Discrete parafermions and quantum-group symmetries

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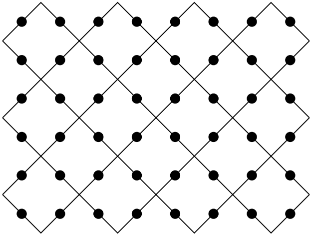
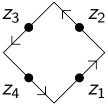
Outline

1. Introduction

Discretely holomorphic functions

- Discrete function: $F(z)$ on midpoints of square lattice $\mathcal{L}$





- Discrete "Cauchy-Riemann" equation:

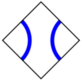
$$e^{\frac{i\pi}{4}}F(z_1) - e^{-\frac{i\pi}{4}}F(z_2) - e^{\frac{i\pi}{4}}F(z_3) + e^{-\frac{i\pi}{4}}F(z_4) = 0$$

- Short-hand notation: $\sum_{\diamond} F(z)\delta z = 0$

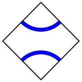
Loop models in Statistical Mechanics

The Temperley-Lieb loop model

- Plaquette configurations:

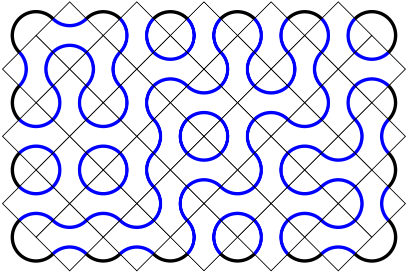


x



y

- Lattice configurations:



- Boltzmann weights: $W(C) = x^{N_x(C)}\ y^{N_y(C)}\ n^{N_\ell(C)}$
- Partition function: $Z = \sum_{\text{config. } C} W(C)$

Loop models in Statistical Mechanics

Correlation functions

- Averaging on Boltzmann weights:

$$\langle f(C) \rangle := \frac{1}{Z} \sum_{C} W(C)\, f(C)\,.$$

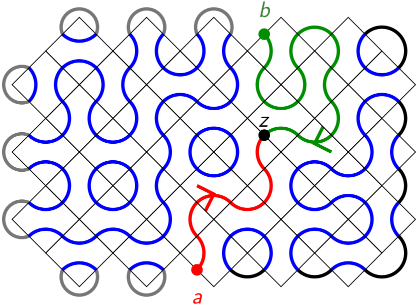
- Two-leg correlation function:

$$G(z_1, z_2) := \frac{1}{Z} \sum_{C|\ z_1, z_2 \in \text{same loop}} W(C)$$

- Phases in scaling limit:
  - Non-critical phase: $G(z_1, z_2) \sim \exp(-|z_1 - z_2|/\xi)$
  - Critical phase: $G(z_1, z_2) \sim |z_1 - z_2|^{-2X_2}$

- "Coulomb-gas" studies $\Rightarrow$ TL model is critical for $0 < n \leq 2$.

Discretely holomorphic observables in loop models



- Pick a pair of boundary points (a, b) $\longrightarrow$ define BC_{ab}.
- Define correlation function:

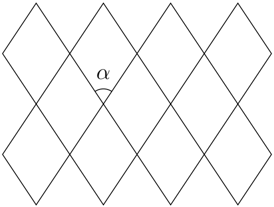
$$F_s(z) := \frac{1}{Z_{ab}} \sum_{C|\ z \in \text{open path}} W(C)\ e^{i\,s\,\theta_{a\to z}(C)}$$

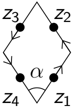
[$\theta_{a\to z}$:= winding angle of red arc from a to z]

- **Theorem:** if $n = 2\sin\frac{\pi s}{2}$ then $\forall \diamond \in \Omega,\ \sum_{\diamond} F_s(z)\delta z = 0.$

Algebraic structure behind discrete holomorphicity?

