Weighted Hurwitz numbers and hypergeometric τ-functions*

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Factorization of elements in S_n

Question: Given a permutation $h \in S_n$ of cycle type

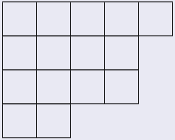
$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\ell(\mu} > 0),$$

what is the number $H^d(\mu)$ of distinct ways it can be written as a product

$$h = (a_1 b_1) \cdots (a_d b_d)$$

of d transpositions ?

Young diagram of a partition. Example $\mu = (5, 4, 4, 2)$



Representation theoretic answer (Frobenius):

$$H^d(\mu) = \sum_{\lambda, |\lambda|=|\mu|} \frac{\chi_\lambda(\mu)}{z_\mu h_\lambda} (\mathrm{cont}_\lambda)^d$$

where $h_\lambda = \left(\det \frac{1}{(\lambda_i - i + j)!}\right)^{-1}$ is the **product of the hook lengths** of the partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda} > 0$,

$$\mathrm{cont}(\lambda) := \sum_{(ij)\in\lambda} (j - i) = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i(\lambda_i - 2i + 1) = \frac{\chi_\lambda((2,(1)^{n-2})h_\lambda}{z_{(2,(1)^{n-2})}}$$

is the **content** sum of the associated Young diagram, $\chi_\lambda(\mu)$ is the **irreducible character** of representation λ evaluated in the conjugacy class μ, and

$$z_\mu := \prod_i i^{m_{(\mu)^i}}(m_i(\mu))! = |\mathrm{aut}(\mu)|$$

Geometric meaning: simple Hurwitz numbers

Hurwitz numbers: Let $H(\mu^{(1)}, \dots, \mu^{(k)})$ be the number of inequivalent branched n-sheeted covers of the Riemann sphere, with k branch points, and ramification profiles $(\mu^{(1)}, \dots, \mu^{(k)})$ at these points.

The **genus** of the covering curve is given by the **Riemann-Hurwitz formula**: $\quad 2-2g = \ell(\lambda) + \ell(\mu) - d, \quad d := \sum_{i=1}^{l} \ell^*(\mu^{(i)})$

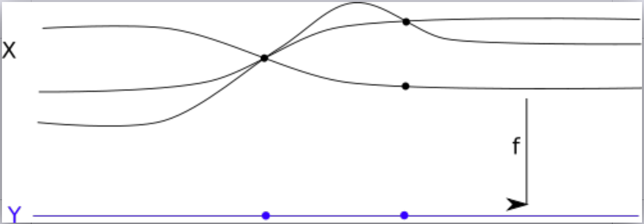
where $\ell^*(\mu) := |\mu| - \ell(\mu)$ is the **colength** of the partition.
The **Frobenius-Schur** formula expresses this in terms of characters:

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \sum_{\lambda, |\lambda|=n=|\mu^{(i)}|} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{z_{\mu^{(i)}}}$$

In particular, choosing only simple ramifications $\mu^{(i)} = (2, (1)^{n-2})$ at $d = k-1$ points and one further arbitrary one μ at a single point, say, 0, we have the **simple Hurwitz number**:

$$H^d(\mu) := H((2,(1)^{n-1}), \dots, (2,(1)^{n-1}), \mu).$$

3-sheeted branched cover with ramification profiles (3) and $(2, 1)$



Double Hurwitz numbers

Double Hurwitz numbers: The double Hurwitz number (Okounkov (2000)), defined as

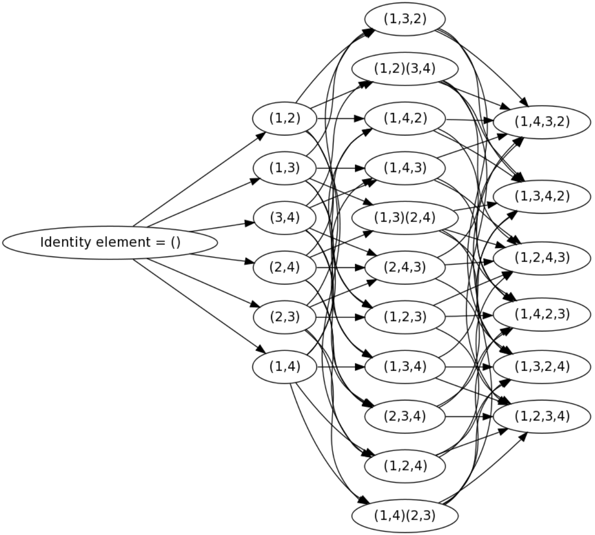
$$\mathrm{Cov}_d(\mu,\nu) = H^d_{\exp}(\mu,\nu)) := H((2,(1)^{n-1}),\ldots,(2,(1)^{n-1}),\mu,\nu).$$

