = \sum_{n\geq 0} c_{rs}^{(n)}\, g^{1-n}$$

Analytical prolongation at small g $(c_{rs} \to g^{r+s-2}\, c_{rs})$

$$c_{rs}(g) = -\sum_{n\geq 1} c_{rs}^{(-n)}\, g^{r+s+n-1}$$

Beisert, Eden, Staudacher

$$c_{rs}^{(-n)} \sim \cos(\tfrac{1}{2}\pi n)\zeta(1+n)$$

- $c_{rs}^{(-n)} = 0$ for $n > 0$ odd: expansion in g^2
- First contribution at $\mathcal{O}(g^6)$: $c_{23}^{(-2)} = 4\zeta(3)$

Matches 4-loop gauge th. calculations!

- 4-gluon amplitude $\to \mathrm{Tr}\, X D^S X$ — Bern, Dixon, Kosover, Smirnov
- Dilatation op. in SU(2) sector — Beisert, Roiban

Analytical structure of the phase

Elementary ex. with finite p at strong coupling

$$E = \sqrt{1 + 16 g^2 \sin^2\left(\tfrac{1}{2}p\right)} \sim g \ , \qquad \Delta\varphi \sim p$$

Classical string solutions: **giant magnons** Hofman, Maldacena

- S_0^{cl} has branch cuts at $p_1 = \pm p_2$: condensate of double poles
  Beisert, Hernandez, Lopez
- 2d: on-shell 2-particle exchange $\rightarrow$ double poles
- Magnons can form stable boundstates Dorey

S_0 has double poles at 2-magnon boundstate exchange

Dorey, Hofman, Maldacena

Twist-two operators: $\mathrm{Tr}\,X\,D^S\,X$

$$\Delta = S + f(g)\log S + \mathcal{O}(S^0)\ , \quad S \to \infty$$

$$f(g) = 8g^2 - \tfrac{8}{3}\pi^2 g^4 + \tfrac{88}{45}\pi^4 g^6 - 16\left(\tfrac{73}{630}\pi^6 + 4\zeta(3)^2\right) g^8 + \cdots$$

At strong coupling: folded string rotating in AdS_5

$$f(g) = 4g - \frac{3\log 2}{\pi} + \mathcal{O}(\tfrac{1}{g})$$

Integral equation for any g: $f = 16g^2\sigma(0)$

Beisert, Eden, Staudacher

$$\sigma(t) = \frac{t}{e^t - 1}\left(K(2gt, 0) - 4g^2\int_0^\infty dt' K(2gt, 2gt')\sigma(t')\right)$$

Numerical and analytical check: **smooth interpolation**

Bena, Benvenuti, Klebanov, Scardicchio
Kotikov, Lipatov

Conclusions

- Non-BPS observables from weak to strong coupling
- Almost a proof of the AdS/CFT correspondence
- Main tool: integrability at any λ
- Understand the origin of the dynamical phase

  Rej, Staudacher, Zieme; Sakai, Satoh
- Study further the analytical structure of the phase
- Finite J: wrapping effects