- Discretely holomorphic observables like F_s exist in various models: TL, $\mathrm{O}(n)$, $\mathbb{Z}_N$ clock models ...
- Rhombic lattice $\Rightarrow$ additional parameter α





Modified Cauchy-Riemann equation:

$$e^{-\frac{i\alpha}{2}}F(z_1)+e^{\frac{i\alpha}{2}}F(z_2)-e^{-\frac{i\alpha}{2}}F(z_3)-e^{\frac{i\alpha}{2}}F(z_4)=0 \qquad (\mathrm{CR}_\alpha)$$

- Observations :
  1. F_s satisfies CR_α when $W \equiv$ **integrable** Boltzmann weights
  2. $\alpha \equiv$ spectral parameter
- Q: general relation discrete holomorphicity $\leftrightarrow$ integrability?

Discrete holomorphicity in Physics and Mathematics

- [Dotsenko,Polyakov 88] : Linear relations for fermions in Ising
- [Smirnov 01–06] : Conf. inv. for interfaces in perco+Ising
- [Cardy,Riva,Rajabpour,YI 06–09] : Discr. holo. in various lattice models, obs. relation to integrability
- [Smirnov,Chelkak,Hongler,Izyurov,Kytölä 09–12] : Scaling limit of interfaces+corr. func. in Ising
- [Duminil-Copin,Smirnov 10] : Proof of connectivity constant for SAW on honeycomb
- [Beaton,de Gier,Guttmann,Jensen 11–12] : Critical boundary parameter for SAW on honeycomb
- [Fendley 12] : Discr. holo. from topological QFT
- [Alam,Batchelor 12] : CR eq $\leftrightarrow$ star-triangle in $\mathbb{Z}_N$ models
- [Hongler,Kytölä,Zahabi 12] : Discr. holo. for non-local currents in Ising, transfer-matrix formalism

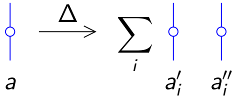
2. The Bernard-Felder construction

Hopf algebras

Bi-algebra structure

- Product $m : \begin{cases} \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \\ a \otimes b \mapsto a.b \end{cases}$

- Coproduct $\Delta : \begin{cases} \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \\ a \mapsto \sum_i a'_i \otimes a''_i \end{cases}$



  - $\Delta(a.b) = \Delta(a).\Delta(b), \quad \Delta(a + \lambda b) = \Delta(a) + \lambda\Delta(b)$
  - $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$

- Example: enveloping algebra of a Lie algebra g
  - g Lie algebra, with bracket $[X_a, X_b] = i\, f_{abc} X_c$
  - $\mathcal{A} := U(g) =$ span(words on alphabet $\{X_a\}$)
  - bracket $\equiv$ commutator ($[a, b] = ab - ba$)
  - Trivial coproduct $\Delta(X_a) = X_a \otimes \mathbf{1} + \mathbf{1} \otimes X_a$

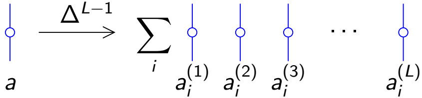
Hopf algebras

Tensor-product representations

- V finite-dimensional vector space
  Map $\pi : \mathcal{A} \to \mathrm{End}(V)$ is a representation of $\mathcal{A}$ iff:
  - π is linear and surjective,
  - π is a morphism: $\quad \pi(ab) = \pi(a)\pi(b)$.
- Coproduct = tool to construct higher-dim. representations:

$$\Delta(a) = \sum_i a_i' \otimes a_i'' \quad \longrightarrow \quad \pi_{12}(a) := \sum_i \pi_1(a_i') \otimes \pi_2(a_i'')$$

- Iterate:



- Example: $\mathcal{A} = U(g)$, for a Lie algebra g

$$\pi^{(L)}(X_a) = \sum_{m=1}^{L} \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underset{\substack{\uparrow \\ m-\text{th}}}{\pi(X_a)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

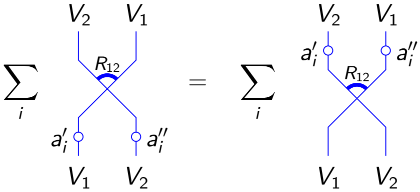
Hopf algebras

The R-matrix

- The two representations $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic.
- Intertwiner $R_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$ such that:

$$\forall a \in \mathcal{A}, \quad R_{12}\ \pi_{12}(a) = \pi_{21}(a)\ R_{12}$$

- Expand coproduct $[\pi_{12}(a) = \sum_i \pi_1(a_i') \otimes \pi_2(a_i'')]$:



- Consistency condition = Yang-Baxter equation:

$$(R_{23} \otimes \mathbf{1}).(\mathbf{1} \otimes R_{13}).(R_{12} \otimes \mathbf{1}) = (\mathbf{1} \otimes R_{12}).(R_{13} \otimes \mathbf{1}).(\mathbf{1} \otimes R_{23})$$

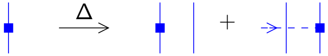
Non-local conserved currents

[Bernard-Felder, 91]

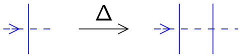
- Generators of $\mathcal{A}$: $\{J_1, J_2 \dots\}$ and $\{\mu_1, \mu_2 \dots\}$.

  Assume the coproduct of $\mathcal{A}$ has the following form:

  $$\Delta(J_k) = J_k \otimes \mathbf{1} + \mu_k \otimes J_k$$



  $$\Delta(\mu_k) = \mu_k \otimes \mu_k$$



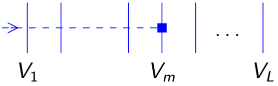
- Iteration of coproduct ⇒ "conserved charges":

  $$Q_k := \Delta^{L-1}(J_k) = \sum_{m=1}^{L} \mu_k \otimes \cdots \otimes \mu_k \otimes \underset{\underset{m}{\uparrow}}{J_k} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

- Non-local currents:

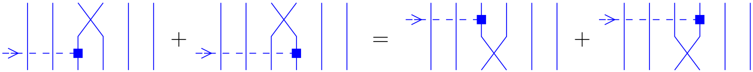
  $$\psi_k(m) := \quad \mu_k \otimes \cdots \otimes \mu_k \otimes \underset{\underset{m}{\uparrow}}{J_k} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

  $\psi_k(m) =$

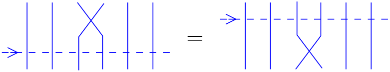


Commutation relations

- From intertwining relations $[R_{12}\ \pi_{12}(a) = \pi_{21}(a)\ R_{12}]$:
  - For $a = J_k$:

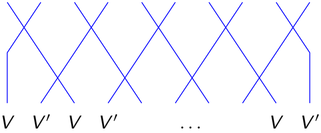


  - For $a = \mu_k$:



- Transfer matrix:

$T =$



- Conservation laws:

$$\forall a \in \mathcal{A}, \quad T.\pi^{(L)}(a) = \pi^{(L)}(a).T$$

The affine quantum group $\mathcal{A} = U_q(\widehat{s\ell_2})$

- Generators: $E_0, E_1, F_0, F_1, T_0, T_1$
  $\{E_0, E_1, F_0, F_1\}$=raising/lowering ops, $\{T_0, T_1\}$=diag. ops.
- Product rules:

$$[T_0, T_1] = 0 \qquad [E_i, F_j] = \delta_{ij} \frac{T_i - T_i^{-1}}{q - q^{-1}}$$

$$T_i E_j T_i^{-1} = q^{2(-1)^{\delta_{ij}}} E_j \qquad T_i F_j T_i^{-1} = q^{2(-1)^{\delta_{ij}+1}} F_j$$

(+higher order rules ...)

- Coproduct rules:

$$\boxed{\begin{aligned}\Delta(E_i) &= E_i \otimes \mathbf{1} + T_i \otimes E_i \\ \Delta(T_i) &= T_i \otimes T_i\end{aligned}} \qquad \Delta(F_i) = F_i \otimes T_i^{-1} + \mathbf{1} \otimes F_i$$

- Introduce $\bar{E}_i := q T_i F_i \quad \Rightarrow \quad \boxed{\Delta(\bar{E}_i) = \bar{E}_i \otimes \mathbf{1} + T_i \otimes \bar{E}_i}$
- BF structure: $\{J_k\} = \{E_0, E_1, \bar{E}_0, \bar{E}_1\} \qquad \{\mu_k\} = \{T_0, T_1\}$.