has the ramification type (μ,ν) at two points, say $(0,\infty)$, and simple ramification $\mu^{(i)} = (2,(1)^{n-2})$ at d other branch points.
Combinatorially: This equals the number of d-step paths in the **Cayley graph** of S_n generated by transpositions, starting at an element $h \in C_\mu$ and ending in the conjugacy class C_ν.
Here $\{C_\mu, |\mu| = n \in \mathbf{C}[S_n])\}$ is defined to be the basis of the group algebra $\mathbf{C}[S_n]$ consisting of the sums over all elements h in the various conjugacy classes of cycle type μ.

$$C_\mu = \sum_{h\in \mathrm{conj}(\mu)} h.$$

Example: Cayley graph for S_4 generated by all transpositions

Transpositioncayleyons4.png 867×779 pixels 14-08-23 10:17 PM



τ-function generating functions for Hurwitz numbers

Define

$$\tau^{mKP(u,z)}(N,\mathbf{t}) := \sum_\lambda r_\lambda^{(u,z)}(N) h_\lambda^{-1} S_\lambda(\mathbf{t})$$
$$\tau^{2DToda(u,z)}(N,\mathbf{t},\mathbf{s}) := \sum_\lambda r_\lambda^{(u,z)}(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s})$$

where
$$r_\lambda^{(u,z)}(N) := \prod_{(ij)\in\lambda} r_{N+j-i}^{(u,z)}, \quad r_j^{(u,z)} := ue^{jz}$$

and
$$\mathbf{t} = (t_1, t_2, \dots), \quad \mathbf{s} = (s_1, s_2, \dots)$$

are the KP and 2D Toda flow variables.

mKP Hirota bilinear relations for $\tau_g^{mKP}(N, \mathbf{t})$, $\mathbf{t} := (t_1, t_2, \dots)$, $N \in \mathbf{Z}$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t}, z)} \tau_g^{mKP}(N, \mathbf{t} - [z^{-1}]) \tau_g^{mKP}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}]) = 0$$

$$\xi(\delta\mathbf{t}, z) := \sum_{i=1}^{\infty} \delta t_i \, z^i, \quad [z^{-1}]_i := \frac{1}{i} z^{-i}, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots)$$

$2D$ Toda Hirota bilinear relations for $\tau_g^{2Toda}(N, \mathbf{t}, \mathbf{s})$, $\mathbf{s} := (s_1, s_2, \dots)$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t}, z)} \tau_g^{2Toda}(N, \mathbf{t} - [z^{-1}], \mathbf{s}) \tau_g^{2Toda}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}], \mathbf{s}) =$$
$$\oint_{z=0} z^{N-N'} e^{-\xi(\delta\mathbf{s}, z)} \tau_g^{2Toda}(N+1, \mathbf{t}, \mathbf{s} - [z]) \tau_g^{2Toda}(N'-1, \mathbf{t}, \mathbf{s} + \delta\mathbf{s} + [z])$$
$$[z]_i := \frac{1}{i} z^i, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots), \ \delta\mathbf{s} := (\delta s_1, \delta s_2, \dots)$$

Hypergeometric τ-functions as generating functions for Hurwitz numbers

For $N = 0$, we have

$$r_\lambda^{(u,z)}(0) = u^{|\lambda|} e^{z\,\mathrm{cont}(\lambda)}$$

Using the **Frobenius character formula**:

$$S_\lambda(\mathbf{t}) = \sum_{\mu,\,|\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{Z_\mu} P_\mu(\mathbf{t})$$

where we restrict to

$$it_i := p_i, \quad is_i := p_i'$$

and the P_μ's are the **power sum symmetric functions**

$$P_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}, \quad p_i := \sum_{a=1}^{n} x_a^i, \quad p_i' := \sum_{a=1}^{n} y_a^i,$$