Evaluation representations of $\mathcal{A} = U_q(\widehat{s\ell_2})$

- Representations are labelled by a complex number u
  Explicit form:

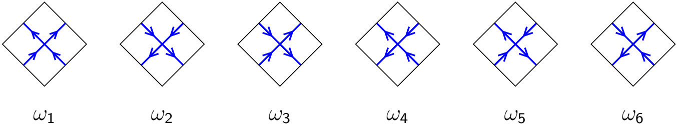
$$\pi_u : \begin{cases} E_0 \mapsto \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} & \bar{E}_0 \mapsto \begin{pmatrix} 0 & u^{-1} \\ 0 & 0 \end{pmatrix} & T_0 \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \\ E_1 \mapsto \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} & \bar{E}_1 \mapsto \begin{pmatrix} 0 & 0 \\ u^{-1} & 0 \end{pmatrix} & T_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \end{cases}$$

- Intertwiner: $R(u/v)\pi_{u,v} = \pi_{v,u}R(u/v)$

$$R(u/v) = \begin{pmatrix} [qu/v] & 0 & 0 & 0 \\ 0 & [u/v] & 1 & 0 \\ 0 & 1 & [u/v] & 0 \\ 0 & 0 & 0 & [qu/v] \end{pmatrix}, \quad [z] = \frac{z - z^{-1}}{q - q^{-1}}$$

Application to the six-vertex model

- Use basis for V_u: $\{\uparrow, \downarrow\}$.
  Plaquette configurations:



- Boltzmann weights:

$$R_{6\mathrm{V}} = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_5 & \omega_4 & 0 \\ 0 & \omega_3 & \omega_6 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}$$

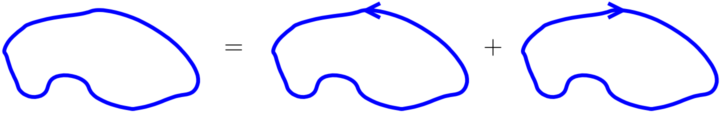
- When $R_{6\mathrm{V}} \equiv R_{U_q(\widehat{s\ell_2})}$, the 6V model is integrable.

3. Mapping to loop models

From the TL model to the 6V model

[Baxter, Kelland, Wu 73]

- Orient each loop independently:



$n = 2\cos 2\pi\lambda$ $\qquad e^{2i\pi\lambda}$ $\qquad e^{-2i\pi\lambda}$

- Partition function:

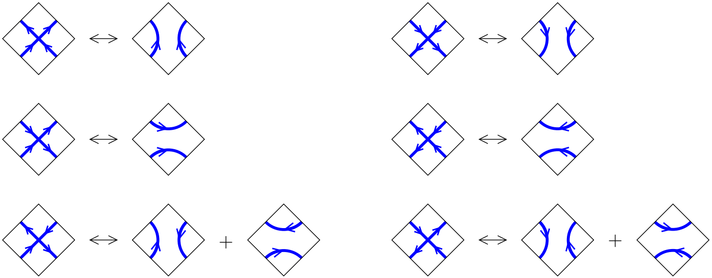
$$Z = \sum_C x^{N_x(C)}\ y^{N_y(C)}\ e^{2i\pi\lambda[N_\ell^+(C) - N_\ell^-(C)]}$$

- Distribute phase factors locally:

$e^{i\alpha\lambda}$ $\qquad e^{-i\alpha\lambda}$

From the TL model to the 6V model (2)

- Vertex configurations:



- Six-vertex weights arising from loop model:

$$\omega_1 = \omega_2 = x, \quad \omega_3 = \omega_4 = y, \quad \begin{cases} \omega_5 = e^{+2i\lambda\alpha} x + e^{-2i\lambda(\pi-\alpha)} y \\ \omega_6 = e^{-2i\lambda\alpha} x + e^{+2i\lambda(\pi-\alpha)} y \end{cases}$$

- Set $q = -e^{2i\lambda\pi}$, $w = e^{-2i\lambda\alpha}$:
  $\omega_1 = \omega_2 = [qw], \quad \omega_3 = \omega_4 = [w] \; \Rightarrow \; \omega_5 = \omega_6 = 1\,.$

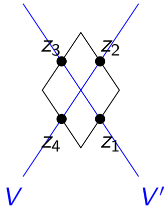
Conserved currents in the 6V model

- $\begin{cases}\Delta(E_0) = E_0 \otimes \mathbf{1} + T_0 \otimes E_0 \\ \Delta(T_0) = T_0 \otimes T_0\end{cases}$ $\Rightarrow$ BF current ψ_0

$$\psi_0(m) = T_0 \otimes T_0 \otimes \cdots \otimes T_0 \otimes \underset{\substack{\uparrow \\ m-\text{th}}}{E_0} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

- Commutation with R-matrix $\Rightarrow$ linear relation:

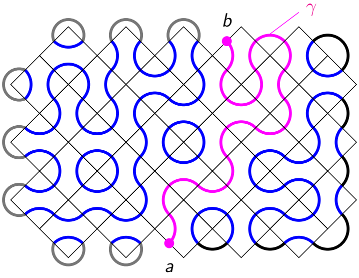
$$\psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) = 0\,.$$



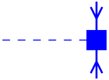
- Similar construction for $E_1, \bar{E}_0, \bar{E}_1 \to \psi_1, \bar{\psi}_0, \bar{\psi}_1$.

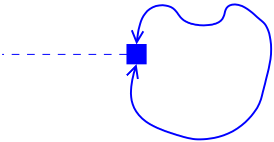
Mapping of conserved currents

What is the meaning of $\langle \psi_0(z) \rangle$ in terms of loops?



$\psi_0(z)$ cannot sit alone on a closed loop

$\psi_0 = u \times$ 

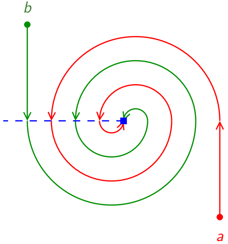
 $= 0$

$$\Rightarrow \langle \psi_0(z) \rangle = \frac{u}{Z} \sum_{C|\ z \in \gamma} W(C) \times (\text{phase factor})$$

Mapping of conserved currents (2)

Identification of phase factors

- $\theta_{b\to z} = \theta_{a\to z} + \pi\,, \qquad q = e^{i\pi(2\lambda-1)}$



- phase factor:

$$e^{i\lambda(\theta_{a\to z}+\theta_{b\to z})} \quad \times \quad q^{\frac{\theta_{a\to z}+\theta_{b\to z}-\pi}{2\pi}} \quad = A\ e^{i(4\lambda-1)\theta_{a\to z}}$$

$\uparrow$ turns $\qquad\qquad$ $\uparrow$ $T_0 \otimes \cdots \otimes T_0$

- $\Rightarrow \langle \psi_0(z) \rangle = \dfrac{uA}{Z} \sum_{C|\ z\in\gamma} W(C)\ e^{i(4\lambda-1)\theta_{a\to z}} = uA \times F_s(z)$

  spin: $s = 4\lambda - 1$ (remember Theorem in Intro)

Mapping of conserved currents (3)

Cauchy-Riemann relation

- Set $u = 1/u' = w^{1/2} \;\Rightarrow\; u/u' = w = e^{-2i\lambda\alpha}$
- Conservation relation:

$$\begin{aligned} & \langle \psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) \rangle = 0 \\ \Leftrightarrow \quad & vF_s(z_1) - uF_s(z_2) - vF_s(z_3) + uF_s(z_4) = 0 \\ \Leftrightarrow \quad & \sum_{\diamond} F_s(z)\,\delta z = 0 \end{aligned}$$

- Conservation of BF current $\Rightarrow$ CR_α relation

Conclusions

- What we have **also** obtained:
  - Boundary CR equation $\leftrightarrow$ integrable K-matrix
  - Discrete parafermions in other models: dilute O(n), chiral Potts (cf R. Weston's talk)
  - Massive regime of chiral Potts: $\bar{\partial} F = m\chi$
- For future work:
  - Observables from $E_0^2, E_0 \otimes E_0$, etc?
  - Find "other half" of CR equations?
  - More relations at roots of unity?

Thank you for your attention!