Hypergeometric τ-functions as generating functions for Hurwitz numbers

$$r_\lambda^{(u,z)} := r_\lambda^{(u,z)}(0) = u^{|\lambda|} e^{z\,\mathrm{cont}(\lambda)}$$

$$\tau^{(u,z)}(\mathbf{t}) := \tau^{KP(u,z)}(0, \mathbf{t}) = \sum_\lambda u^{|\lambda|} h_\lambda^{-1} e^{z\,\mathrm{cont}(\lambda)} S_\lambda(\mathbf{t})$$
$$= \sum_{n=0}^\infty u^n \sum_{d=0}^\infty \frac{z^d}{d!} \sum_{\mu, |\mu|=n} H^d(\mu) P_\mu(\mathbf{t})$$
$$\tau^{2D(u,z)}(\mathbf{t}, \mathbf{s}) := \tau^{2DToda(u,z)}(0, \mathbf{t}, \mathbf{s}) = \sum_\lambda u^{|\lambda|} e^{z\,\mathrm{cont}(\lambda)} S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s})$$
$$= \sum_{n=0}^\infty u^n \sum_{d=0}^\infty \frac{z^d}{d!} \sum_{\mu, \nu, |\mu|=\nu|=n} H^d_{\exp}(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s})$$

These are therefore **generating functions** for the **single and double Hurwitz numbers.**

Fermionic representation of KP and 2D Toda τ-functions

$$\tau^{mKP(u,z)}(N,\mathbf{t}) = \langle N|\hat{\gamma}_+(\mathbf{t})u^{\hat{F}_1}e^{z\hat{F}_2}\hat{\gamma}_-(1,0,0\dots)|N\rangle$$
$$\tau^{2DToda(u,z)}(N,\mathbf{t},\mathbf{s}) = \langle N|\hat{\gamma}_+(\mathbf{t})u^{\hat{F}_1}e^{z\hat{F}_2}\hat{\gamma}_-(\mathbf{s})|N\rangle$$

where the fermionic creation and annihiliation operators $\{\psi_i,\psi_i^\dagger\}_{i\in\mathbf{Z}}$ satisfy the usual **anticommutation relations** and **vacuum state** $|0\rangle$ vanishing conditions

$$[\psi_i,\,\psi_j^\dagger]_+ = \delta_{ij} \quad \psi_i|0\rangle = 0, \quad \text{for } i<0, \quad \psi_i^\dagger|0\rangle = 0, \quad \text{for } i \geq 0,$$

$$\hat{F}_k := \frac{1}{k}\sum_{j\in\mathbf{Z}} j^k : \psi_j\psi_j^\dagger$$

$$\hat{\gamma}_+(\mathbf{t}) = e^{\sum_{i=1}^\infty t_iJ_i}, \quad \hat{\gamma}_-(\mathbf{s}) = e^{\sum_{i=1}^\infty s_iJ_i}, \quad J_i = \sum_{k\in\mathbf{Z}}\psi_k\psi_{k+i}^\dagger, \quad i\in\mathbf{Z}.$$

Question: How general is this?

Is this just a unique case? Or are there other KP or 2D Toda τ-functions that are **generating functions** for enumerative geometrical / combinatorial invariants?

Answer: Very general

There is an infinite dimensional variety of such τ-functions. This particular class consists of τ-functions of **hypergeometric type**:

$$\tau(N, \mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_\lambda(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s})$$

where $r_\lambda(N)$ is given by a **content product formula**

$$r_\lambda(N) = \prod_{(ij)\in\lambda} r_{N+j-i}$$

for an infinite sequence $\{r_i\}_{i\in\mathbf{Z}}$ of (real or complex) numbers.

Weighted Hurwitz numbers and their transforms

Every such τ-function can be used as a generating function for enumerative geometric/combinatorial invariants of the Hurwitz type. Moreover, by application of suitable **symmetries**, these can be transformed into other τ-functions, that are *not* of this class, but which are **generating functions** for:

- **Gromov-Witten invariants** (intersection indices on moduli spaces of marked Riemann surfaces). (Related to Hurwitz numbers by the **ELSV formula**.)
- **Hodge integrals** (i.e. GW combined with Hodge classes) (Also related to Hurwitz numbers by the ELSV formula.)
- **Donaldson-Thomas invariants** (e.g. of toric Calabi-Yau manifolds)
- This also underlies the (Eynard-Orantin) programme of **Topological recursion**.

Weight generating functions

In all cases we have a **weight generating function**

$$G(z) = 1 + \sum_{i=1}^{\infty} G_i z^i \quad (= \exp^z \text{ for single and double Hurwitz numbers})$$

and a **content product formula**

$$r_j^G := G(jz), \quad r_\lambda^G = \prod_{(ij)\in\lambda} G((j-i)z), \quad T_j = \ln(\prod_{i=1}^{j} r_j),$$

Hypergeometric 2D Toda τ-function: generalized Hurwitz generating function

$$\tau^G(\mathbf{t}, \mathbf{s}) = \sum_\lambda r_\lambda^G S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) = \sum_{k=0}^{\infty} \sum_{\substack{\mu,\nu,\\ |\mu|=|\nu|}} F_G^d(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}) z^d.$$

Fermionic representation of hypergeometric 2D Toda τ-functions

$$\tau^{G(z),2DToda}(N,\mathbf{t},\mathbf{s}) = \langle N|\hat{\gamma}_+(\mathbf{t})e^{\sum_{i\in\mathbf{Z}} T_i :\psi_i\psi_i^\dagger:}\hat{\gamma}_-(\mathbf{s})|N\rangle$$

where the fermionic creation and annihiliation operators $\{\psi_i, \psi_i^\dagger\}_{i\in\mathbf{Z}}$ satisfy the usual **anticommutation relations** and **vacuum state** $|0\rangle$ vanishing conditions

$$[\psi_i,\, \psi_j^\dagger]_+ = \delta_{ij} \quad \psi_i|0\rangle = 0, \quad \text{for } i<0, \quad \psi_i^\dagger|0\rangle = 0, \quad \text{for } i\geq 0,$$

$$\hat{\gamma}_+(\mathbf{t}) = e^{\sum_{i=1}^\infty t_i J_i}, \quad \hat{\gamma}_-(\mathbf{s}) = e^{\sum_{i=1}^\infty s_i J_i}, \quad J_i = \sum_{k\in\mathbf{Z}} \psi_k\psi_{k+i}^\dagger, \quad i\in\mathbf{Z}.$$

Definition (Paths in the Caley graph and signature)

A **d-step path in the Cayley graph of S_n** (generated by all transpositions) is an **ordered sequence**

$$(h,\ (a_1\, b_1)h,\ (a_2\, b_2)(a_1\, b_1)h,\ \dots,\ (a_d\, b_d)\cdots(a_1\, b_1)h)$$

of $d+1$ elements of S_n. If $h \in \mathrm{cyc}(\nu)$ and $g \in \mathrm{cyc}(\mu)$, the path will be referred to as going *from* $\mathrm{cyc}(\nu)$ *to* $\mathrm{cyc}(\mu)$.

If the sequence $b_1, b_2, \dots, b_d$ is either weakly or strictly increasing, then the path is said to be **weakly** (resp. **strictly**) **monotonic**.

The **signature** of the path $(a_d\, b_d)\cdots(a_1\, b_1)h$ is the partition λ of weight $|\lambda| = d$ whose parts are equal to the number of times each particular number b_i appears in the sequence $b_1, b_2, \dots, b_d$, expressed in weakly decreasing order.

Definition (Jucys-Murphy elements)

The **Jucys-Murphy elements** $(\mathcal{J}_1, \dots, \mathcal{J}_n)$

$$\mathcal{J}_b := \sum_{a=1}^{b-1} (ab), \quad b = 2, \dots n, \quad \mathcal{J}_1 := 0$$

are a set of **commuting elements of the group algebra $\mathbf{C}[S_n]$**

$$\mathcal{J}_a \mathcal{J}_b = \mathcal{J}_b \mathcal{J}_a.$$

Definition (Two bases of the center $\mathbf{Z}(\mathbf{C}(S_n))$ of the group algebra)

Cycle sums: $$C_\mu := \sum_{h \in \text{cyc}(\mu)} h$$

Orthogonal idempotents: $$F_\lambda := h_\lambda \sum_{\mu, |\mu| = |\lambda| = n} \chi_\lambda(\mu) C_\mu, \quad F_\lambda F_\mu = F_\lambda \delta_{\lambda\mu}$$

Theorem (Jucys, Murphy)

If

$$f \in \Lambda_n, \quad f(\mathcal{J}_1, \dots, \mathcal{J}_n) \in \mathbf{Z}(\mathbf{C}[S_n].$$

and

$$f(\mathcal{J}_1, \dots, \mathcal{J}_n)F_\lambda = f(\{j-i\})F_\lambda, \quad (ij) \in \lambda.$$

Let

$$G(z, \mathbf{x}) = \prod_{a=1}^{\infty} G(zx_a) \in \Lambda, \qquad \hat{G}(z\mathcal{J}) := G(z, \mathcal{J}) \in \mathbf{Z}(\mathbf{C}[S_n])$$

then

Corollary

$$\hat{G}(z\mathcal{J})F_\lambda = \prod_{(ij)\in\lambda} G(z(j-i))F_\lambda$$

Weighted path enumeration

Let $m^\lambda_{\mu\nu}$ be the number of paths $(a_1 b_1)\cdots(a_{|\lambda|}b_{|\lambda|})h$ of signature λ starting at an element in the conjugacy class cyc μ with cycle type μ and ending in cyc ν.

Definition

The **weighting factor** for paths of signature λ, $|\lambda| = d$ is defined to be

$$G_\lambda := \prod_{i=1}^{\ell(\lambda)} G_{\lambda_i}.$$

Then
$$G(z,\mathcal{J})C_\mu = \sum_{d=1}^{\infty} Z_\nu F^d_G(\mu,\nu) C_\nu z^d,$$

where
$$F^d_G(\mu,\nu) = \frac{1}{n!}\sum_{\lambda,\ |\lambda|=d} G_\lambda m^\lambda_{\mu\nu}$$

is the **weighted sum** over all such d-step paths, with weight G_λ.

Theorem (Hypergeometric τ-functions as generating function for weighted paths)

Combinatorially,

$$\tau^G(\mathbf{t},\mathbf{s}) = \sum_{\lambda} r_\lambda^G S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) = \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu, \\ |\mu|=|\nu|}} F_G^d(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}) z^d.$$

is the generating function for the numbers $F_G^d(\mu,\nu)$ of weighted d-step paths in the Cayley graph, starting at an element in the conjugacy class of cycle type μ and ending at the conjugacy class of type ν, with weights of all **weakly monotonic paths of type λ** *given by G_λ.*

Suppose the **generating function** $G(z)$ and its **dual** $\tilde{G}(z) := \frac{1}{G(-z)}$ can be represented as infinite products

$$G(z) = \prod_{i=1}^{\infty}(1 + zc_i), \quad \tilde{G}(z) = \prod_{i=1}^{\infty} \frac{1}{1 - zc_i}.$$

Define the **weight for a branched covering having a pair of branch points with ramification profiles of type (μ, ν), and k additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$** to be:

$$W_G(\mu^{(1)}, \dots, \mu^{(k)}) := m_\lambda(\mathbf{c}) = \frac{1}{|\mathrm{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \le i_1 < \cdots < i_k} c_{i_\sigma(1)}^{\ell^*(\mu^{(1)})} \cdots c_{i_\sigma(k)}^{\ell^*(\mu^{(k)})},$$

$$W_{\tilde{G}}(\mu^{(1)}, \dots, \mu^{(k)}) := f_\lambda(\mathbf{c}) = \frac{(-1)^{\ell^*(\lambda)}}{|\mathrm{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \le i_1 \le \cdots \le i_k} c_{i_\sigma(1)}^{\ell^*(\mu^{(1)})}, \cdots c_{i_\sigma(k)}^{\ell^*(\mu^{(k)})},$$

where the partition λ of length k has **parts $(\lambda_1, \dots, \lambda_k)$ equal to the colengths $(\ell^*(\mu^{(1)}), \dots, \ell^*(\mu^{(k)}))$**, arranged in weakly decreasing order, and $|\mathrm{aut}(\lambda)|$ is the product of the factorials of the multiplicities of the parts of λ.

Definition (Weighted geometrical Hurwitz numbers)

The **weighted geometrical Hurwitz numbers** for n-sheeted branched coverings of the Riemann sphere, having a pair of branch points with ramification profiles of type (μ, ν), and k additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$ are defined to be

$$H^d_G(\mu,\nu) := \sum_{k=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{k} \ell^*(\mu^{(i)})=d}} W_G(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

$$H^d_{\tilde{G}}(\mu,\nu) := \sum_{k=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{d} \ell^*(\mu^{(i)})=d}} W_{\tilde{G}}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

where $\sum'$ denotes the sum over all partitions other than the cycle type of the identity element.

Theorem (Hypergeometric τ-functions as generating function for weighted branched covers)

Geometrically,

$$\tau^G(\mathbf{t},\mathbf{s}) = \sum_\lambda r_\lambda^G S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) = \sum_{d=0}^\infty \sum_{\substack{\mu,\nu,\\ |\mu|=|\nu|}} H_G^d(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}) z^d.$$

is the generating function for the numbers $H_G^d(\mu,\nu)$ of such weightedn-fold branched coverings of the sphere, with a pair of specified branch points having ramification profiles (μ,ν) and genus given by the **Riemann-Hurwitz formula**

$$2 - 2g = \ell(\mu) + \ell(\nu) - d.$$

Corollary (combinatorial-geometrical equivalence)

$$H_G^d(\mu,\nu) = F_G^d(\mu,\nu)$$

Example: Belyi curves: strongly monotone paths

$$G(z) = E(z) := 1 + z, \quad E(z, \mathcal{J}) = \prod_{a=1}^{n} (1 + z\mathcal{J}_a)$$

$$E_1 = 1, \quad G_j = E_j = 0 \text{ for } j > 1,$$

$$r_j^E = 1 + zj, \quad r_\lambda^E(z) = \prod_{((ij)\in\lambda} (1 + z(j - i)),$$

$$T_j^E = \sum_{k=1}^{j} \ln(1 + kz), \quad T_{-j}^E = -\sum_{k=1}^{j-1} \ln(1 - kz), \quad j > 0.$$

Example: Belyi curves: strongly monotone paths

The coefficients $F^d_E(\mu,\nu)$ are double Hurwitz numbers for **Belyi curves**, which enumerate n-sheeted branched coverings of the Riemann sphere having three ramification points, with ramification profile types μ and ν at 0 and ∞, and a single additional branch point, with $n-d$ preimages.

The **genus** of the covering curve is again given by the **Riemann-Hurwitz formula**:

$$2-2g=\ell(\lambda)+\ell(\mu)-d.$$

Combinatorially, $F^d_E(\mu,\nu)$ enumerates d-step paths in the **Cayley graph** of S_n from an element in the conjugacy class of cycle type μ to the class of cycle type ν, that are **strictly monotonically increasing** in their second elements.

Example: Composite Hurwitz numbers: multimonotone paths

$$G(z) = E^k(z) := (1+z)^k, \quad E^k(z, \mathcal{J}) = \prod_{a=1}^{n} (1 + z\mathcal{J}_a)^k, \quad E_i^k = \binom{k}{i}$$

$$r_j^{E^k} = (1+zj)^k \quad r_\lambda^{E^k}(z) = \prod_{(ij)\in\lambda} (1 + z(j-i))^k,$$

$$T_j^{E^k} = k \sum_{i=1}^{j} \ln(1+iz), \quad T_{-j}^{E^k} = -k \sum_{i=1}^{j-1} \ln(1-iz), \quad j > 0.$$

Composite Hurwitz numbers: multimonotone paths)cont'd)

The coefficients $F^d_{E^k}(\mu,\nu)$ are double Hurwitz numbers that enumerate branched coverings of the Riemann sphere with ramification profile types μ and ν at 0 and ∞, and k additional branch points, such that the sum of the colengths of the ramification profile type is equal to k.

The **genus** is again given by the **Riemann-Hurwitz formula**:

$$2-2g=\ell(\lambda)+\ell(\mu)-d.$$

Combinatorially, $F^d_{E^k}(\mu,\nu)$ enumerates d-step paths in the Cayley graph of S_n, formed from consecutive transpositions, from an element in the conjugacy class of cycle type μ to the class of cycle type ν, that consist of a sequence of k **strictly monotonically increasing subsequences** in their second elements.

Example: Signed Hurwitz numbers: weakly monotone paths

$$G(z) = H(z) := \frac{1}{1-z}, \quad H(z, \mathcal{J}) = \prod_{a=1}^{n} (1 - z\mathcal{J}_a)^{-1}, \quad H_i = 1, \ i \in \mathbf{N}^+$$

$$r_j^H = (1 - zj)^{-1}, \quad r_\lambda^H(z) = \prod_{(ij)\in\lambda} (1 - z(j-i))^{-1},$$

$$T_j^H = -\sum_{i=1}^{j} \ln(1 - iz), \quad T_{-j}^E = \sum_{i=1}^{j-1} \ln(1 + iz), \quad j > 0.$$

Signed Hurwitz numbers: weakly monotone paths (cont'd)

The coefficients $H^d_H(\mu,\nu)$ are double Hurwitz numbers that enumerate n-sheeted branched coverings of the Riemann sphere curves with branch points at 0 and ∞ having ramification profile types μ and ν, and an arbitrary number of further branch points, such that the sum of **the complements of their ramification profile lengths** (i.e., the "defect" in the Riemann Hurwitz formula) **is equal to d**. The latter are counted with a sign, which is $(-1)^{n+d}$ times the parity of the number of branch points .

The **genus** g is again given by the **Riemann-Hurwitz formula**:

$$2-2g = \ell(\lambda)+\ell(\mu)-d$$

Combinatorially, $H^d_H(\mu,\nu) = F^d_H(\mu,\nu)$ enumerates d-step paths in the Cayley graph of S_n from an element in the conjugacy class of cycle type μ to the class cycle type ν, that are **weakly monotonically increasing** in their second elements.

Weight generating functions for Quantum Hurwitz numbers

$$G(z) = E(q,z) := \prod_{k=0}^{\infty}(1+q^k z) = \sum_{k=0}^{\infty} E_k(q) z^k,$$

$$= e^{-\mathrm{Li}_2(q,-z)}, \quad \mathrm{Li}_2(q,z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)} \text{ (quantum dilogarithm)}$$

$$E_i(q) := \prod_{j=0}^{i} \frac{q^j}{1-q^j},$$

$$E(q,\mathcal{J}) = \prod_{a=1}^{n} \prod_{i=0}^{\infty} (1 + q^i z \mathcal{J}_a),$$

$$r_j^{E(q)} = \prod_{k=0}^{\infty}(1+q^k z j), \quad r_\lambda^{E(q)}(z) = \prod_{k=0}^{\infty} \prod_{(ij)\in\lambda}^{n} (1 + q^k z(j-i)),$$

$$T_j^{E(q)} = -\sum_{i=1}^{j} \mathrm{Li}_2(q,-zi).$$

Symmetrized monotone monomial sums

Using the sums:

$$\sum_{\sigma\in S_k}\sum_{0\le i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$
$$=\sum_{\sigma\in S_k}\frac{x_{\sigma(1)}^{k-1}x_{\sigma(2)}^{k-2}\cdots x_{\sigma(k-1)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$
$$\sum_{\sigma\in S_k}\sum_{1\le i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$
$$=\sum_{\sigma\in S_k}\frac{x_{\sigma(1)}^{k}x_{\sigma(2)}^{k-1}\cdots x_{\sigma(k)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$

Theorem (Quantum Hurwitz numbers (cont'd))

$$\tau^{E(q,z)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} z^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^d_{E(q)}(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \quad \textit{where}$$

$$H^d_{E(q)}(\mu,\nu) := \sum_{d=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

with $$W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{k!} \sum_{\sigma\in S_k} \sum_{0\le i_1<\cdots<i_k}^{\infty} q^{i_1\ell^*(\mu^{(\sigma(1))})} \cdots q^{i_k\ell^*(\mu^{(\sigma(k))})}$$

$$= \frac{1}{k!} \sum_{\sigma\in S_k} \frac{q^{(k-1)\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k-1))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})}) \cdots (1-q^{\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k))})})}$$

are the **weighted (quantum) Hurwitz numbers** *that count the number of branched coverings with genus g given by the* **Riemann-Hurwitz formula***:* $2-2g = \ell(\lambda)+\ell(\mu)-k$. *and sum of colengths d.*

Corollary (Quantum Hurwitz numbers and quantum paths)

The **weighted sum** *over d-step paths in the Cayley graph from an element of the conjugacy class μ to one in the class ν*

$$F^d_{E(q)}(\mu,\nu) := \frac{1}{n!}\sum_{\lambda,\ |\lambda|=d} E(q)_\lambda m^\lambda_{\mu\nu}, \quad E(q)_\lambda = \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{i}\frac{q^j}{1-q^j}$$

is equal to the **weighted Hurwitz number**

$$F^d_{E(q)}(\mu,\nu) = H^d_{E(q)}(\mu,\nu)$$

counting **weighted n-sheeted branched coverings** *of* $\mathbf{P}^1$ *with a pair of branched points of ramification profiles μ and ν, and any number of further branch points, and genus determined by the Riemann-Hurwitz formula*

$$2-2g = \ell(\mu)+\ell(\nu)-d$$

and these are generated by the τ function $\tau^{E(q,z)}(\mathbf{t},\mathbf{s})$.

Bosonic gases

A slight modification consists of replacing the generating function $E(q,z)$ by

$$E'(q,z) := \prod_{k=1}^{\infty}(1+q^k z).$$

The effect of this is simply to replace the weighting factors

$$\frac{1}{1-q^{\ell^*(\mu)}} \quad \text{by} \quad \frac{1}{q^{-\ell^*(\mu)}-1}.$$

If we identify

$$q := e^{-\beta\hbar\omega}, \quad \beta = k_B T,$$

where ω_0 is the lowest frequency excitation in a **gas of identical bosonic particles** and assume the energy spectrum of the particles consists of integer multiples of $\hbar\omega$

$$\epsilon_k = k\hbar\omega,$$

Expectation values of Hurwitz numbers

The relative probability of occupying the energy level ϵ_k is

$$\frac{q^k}{1-q^k} = \frac{1}{e^{\beta\epsilon_k}-1},$$

the **energy distribution of a bosonic gas**.

We may associate the branch points to the states of the gas and view the Hurwitz numbers $H(\mu^{(1)}, \dots \mu^{(l)})$ as **random variables**, with the state energies proportional to the sums over the colengths

$$\epsilon_{\ell^*(\mu^{(i)})} = \hbar\ell^*(\mu^{(i)})\beta\omega_0,$$

and weight

$$\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}} = \frac{1}{e^{\beta\epsilon_{\ell^*(\mu^{(i)})}}-1}$$

Expectation values of Hurwitz numbers

the normalized weighted Hurwitz numbers are expectation values

$$\bar{H}^d_{E'(q)}(\mu,\nu) := \frac{1}{\mathbf{Z}^d_{E'(q)}} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

$$\text{where } W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) = \frac{1}{k!}\sum_{\sigma\in S_k} W(\mu^{(\sigma(1)})\cdots W(\mu^{(\sigma(1)},\cdots,\mu^{(\sigma(k)})$$

$$W(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{e^{\beta\sum_{i=1}^k \epsilon(\mu^{(i)})}-1},$$

$$\mathbf{Z}^d_{E'(q)} := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}).$$

is the **canonical partition function** for total energy $d\hbar\omega$.

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